

NON-INTERSECTION BODIES, ALL OF WHOSE CENTRAL SECTIONS ARE INTERSECTION BODIES

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ABSTRACT. We construct symmetric convex bodies that are not intersection bodies, but all of their central hyperplane sections are intersection bodies. This result extends the studies by Weil in the case of zonoids and by Neyman in the case of subspaces of L_p .

1. INTRODUCTION

The concept of an intersection body was introduced by Lutwak [Lu] in 1988. Let K and L be origin symmetric star bodies in \mathbb{R}^n . Following [Lu], we say that K is the *intersection body of L* if the radius of K in every direction is equal to the volume of the central hyperplane section of L perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\|\xi\|_K^{-1} = \text{vol}_{n-1}(L \cap \xi^\perp).$$

The closure in the radial metric of the class of intersection bodies of star bodies gives the class of *intersection bodies*.

Intersection bodies played an important role in the solution of the Busemann-Petty problem (see [K5, Chapter 5] for the solution and [K5, Chapter 1] for the historical details). Posed in 1956, [BP], the Busemann-Petty problem asks the following. Let K and L be two origin-symmetric convex bodies in \mathbb{R}^n so that the $(n-1)$ -dimensional volume of every central hyperplane section of K is smaller than the same for L . Does it follow that the n -dimensional volume of K is smaller than the n -dimensional volume of L ? The answer turns out to be affirmative for dimensions $n \leq 4$ and negative for $n \geq 5$.

The connection between intersection bodies and the Busemann-Petty problem was found by Lutwak [Lu]. First, the answer to the problem is affirmative if K is an intersection body and L is any origin-symmetric star body. On the other hand, if L is an origin-symmetric convex body that is not an intersection body, one can perturb L to construct a counterexample to the Busemann-Petty problem. Hence, a solution to the problem in \mathbb{R}^n is affirmative if and only if every infinitely smooth origin-symmetric convex body in \mathbb{R}^n is an intersection body, which is the case for dimensions $n \leq 4$. Examples of non-intersection bodies in dimensions 5 and higher were constructed in [Ga1], [Zh1], [GKS], [K3].

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In this paper we are interested in the following problem. Does there exist a convex body K that is not an intersection body, but its section by a central hyperplane is an intersection body? We construct an example of such a body for dimensions $n \geq 5$. Our result can also be considered as a new way of constructing non-intersection bodies.

This paper was motivated by results of W.Weil [W] and A.Neyman [N]. In 1982 W.Weil, [W], showed that it is not possible to characterize zonoids by means of their projections. He constructed a convex body in \mathbb{R}^n ($n \geq 3$) that is not a zonoid but all its projections onto hyperplanes are zonoids. A.Neyman in [N] showed that there are n -dimensional normed spaces that do not embed in L_p , but all their $(n-1)$ -dimensional subspaces embed in L_p for $p > 0$. He used this to prove that for $p > 0$, $p \neq 2$, L_p is not characterized by a finite number of equations. Let us note that M.Burger in [B] used another approach to show that, for $n \geq 3$, zonoids cannot be characterized by a finite number of piecewise inequalities. A.Koldobsky in [K4] introduced the concept of embedding of normed spaces in L_p , $p < 0$, and proved that intersection bodies are the unit balls of spaces that embed in L_{-1} . Therefore, our result can be considered as an extension of Neyman's example to negative p .

2. MAIN RESULTS

Our main tool is the Fourier transform of distributions. The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function ϕ from the Schwartz space \mathcal{S} of rapidly decreasing infinitely differentiable functions on \mathbb{R}^n . For any even distribution f , we have $(\hat{f})^\wedge = (2\pi)^n f$.

A distribution is *positive definite* if its Fourier transform is a positive distribution in the sense that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function ϕ ; see, for example, [GV, p. 152].

Let K be a convex origin-symmetric body in \mathbb{R}^n . Our definition of a convex origin-symmetric body assumes that the origin is an interior point of K . The *radial function* of K is given by

$$\rho_K(x) = \max\{a > 0 : ax \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

The Minkowski *norm* of K is defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}.$$

Clearly $\rho_K(x) = \|x\|_K^{-1}$.

