

THE EFFECT OF NOISE ON THE CHAFEE-INFANTE EQUATION: A NONLINEAR CASE STUDY

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ABSTRACT. We investigate the effect of perturbing the Chafee-Infante scalar reaction diffusion equation, $u_t - \Delta u = \beta u - u^3$, by noise. While a single multiplicative Itô noise of sufficient intensity will stabilise the origin, its Stratonovich counterpart leaves the dimension of the attractor essentially unchanged. We then show that a collection of multiplicative Stratonovich terms can make the origin exponentially stable, while an additive noise of sufficient richness reduces the random attractor to a single point.

1. INTRODUCTION

In this paper we investigate the effect of noise on the Chafee-Infante scalar reaction diffusion equation

$$(1.1) \quad u_t - \Delta u = \beta u - u^3 \quad \text{for } x \in D, \quad \text{with } u|_{\partial D} = 0,$$

where D is a smooth bounded domain in \mathbb{R}^m . We show that the effect of the noise is highly dependent on the precise way in which it is included in the model.

We choose the Chafee-Infante equation since it is the canonical example of those infinite-dimensional gradient systems in which the structure of the global attractor can be fully described, and we recall this theory very briefly in Section 2.

We then consider the effect of adding a single multiplicative noise: in Section 3 we show that an Itô noise $+\sigma u dW_t$ of sufficient intensity will stabilise the origin, while in Section 4 we show that a Stratonovich noise $+\sigma u \circ dW_t$ leaves the complexity of the system, as measured by the dimension of its attractor, essentially unchanged whatever the value of σ .

The order-preserving property of the deterministic model is retained by its stochastic counterparts, and this is the key to the two new results presented here: in Section 5 we adapt the linear result of Caraballo & Robinson [10] to show that the zero solution can be made exponentially stable by the addition of a number of multiplicative noise terms (in the Stratonovich sense) and in Section 6 we give a

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simple proof that the addition of a rich enough additive noise reduces the random attractor to a single random point (cf. Chueshov & Scheutzow [11]).

2. THE DETERMINISTIC CHAFEE-INFANTE EQUATION

The Chafee-Infante equation is probably the best understood deterministic parabolic PDE, and one of the few for which we have a full understanding of the structure of its attractor (see Hale [23] and Henry [24], for example).

Existence and uniqueness results for the deterministic equation are proved in Marion [28] and Robinson [30]: given an initial condition $u_0 \in L^2(D)$ there exists a unique weak solution $u(t; u_0)$ such that for any $T > 0$

$$u(t; u_0) \in L^2(0, T; H_0^1(D)) \cap L^4((0, T) \times D) \cap C([0, T]; L^2(D)),$$

which we can use to define a semigroup $S(t)$ on $L^2(D)$, via $S(t)u_0 = u(t; u_0)$.

It is shown in both of the above references (and Temam [32]) that the equation also enjoys the existence of a global attractor \mathcal{A} , that is, a compact invariant set that attracts the orbits of all bounded sets, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}$ and

$$\text{dist}(S(t)B, \mathcal{A}) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty,$$

where B is any bounded subset of $L^2(D)$ and $\text{dist}(A, B)$ is the Hausdorff semidistance between A and B ,

$$\text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|.$$

Since the equation defines a gradient system, the attractor consists of the collection of all the stationary points and their unstable manifolds (see [23], [30], or [32]).

For the case of a one-dimensional domain it is known that as β passes through each successive eigenvalue λ_n of the Laplacian on $[0, L]$ another direction becomes unstable and two new stationary points appear in a pitchfork bifurcation. It follows that for $\lambda_n < \beta < \lambda_{n+1}$ the dimension of the attractor is n . Since $\lambda_n \sim n^2$, it follows that $d(A) \sim \beta^{1/2}$. (Here we use the upper box-counting or ‘fractal’ dimension.)

In a smooth m -dimensional domain D a similar result is valid for the dimension of the attractor, namely that $d(A) \sim \beta^{m/2}$; see Temam [32] for the upper bound and Babin & Vishik [5] for the lower bound.