The main result of this paper is the following.

Theorem 2.1. *There exists a convex body K in \mathbb{R}^n , $n \geq 5$, that is not an intersection body, but for every $(n-1)$ -dimensional subspace V of \mathbb{R}^n , $K \cap V$ is an intersection body.*

To construct an example of such a body we use a connection between the Fourier transform and intersection bodies. A.Koldobsky in [K2] proved that an origin-symmetric star body K in \mathbb{R}^n is an intersection body if and only if $\|x\|_K^{-1}$ is a positive definite distribution.

We will use Lemma 3.16 from [K5, p. 58]. It states the following.

Lemma 2.2. *Let $k \in \mathbb{N} \cup \{0\}$ and $f \in C^{2k}(S^{n-1})$, where f is even, $q \leq 2k$, and q is not an odd integer. Let $x = (r, \theta)$ be polar coordinates in \mathbb{R}^n , so that $f_\varepsilon(\theta)r^p = f_\varepsilon\left(\frac{x}{|x|_2}\right)|x|_2^{-p}$. Then:*

- (i) *The Fourier transform of the distribution $f(\theta)r^{-n+q+1}$ is a homogeneous of degree $-1-q$ continuous on $\mathbb{R}^n \setminus \{0\}$ function. If $q < 2k$, then for every $x \in \mathbb{R}^n$,*

$$|x|_2^{2k} (f(\theta)r^{-n+q+1})^\wedge(x) = \frac{(-1)^k \pi}{-2\Gamma(2k-q)\sin(\pi(2k-q-1)/2)} \times \int_{S^{n-1}} |(x, \xi)|^{2k-q-1} \Delta^k (f(\theta)r^{-n+q+1})(\xi) d\xi.$$

If $q = 2k$, then

$$(f(\theta)r^{-n+q+1})^\wedge(x) = (-1)^k \pi |x|_2^{-1-2k} \times \int_{S^{n-1} \cap (x/|x|_2)^\perp} \Delta^k (f(\theta)r^{-n+q+1})(\xi) d\xi,$$

where Δ is the Laplace operator in \mathbb{R}^n .

- (ii) *If $f \in C^\infty(S^{n-1})$, then there exist an even function $g \in C^\infty(S^{n-1})$ so that for every $x = t\xi \in \mathbb{R}^n$, $t \neq 0$, $\xi \in S^{n-1}$,*

$$(f(\theta)r^{-n+q+1})^\wedge(x) = g(\xi)t^{-1-q},$$

so the Fourier transform of $f(\theta)r^{-n+q+1}$ is an infinitely smooth function on $\mathbb{R}^n \setminus \{0\}$.

Fix a point x_0 on the unit sphere S^{n-1} . Define a function f_ε as follows:

$$f_\varepsilon(x) = \begin{cases} 2e^{-\frac{|x-x_0|^2}{\varepsilon^2-|x-x_0|^2}} & \text{if } |x-x_0| < \varepsilon, \\ 2e^{-\frac{|x+x_0|^2}{\varepsilon^2-|x+x_0|^2}} & \text{if } |x+x_0| < \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly f_ε is an infinitely differentiable function. Define a body K by

$$(1) \quad \|x\|_K^{-1} = ((1 - f_\varepsilon(\theta))r^{-n+1})^\wedge(x), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

The function $\|x\|_K^{-1}$ is infinitely smooth on $\mathbb{R}^n \setminus \{0\}$ by Lemma 2.2. It will be shown in Lemma 2.4 that $(f_\varepsilon(\theta)r^{-n+1})^\wedge(x)$ is of the order ε^{n-2} uniformly with respect to $x \in S^{n-1}$ and the Fourier transform of r^{-n+1} is strictly greater than zero, therefore $\|x\|_K$ is positive for a small ε .