3. LINEAR STABILISATION VIA A MULTIPLICATIVE ITÔ NOISE

The effect of perturbing the equation by a multiplicative noise in the Itô sense,

$$(3.1) \quad du = [\Delta u + \beta u - u^3] dt + \sigma u dW_t,$$

is to introduce a somewhat ‘artificial’ stabilising effect. This is most easily seen by considering the equivalent Stratonovich equation

$$(3.2) \quad du = [\Delta u + (\beta - \frac{1}{2}\sigma^2)u - u^3] dt + \sigma u \circ dW_t,$$

where the linear stability has clearly been enhanced by the new $-\frac{1}{2}\sigma^2 u$ term. Making the change of variable $v = ue^{-\sigma W_t}$ produces a family of nonautonomous equations parametrised by ω :

$$v_t = \Delta v + (\beta - \frac{1}{2}\sigma^2)v - e^{2\sigma W_t(\omega)}v^3.$$

Taking the inner product of this equation with v in $L^2(D)$ we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |v|^2 &= -|Dv|^2 + (\beta - \frac{1}{2}\sigma^2)|v|^2 - e^{2\sigma W_t} \|v\|_{L^4}^4 \\ &\leq -\lambda_1 |v|^2 + (\beta - \frac{1}{2}\sigma^2)|v|^2. \end{aligned}$$

It follows that we have

$$|v(t)|^2 \leq |v(0)|^2 e^{-\gamma t}, \quad \text{where } \gamma = 2(\lambda_1 - \beta) + \sigma^2,$$

and hence

$$|u(t)|^2 \leq |u(0)|^2 e^{2\sigma W_t} e^{-\gamma t}.$$

Since \mathbb{P} -almost surely (\mathbb{P} -a.s.)

$$\lim_{t \rightarrow \infty} \frac{|W_t|}{t} = 0,$$

it follows that for $\sigma^2 > 2(\beta - \lambda_1)$ the origin becomes (pathwise) exponentially stable.

We note here that the addition of nonlinear multiplicative Itô noise can also have a stabilising effect (see e.g. Caraballo, Liu, & Mao [9]). However, here and in Section 5 our aim is to demonstrate that stabilisation can be obtained using a very simple multiplicative noise, namely a linear one.

4. A SINGLE MULTIPLICATIVE STRATONOVICH NOISE

The above observation that an Itô noise produces a somewhat artificial stabilising effect led us to study the effect of a single multiplicative *Stratonovich* term on the Chafee-Infante equation

$$(4.1) \quad du = [\Delta u + \beta u - u^3] dt + \sigma u \circ dW_t$$

(W_t is a two-sided one-dimensional Brownian motion) using the framework of random dynamical systems, which we now recall.

4.1. Random dynamical systems and random attractors. In the interest of brevity we only state the definitions here (for more background on random dynamical systems see Arnold [2]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\vartheta_t : \Omega \rightarrow \Omega, t \in \mathbb{R}\}$ a family of measure preserving transformations such that $(t, \omega) \mapsto \vartheta_t \omega$ is measurable, $\vartheta_0 = \text{id}$, and $\vartheta_{t+s} = \vartheta_t \vartheta_s$ for all $s, t \in \mathbb{R}$. The flow ϑ_t together with the corresponding probability space

$$(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t \in \mathbb{R}})$$

is called a (*measurable*) *dynamical system*.

A continuous *random dynamical system* (RDS) on a Polish space (X, d) with Borel σ -algebra \mathcal{B} over ϑ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable map

$$\begin{aligned} \varphi : \mathbb{R}^+ \times \Omega \times X &\rightarrow X \\ (t, \omega, x) &\mapsto \varphi(t, \omega)x \end{aligned}$$

such that \mathbb{P} -a.s.

- i) $\varphi(0, \omega) = \text{id}$ on X ,
- ii) $\varphi(t + s, \omega) = \varphi(t, \vartheta_s \omega) \circ \varphi(s, \omega)$ for all $t, s \in \mathbb{R}^+$ (cocycle property),
and
- iii) $\varphi(t, \omega) : X \rightarrow X$ is continuous.

A *random attractor* for an RDS φ is a random set $\omega \mapsto \mathcal{A}(\omega)$ such that

- (i) \mathcal{A} is a random compact set, that is, \mathbb{P} -a. s., $\mathcal{A}(\omega)$ is compact, and for all $x \in X$ the map $\omega \mapsto \text{dist}(x, \mathcal{A}(\omega))$ is measurable with respect to \mathcal{F} ,
- (ii) \mathbb{P} -a. s. $\varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\vartheta_t \omega)$ for all $t \geq 0$, and
- (iii) for every $D \subset X$ bounded, \mathbb{P} -a. s.,

$$\lim_{t \rightarrow \infty} \text{dist}(\varphi(t, \vartheta_{-t} \omega)D, \mathcal{A}(\omega)) = 0.$$

4.2. Our equation as a random dynamical system, and its attractor. In order to cast our equation as a random dynamical system we let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space generating the two-sided Wiener process W_t , and define a shift ϑ_t on Ω by

$$W_t(\vartheta_s \omega) = W_{t+s}(\omega) - W_s(\omega),$$

the additional subtracted term ensuring that $W_t(\vartheta_s \omega)$ is still a Brownian motion.