Lemma 2.3. *For any $\varepsilon > 0$, K is not an intersection body.*

Proof. Since f_ε is an even function, we have

$$(\|x\|_K^{-1})^\wedge = (2\pi)^n (1 - f_\varepsilon(\theta))r^{-n+1},$$

which is negative for θ in some neighborhood of x_0 . Therefore, by [K2, Theorem 1], K is not an intersection body. □

Lemma 2.4. *There exist numbers D_1, D_2, D_3 that do not depend on ε so that for every $x \in S^{n-1}$*

$$\begin{aligned} \left| (f_\varepsilon(\theta)r^{-n+1})^\wedge(x) \right| &\leq D_1\varepsilon^{n-2}, \\ \left| \frac{\partial}{\partial x_i} (f_\varepsilon(\theta)r^{-n+1})^\wedge(x) \right| &\leq D_2|\ln \varepsilon| \cdot \varepsilon^{n-3}, \end{aligned}$$

and

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} (f_\varepsilon(\theta)r^{-n+1})^\wedge(x) \right| \leq D_3\varepsilon^{n-4}.$$

Proof. First we show the estimate for the second derivative. Using the connection between the Fourier transform and differentiation and Lemma 2.2 with $q = 2$ and $k = 1$, we get

$$\begin{aligned} \frac{\partial^2}{\partial x_i \partial x_j} (f_\varepsilon(\theta)r^{-n+1})^\wedge(x) &= - \left(f_\varepsilon \left(\frac{y}{|y|} \right) y_i y_j |y|^{-n+1} \right)^\wedge(x) \\ &= - \left(f_\varepsilon \left(\frac{y}{|y|} \right) \frac{y_i y_j}{|y|^2} |y|^{-n+3} \right)^\wedge(x) = - (g_\varepsilon(\theta)|y|^{-n+3})^\wedge(x) \\ (2) \qquad &= \pi|x|_2^{-3} \int_{S^{n-1} \cap (x/|x|_2)^\perp} \Delta (g_\varepsilon(\theta)|y|^{-n+3})(\xi) d\xi, \end{aligned}$$

where

$$g_\varepsilon(\theta) = f_\varepsilon \left(\frac{y}{|y|} \right) \frac{y_i y_j}{|y|^2}, \quad \theta = \frac{y}{|y|} \in S^{n-1}.$$

Note that the function $g_\varepsilon(\theta)$ is supported in $B_\varepsilon(x_0)$ and $B_\varepsilon(-x_0)$, where

$$B_\varepsilon(x_0) = \{x \in S^{n-1} : |x - x_0| < \varepsilon\}.$$

The volume of these balls is of the order ε^{n-1} .

Now we want to show that $\frac{\partial^2}{\partial x_i \partial x_j} (f_\varepsilon(\theta)r^{-n+1})^\wedge(x)$ can be made as small as desirable if ε is small. First let us show that $\frac{\partial^2}{\partial y_i^2} \left(g_\varepsilon \left(\frac{y}{|y|} \right) \right)$ is of the order ε^{-2} .

To prove this, consider the function

$$f_1(x) = \begin{cases} 2e^{-\frac{|x-x_0|^2}{1-|x-x_0|^2}} & \text{if } |x - x_0| < 1, \\ 2e^{-\frac{|x+x_0|^2}{1-|x+x_0|^2}} & \text{if } |x + x_0| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $x \in S^{n-1}$.

The function f_1 is infinitely differentiable, so all its derivatives are bounded. Also $f_1 \left(\frac{x/|x|-x_0}{\varepsilon} + x_0 \right) = f_\varepsilon(x/|x|)$. Therefore,

$$\left| \frac{\partial^k}{\partial x_i^k} f_\varepsilon(x/|x|) \right| = \left| \frac{\partial^k}{\partial x_i^k} f_1 \left(\frac{x/|x|-x_0}{\varepsilon} + x_0 \right) \right| \leq C_k \varepsilon^{-k},$$

where C_k depends on k but not on x .

The same is true for the derivatives of $g_\varepsilon(\theta) = f_\varepsilon \left(\frac{y}{|y|} \right) \frac{y_i^2}{|y|^2}$, i.e.

$$\left| \frac{\partial^k}{\partial y_i^k} g_\varepsilon \left(\frac{y}{|y|} \right) \right| \leq \tilde{C}_k \varepsilon^{-k}.$$

So $\Delta (g_\varepsilon(\theta)|y|^{-n+3})(\xi)$ is also of the order ε^{-2} since the main term when ε is small is $\frac{\partial^2}{\partial y_i^2} \left(g_\varepsilon \left(\frac{y}{|y|} \right) \right) |y|^{-n+3}$.

Therefore,

$$\begin{aligned} & \left| \int_{S^{n-1} \cap (x/|x|_2)^\perp} \Delta (g_\varepsilon(\theta)|y|^{-n+3})(\xi) d\xi \right| \\ & \leq \tilde{C}_2 \sup |\Delta (g_\varepsilon(\theta)|y|^{-n+3})| \cdot \int_{(B_\varepsilon(x_0) \cup B_\varepsilon(-x_0)) \cap (x/|x|_2)^\perp} d \\ & = O(\varepsilon^{-2} \varepsilon^{n-2}) = O(\varepsilon^{n-4}), \end{aligned}$$

since the volume of the balls $B_\varepsilon(x_0)$ and $B_\varepsilon(-x_0)$ is of the order ε^{n-1} and the volume of their intersection with a hyperplane is of the order ε^{n-2} .

So, using equality (2) we get $\frac{\partial^2}{\partial x_i \partial x_j} (f_\varepsilon(\theta)r^{-n+1})^\wedge(x) = O(\varepsilon^{n-4})$, and for $n > 4$ it can be made as small as desirable uniformly with respect to $x \in S^{n-1}$.

Using the same argument, we prove that $(f_\varepsilon(\theta)r^{-n+1})^\wedge(x) = O(\varepsilon^{n-2})$.

To get the estimate for $\frac{\partial}{\partial x_i} (f_\varepsilon(\theta)r^{-n+1})^\wedge(x)$ we again use Lemma 2.2 and the connection between the Fourier transform and differentiation. Take α close to 1. Then, if $g_\varepsilon(\theta) = f_\varepsilon \left(\frac{y}{|y|} \right) \frac{y_i}{|y|}$, $\theta = \frac{y}{|y|} \in S^{n-1}$,

$$\begin{aligned} & \frac{\partial}{\partial x_i} (f_\varepsilon(\theta)r^{-n+\alpha+1})^\wedge(x) = - (g_\varepsilon(\theta)|y|^{-n+\alpha+2})^\wedge(x) \\ (3) \quad & = \frac{-\pi|x|_2^{-2} \int_{S^{n-1}} |(x, \xi)|^{1-\alpha} \Delta(g_\varepsilon(\theta)|y|^{-n+\alpha+1})(\xi) d\xi}{2\Gamma(2-\alpha) \sin \frac{\pi(1-\alpha)}{2}}. \end{aligned}$$

When α approaches 1, the numerator and denominator on the right-hand side approach zero. Indeed, let us show that the limit of the numerator is zero:

$$\lim_{\alpha \rightarrow 1} \int_{S^{n-1}} |(x, \xi)|^{1-\alpha} \Delta(g_\varepsilon(\theta)|y|^{-n+\alpha+1})(\xi) d\xi = \int_{S^{n-1}} \Delta(g_\varepsilon(\theta)|y|^{-n+2})(\xi) d\xi.$$

Recall the relation between the spherical Laplacian Δ_S and Euclidean Laplacian Δ (see, for example, [Gr, p. 7]): if f is a homogeneous function of degree m , then on the sphere

$$\Delta_S f_1 = \Delta f_1 - m(m+n-2)f_1.$$

Since $g_\varepsilon(\theta)|y|^{-n+2}$ has degree of homogeneity $-n+2$, the previous formula implies $\Delta(g_\varepsilon(\theta)|y|^{-n+2})(\xi) = \Delta_S(g_\varepsilon(\theta)|y|^{-n+2})(\xi)$. Due to the fact that Δ_S is a self-adjoint operator, [Gr, p. 7], we have

$$\int_{S^{n-1}} \Delta_S(g_\varepsilon(\theta)|y|^{-n+2})(\xi) d\xi = 0.$$

Now to compute the limit of (3) as $\alpha \rightarrow 0$, apply l'Hôpital's rule:

$$\frac{\partial}{\partial x_i} (f_\varepsilon(\theta)r^{-n+2})^\wedge(x) = |x|_2^{-2} \int_{S^{n-1}} \ln |(x, \xi)| \Delta(g_\varepsilon(\theta)|y|^{-n+2})(\xi) d\xi.$$