Existence and uniqueness results due to Pardoux [29] guarantee that for each initial condition $u_0 \in L^2(D)$ and $T > 0$, there exists a unique strong solution

$$u(t; u_0) \in L^2(\Omega \times (0, T); H_0^1(D)) \cap L^4(\Omega \times (0, T) \times D) \cap L^2(\Omega; C(0, T; L^2(D))).$$

We can use the resulting solution to define a random dynamical system φ on the phase space $L^2(D)$ by setting $\varphi(t, \omega)u_0 = u(t; u_0)$.

The computations presented in Caraballo et al. [7], which are relatively standard, show that (4.1) has a random attractor. We then used the stochastic extension of the deterministic theory (due to Debussche [20]) to show that the Hausdorff dimension of the attractor (which is \mathbb{P} -a.s. constant; see Crauel & Flandoli [16]) is bounded by d when

$$\beta < \frac{1}{d} \sum_{j=1}^d \lambda_j,$$

where λ_j are the eigenvalues of the Laplacian arranged in increasing order. It follows from recent work of Langa & Robinson [26] that these calculations also provide the same upper bound on the upper box-counting dimension of the attractor (which is also constant \mathbb{P} -a.s. [26]). Since $\lambda_n \sim n^{2/m}$, this implies that $d(\mathcal{A}) \leq c\beta^{m/2}$.

In a subsequent paper [8] we adapted a proof of Da Prato & Debussche [17] to show that provided $m \leq 5$ (a technical condition) the unstable manifold near the origin has dimension at least d when $\beta > \lambda_d$. This leads to a lower bound on the dimension of the same order as the upper bound, and hence shows that the dimension of the random attractor is of the same order as its deterministic counterpart, namely

$$d(\mathcal{A}(\omega)) \sim \beta^{m/2}.$$

In this sense the addition of a single multiplicative Stratonovich noise has no effect on the asymptotic complexity of the dynamics.

5. EXPONENTIAL STABILITY OF THE ZERO SOLUTION VIA A NUMBER OF STRATONOVICH MULTIPLICATIVE NOISE TERMS

Generalising the finite-dimensional result of Arnold, Crauel, & Wihstutz [4], Caraballo and Robinson [10] recently showed that a linear PDE $u_t = Au$ can be stabilised by a collection of multiplicative Stratonovich noisy terms if and only if the trace of the linear partial differential operator A is negative.

In this section we show that the *nonlinear* equation (1.1) can be stabilised by adding a similar collection of noisy terms:

$$(5.1) \quad du = [\Delta u + \beta u - u^3] dt + \sum_{i=1}^d B_i u \circ dW_t^i.$$

Essentially we show that solutions of (5.1) can be bounded using appropriate *positive* solutions of the linear equation

$$(5.2) \quad du = [\Delta u + \beta u] dt + \sum_{i=1}^d B_i u \circ dW_t^i.$$

Since (5.2) can be stabilised via a suitable choice of $\{B_i\}$, so can (5.1). The proof makes essential and continual use of the order-preserving properties of (5.1).

To begin with, we recall the stabilisation result for linear equations given in [10]: we suppose that \mathbb{A} is a linear operator with a sequence of eigenvalues μ_j with corresponding eigenfunctions e_j that form an orthonormal basis of a Hilbert space H . We also assume that the eigenvalues μ_j are bounded above, and order them so that $\mu_1 \geq \mu_2 \geq \dots$

Theorem 5.1. *If the trace of \mathbb{A} is negative, then there exist bounded linear operators $B_k : H \rightarrow H$, $k = 1, \dots, d$, such that the zero solution of*

$$du = \mathbb{A}u dt + \sum_{j=1}^d B_k u \circ dW_k$$

is exponentially stable with probability one.

The proof is simple: choose N such that $\sum_{j=1}^N \mu_j < 0$, and consider the projection of $u_t = \mathbb{A}u$ onto the two complementary subspaces P and Q spanned by $\{e_j\}_{j=1}^N$ and $\{e_j\}_{j=N+1}^\infty$ respectively. The Q components of the equation converge exponentially to zero, while the solutions of the finite-dimensional ODE $\dot{p} = -Ap$ (where p denotes the orthogonal projection of u onto P) can be stabilised [4] by adding a collection of noisy terms $+B_k p \circ dW_k$, where the B_k are $N \times N$ skew-symmetric matrices. These matrices correspond to linear operators $B_k : H \rightarrow H$.

In our case we will choose $H = L^2(D)$; we will also denote by $-A$ the linear operator in H associated to the Laplacian. We then take $\mathbb{A} = -A + \beta I$, which clearly satisfies the conditions of Theorem 5.1, and let N be the smallest integer such that $\sum_{j=1}^N (\beta - \lambda_j) < 0$. It follows that there exist linear operators $B_k : H \rightarrow H$ such that the zero solution of

$$(5.3) \quad du = [-Au + \beta u] dt + \sum_{j=1}^d B_k u \circ dW_t^k$$

is exponentially stable with probability one.

We now show how to use this to deduce stabilisation of the nonlinear equation via the addition of the same noisy terms.

Theorem 5.2. *There exist bounded linear operators $B_k : H \rightarrow H$, and independent real Wiener processes W_t^k , $k = 1, \dots, d$, such that the zero solution of*

$$(5.4) \quad du = (-Au + \beta u - u^3) dt + \sum_{j=1}^d B_k u \circ dW_t^k$$

is exponentially stable with probability one.

Before giving the proof, we note that by considering the equivalent Itô form of (5.4) it follows from results of Pardoux [29] that for each $u_0 \in L^2(D)$ and $T > 0$, there exists a unique strong solution satisfying

$$u(t; u_0) \in L^2(\Omega \times (0, T); H_0^1(D)) \cap L^2(\Omega; C(0, T; L^2(D))).$$

Proof. The stabilisation of the zero solution follows once we show that solutions of (5.4) can be bounded by appropriate solutions of the linear equation (5.3).

The key observation is that we can bound from above an arbitrary solution of (5.4) pointwise by a *positive* solution of the same equation (and from below by a negative one). Indeed, if $u_0(x) \leq U_0(x)$, then the order-preserving property of (5.4) (see Kotelenetz [25] or Chueshov & Vuillermot [12]) guarantees that

$$(5.5) \quad u(t, x, \omega; u_0) \leq u(t, x, \omega; U_0) \quad \text{for almost all } (t, x, \omega) \in [0, +\infty) \times [0, L] \times \Omega,$$

where with the obvious notation $u(t, x, \omega; u_0)$ denotes the solution of (5.4) with $u(0, x) = u_0(x)$ and noise ω . It follows that

$$u(t, x, \omega; -|u_0|) \leq u(t, x, \omega; u_0) \leq u(t, x, \omega; |u_0|).$$

Now, positive solutions of the nonlinear equation enjoy a comparison principle with those of the linear equation (see Chueshov & Vuillermot [12] again): if $u_0 \geq 0$, then

$$u(t, x, \omega; u_0) \leq u_L(t, x, \omega; u_0),$$

where $u_L(t, x, \omega)$ is the solution of the corresponding linear stochastic PDE

$$du = (-Au + \beta u) dt + \sum_{j=1}^d B_j u \circ dW_t^j.$$

It follows that

$$u_L(t, x, \omega; -|u_0|) \leq u(t, x, \omega; u_0) \leq u_L(t, x, \omega; |u_0|),$$

which can be rewritten, since u_L solves a linear equation, as

$$|u(t, x, \omega; u_0)| \leq u_L(t, x, \omega; |u_0|).$$

Since solutions of (5.3) tend exponentially to zero with probability one, so do all solutions of (5.4). □

6. COLLAPSE OF THE RANDOM ATTRACTOR PRODUCED BY ADDITIVE NOISE

In this final section we show that the addition of a sufficiently rich additive white noise will reduce the random attractor of the equation to a single (random) point.

Such behaviour was originally demonstrated for the one-dimensional ordinary differential equation

$$dx = [\alpha x - x^3] dt + \epsilon dW_t \quad \text{with} \quad \alpha > 0$$

by Crauel & Flandoli [15], and has recently been shown by Tearne [31] for a general gradient ODE of the form

$$dx = -\nabla V(x) + \epsilon dW_t,$$

where $x \in \mathbb{R}^m$, W_t is an m -dimensional Brownian motion, and ϵ is sufficiently small (note that this is not in general an order-preserving system).