Recall that the function $g_\varepsilon(\theta)$ is supported in the balls $B_\varepsilon(x_0)$ and $B_\varepsilon(-x_0)$. Then

$$\begin{aligned} & \left| \int_{S^{n-1}} \ln |(x, \xi)| \Delta(g_\varepsilon(\theta)|y|^{-n+2})(\xi) d\xi \right| \leq \sup |\Delta(g_\varepsilon(\theta)|y|^{-n+2})(\xi)| \\ & \quad \times \int_{B_\varepsilon(x_0) \cup B_\varepsilon(-x_0)} |\ln |(x, \xi)|| d\xi. \end{aligned}$$

Now we want to estimate the latter integral. Note that it is enough to estimate just $\int_{B_\varepsilon(x_0)} |\ln |(x, \xi)|| d\xi$. Consider two cases. First, suppose that x is not perpendicular to any $y \in B_{2\varepsilon}(x_0)$. In this case one can check that $|(x, \xi)| > \varepsilon/2$ and therefore

$$\int_{B_\varepsilon(x_0)} |\ln |(x, \xi)|| d\xi \leq |\ln(\varepsilon/2)| \cdot \text{vol}(B_\varepsilon(x_0)) = O(|\ln \varepsilon| \cdot \varepsilon^{n-1}),$$

since the volume of the ball $B_\varepsilon(x_0)$ is of the order ε^{n-1} .

In the second case there exists $y \in B_{2\varepsilon}(x_0)$ such that $x \perp y$. Consider the ball $B_{4\varepsilon}(y)$. Clearly, $B_{2\varepsilon}(x_0) \subset B_{4\varepsilon}(y)$, therefore

$$\int_{B_\varepsilon(x_0)} |\ln |(x, \xi)|| d\xi \leq \int_{B_{4\varepsilon}(y)} |\ln |(x, \xi)|| d\xi.$$

Let us make a change of coordinates from $\xi \in S^{n-1}$ to $\zeta \in S^{n-2}$ and $t \in [-\pi/2, \pi/2]$ such that $\xi = y\sqrt{1-t^2} + \zeta t$. The Jacobian is equal to $\frac{t^{n-2}}{\sqrt{1-t^2}}$. If $\xi \in B_{4\varepsilon}(y)$, then $t \in [0, 4\varepsilon\sqrt{1-4\varepsilon^2}] \subset [0, 4\varepsilon]$. Using the fact that x is perpendicular to y , we get

$$\begin{aligned} \int_{B_{4\varepsilon}(y)} |\ln |(x, \xi)|| d\xi &\leq \int_{S^{n-2}} \int_0^{4\varepsilon} |\ln |(x, \zeta)t|| \frac{t^{n-2}}{\sqrt{1-t^2}} dt d\zeta \\ &= \int_{S^{n-2}} \left(\int_0^{4\varepsilon} |\ln |(x, \zeta)|| \frac{t^{n-2}}{\sqrt{1-t^2}} dt + \int_0^{4\varepsilon} |\ln t| \frac{t^{n-2}}{\sqrt{1-t^2}} dt \right) d\xi \\ &\leq 2 \int_{S^{n-2}} \left(\int_0^{4\varepsilon} |\ln |(x, \zeta)|| t^{n-2} dt + \int_0^{4\varepsilon} |\ln t| t^{n-2} dt \right) d\xi \end{aligned}$$

for a small ε . The first integral can be estimated in the following way:

$$\int_{S^{n-2}} |\ln |(x, \zeta)|| d\xi \int_0^{4\varepsilon} t^{n-2} dt = O(\varepsilon^{n-1}).$$

Using integration by parts in the second integral, we get that

$$\begin{aligned} \int_0^{4\varepsilon} |\ln t| t^{n-2} dt &= \frac{1}{n-1} \ln t \cdot t^{n-1} \Big|_0^{4\varepsilon} - \frac{1}{n-1} \int_0^{4\varepsilon} t^{n-2} dt \\ &= O(|\ln \varepsilon| \cdot \varepsilon^{n-1}). \end{aligned}$$