Here we prove a similar result for the equation

$$(6.1) \quad du = [\Delta u + \beta u - u^3] dt + \sqrt{C} dW_t, \quad x \in D = [0, L],$$

where $W_t, t \in \mathbb{R}$, is a two-sided cylindrical Wiener process on $H = L^2(D)$ and C is a bounded linear operator with bounded inverse on H . Note that here we have to restrict ourselves to a one-dimensional domain.

Our argument, which we will make precise below, could be generalised to treat more abstract problems (cf. Chueshov & Scheutzow [11]) but the underlying idea is simple: Results of Arnold & Chueshov [3] on the structure of random attractors in order-preserving systems guarantee the existence of two random fixed points \underline{a} and \bar{a} that are contained in the attractor and are such that

$$\underline{a}(\omega) \leq u \leq \bar{a}(\omega) \quad \text{for all} \quad u \in \mathcal{A}(\omega).$$

Corresponding to these random fixed points there are invariant measures $\delta_{\underline{a}(\omega)}$ and $\delta_{\bar{a}(\omega)}$. Since the noise in (6.1) is sufficiently rich to guarantee that the equation has a *unique* invariant measure (e.g., Da Prato, Debussche, & Goldys [18]) it follows that the laws of $\underline{a}(\omega)$ and $\bar{a}(\omega)$ must coincide. It is only a small step from this, using the fact that $\underline{a}(\omega) \leq \bar{a}(\omega)$, to the deduction that $\underline{a}(\omega) = \bar{a}(\omega) = a(\omega)$, and hence that $\mathcal{A}(\omega) = \{a(\omega)\}$, i.e., the attractor is a single point.

We now recall the formal existence and uniqueness results for (6.1), and give a rigorous proof that the random attractor is a point.

We take $(\Omega, \mathcal{F}, \mathbb{P})$ to be the probability space that generates the cylindrical Wiener process W_t , and define a shift ϑ_t on Ω by $W_t(\vartheta_s\omega) = W_{t+s}(\omega) - W_s(\omega)$ as in Section 4.

Under these assumptions, it is known (Da Prato and Zabczyk [19]) that for each $u_0 \in L^2(D)$ and $T > 0$ there exists a unique solution $u(t; u_0)$ for (6.1), with

$$u(t; u_0) \in L^2(\Omega \times (0, T); H_0^1(D)) \cap L^4(\Omega \times (0, T) \times D) \cap L^2(\Omega; C(0, T; L^2(D))).$$

It follows that the solutions of (6.1) generate a random dynamical system on $L^2(D)$ if we define

$$\varphi(t, \omega)u_0 = u(t; \omega, u_0),$$

where $u(t; \omega, u_0)$ is the solution of (6.1) with noise ω and initial condition $u(0) = u_0$.

Theorem 6.1. *The random attractor for (6.1) consists of a single point, i.e., there exists a random variable $a : \Omega \rightarrow H$ with*

$$\varphi(t, \omega)a(\omega) = a(\vartheta_t\omega) \quad \text{for every } t \geq 0 \quad \mathbb{P}\text{-a.s.}$$

such that $\mathcal{A}(\omega) = \{a(\omega)\}$.

Proof. The existence of a random attractor $\mathcal{A}(\omega)$ for (6.1) can be proved using standard techniques (see, e.g., Allouba and Langa [1], Debussche [20] or Yuhong Li [27]). From Theorem 3.3 in Crauel [13] we know that $\omega \mapsto \mathcal{A}(\omega)$ is measurable with respect to the past \mathcal{F}^- , which is the σ -algebra

$$\mathcal{F}^- = \sigma\{\omega \mapsto \varphi(s, \vartheta_{-t}\omega)x : x \in X, 0 \leq s \leq t\}.$$

Theorem 5.8 in Chueshov and Vuillermot [12] guarantees that the random dynamical system associated with (6.1) is order-preserving, i.e., if $u \leq v$, then \mathbb{P} -a. s., for every $t \geq 0$, $\varphi(t, \omega)u \leq \varphi(t, \omega)v$. As a consequence (Theorem 2 in Arnold & Chueshov [3]) there exist equilibria \underline{a} and \bar{a} such that, \mathbb{P} -a.s.,

$$\underline{a}(\omega) \leq u \leq \bar{a}(\omega) \quad \text{for all} \quad u \in \mathcal{A}(\omega).$$

We now wish to associate invariant measures with the two extremal equilibria. In order to do this we first show that they are measurable with respect to the past \mathcal{F}^- : it will then follow from Crauel [14] that for each equilibrium $a(\omega)$, $\rho = \mathbb{E}(\delta_{a(\omega)})$ is

a P_t -invariant Markov measure, i.e., $\rho \in Pr(H)$ (it is a probability measure on H) such that $P_t\rho = \rho$ for every $t \geq 0$, where $(P_t)_{t \geq 0}$ is the canonical Markov semigroup on $Pr(H)$ associated with φ .

Since $\mathcal{A}(\omega)$ is a closed random set that is measurable with respect to the past \mathcal{F}^- , there exists a countable family $\{a_n\}_{n \in \mathbb{N}}$ of \mathcal{F}^- -measurable selections, such that $\mathcal{A}(\omega) = \text{cl} \{a_n : n \in \mathbb{N}\}$ a.s. We now define a sequence $(b_n)_{n \in \mathbb{N}}$, which need not be selections of $\mathcal{A}(\omega)$, by $b_1 = a_1$, and $b_{n+1} = \max\{a_{n+1}, b_n\}$, or, more explicitly,

$$b_{n+1}(\omega) = \begin{cases} a_{n+1}(\omega) & \text{if } a_{n+1}(\omega) \geq b_n(\omega), \\ b_n(\omega) & \text{otherwise,} \end{cases}$$

for $n \geq 1$. Then every b_n is \mathcal{F}^- -measurable, and the sequence is increasing. Furthermore we have $a_k \leq b_n \leq \bar{a}$ \mathbb{P} -a.s. for all $k \leq n$, and $n \in \mathbb{N}$. We claim that b_n converges to \bar{a} \mathbb{P} -a.s. In fact, if b_n would be bounded away from \bar{a} with positive probability, we would get that a_n is bounded away from \bar{a} with positive probability, hence $\sup a_n < \bar{a}$ with positive probability, which would contradict $\mathcal{A} = \text{cl} \{a_n : n \in \mathbb{N}\}$ \mathbb{P} -a.s., and $\bar{a} \in \mathcal{A}$. Consequently, $\bar{a} = \lim b_n$ is \mathcal{F}^- -measurable.

A similar argument implies that \underline{a} is \mathcal{F}^- -measurable as well.

We therefore obtain two P_t -invariant Markov measures $\mathbb{E}(\delta_{\bar{a}})$ and $\mathbb{E}(\delta_{\underline{a}})$. These are just the laws of the equilibria, i.e., for any Borel set $B \in \mathcal{B}(H)$

$$\mathbb{E}(\delta_{\bar{a}})(B) = \mathcal{L}(\bar{a})(B) = \mathbb{P}(u \in B).$$

Now, Da Prato et al. [18] (Section 6) showed that the Markov semigroup associated with (6.1) has a *unique* invariant measure,¹ which means that $\mathcal{L}(\bar{a}) = \mathcal{L}(\underline{a})$, and so in particular we must have $\mathbb{E}(\underline{a}) = \mathbb{E}(\bar{a})$. Since \mathbb{P} -a.s. $\underline{a} \leq \bar{a}$ and \underline{a} and \bar{a} are real functions defined on $[0, L]$, we must have $\underline{a} = \bar{a}$, \mathbb{P} -a.s.

Setting $a(\omega) = \underline{a}(\omega) = \bar{a}(\omega)$, it follows that \mathbb{P} -a.s. $\mathcal{A}(\omega) = \{a(\omega)\}$, i.e., the attractor consists of a single random point. \square

7. CONCLUSION

Here we have aimed to draw attention to the very different effects that different types of noise can have on the asymptotic behaviour of deterministic systems. Of course, all the above analysis could be carried out for more general systems, but we believe that treating a simple canonical model helps to clarify the arguments.

In particular, although elementary, it seems worthwhile to emphasise the somewhat artificial stabilisation effect produced by the multiplicative Itô noise $+\sigma u dW_t$ discussed in Section 3. That the intensity of the corresponding Stratonovich noise $+\sigma u \circ dW_t$ has no effect on the dimension of the random attractor is remarkable.

We would also like to highlight the possibilities for detailed analysis afforded by order-preserving systems, as demonstrated by the Stratonovich stabilisation and ‘attractor collapse through additive noise’ results of Sections 5 and 6.

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¹For several other possible choices of suitable conditions on the Wiener process to ensure the existence of a unique invariant measure see Bakhtin and Mattingly [6], E and Liu [21], Eckmann and Hairer [22], among others.

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