Therefore,

$$\int_{B_\varepsilon(x_0) \cup B_\varepsilon(-x_0)} |\ln |(x, \xi)|| d\xi = O(|\ln \varepsilon| \cdot \varepsilon^{n-1})$$

and

$$\frac{\partial}{\partial x_i} (f_\varepsilon(\theta)r^{-n+\alpha+1})^\wedge(x) = O(|\ln \varepsilon| \cdot \varepsilon^{n-3}).$$

□

Lemma 2.5. *If $n \geq 5$, the body K is convex for small enough ε .*

Proof. Geometrically the proof is based on the idea of adding in radial metric of an arbitrary small function to a radial function of the body which is uniformly convex, i.e. the curvature is a constant bounded from zero. In our case this body is the Euclidean ball. So, due to this property of a uniformly convex body, we can always choose a function in such a way that the body we get is still convex.

Let $|\cdot|_2$ be the Euclidean norm. By [GS, p. 363] the Fourier transform of $|x|_2^q$, $q \in (-n, 0)$, equals

$$(|x|_2^q)^\wedge(t) = 2^{q+n} \pi^{n/2} \frac{\Gamma(\frac{q+n}{2})}{\Gamma(\frac{-q}{2})} |t|_2^{-n-q}.$$

Using this formula and the definition of the body K

$$\begin{aligned} \|x\|_K^{-1} &= ((1 - f_\varepsilon(\theta))r^{-n+1})^\wedge(x) \\ (4) \qquad &= C_n |x|_2^{-1} - (f_\varepsilon(\theta)r^{-n+1})^\wedge(x), \end{aligned}$$

where $C_n = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n-1}{2})}$.

Let K_W be the section of K by a 2-dimensional central plane W with an orthonormal basis ξ_1, ξ_2 . So, if $x \in W \cap S^{n-1}$, then $x = \xi_1 \cos \phi + \xi_2 \sin \phi$, $\phi \in [0, 2\pi]$. To show that K is convex, it is enough to show that K_W is convex for any W .

Consider a function

$$\rho_W(\phi) = (f_\varepsilon(\theta)r^{-n+1})^\wedge(\xi_1 \cos \phi + \xi_2 \sin \phi).$$

By the definition of K , the radial function of K_W is given by

$$\rho(\phi) = C_n - \rho_W(\phi).$$

To prove that K_W is convex, we need to show that for small ε

$$J(W, \varepsilon, \phi) = 2(\rho')^2 - \rho''\rho + \rho^2 > 0$$

for every W and ϕ ; see [Ga2, p. 25].

Computing the derivatives,

$$\begin{aligned} \rho'(\phi) &= -\frac{d}{d\phi}(\rho_W(\phi)), \\ \rho''(\phi) &= -\frac{d^2}{d^2\phi}(\rho_W(\phi)). \end{aligned}$$

To estimate ρ' and ρ'' , consider

$$\begin{aligned} \left| \frac{d}{d\phi}(\rho_W(\phi)) \right| &= \left| \sum_{i=1}^n \frac{\partial}{\partial x_i} (f_\varepsilon(\theta)r^{-n+1})^\wedge(x) \right| \left| \frac{dx_i}{d\phi} \right| \\ &\leq 2 \left| \sum_{i=1}^n \frac{\partial}{\partial x_i} (f_\varepsilon(\theta)r^{-n+1})^\wedge(x) \right|, \end{aligned}$$

since $x = \xi_1 \cos \phi + \xi_2 \sin \phi$ and

$$\left| \frac{dx_i}{d\phi} \right| = |-\xi_{1,i} \sin \phi + \xi_{2,i} \cos \phi| \leq 2.$$

Similarly,

$$\left| \frac{d^2}{d\phi^2}(\rho_W(\phi)) \right| \leq 4 \left| \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (f_\varepsilon(\theta)r^{-n+1})^\wedge(x) \right|.$$

By Lemma 2.4 we have $\rho' = O(|\ln \varepsilon| \varepsilon^{n-3})$ and $\rho'' = O(\varepsilon^{n-4})$. Since these estimates are uniform with respect to ϕ and W , it follows that, for small enough ε , $J(W, \varepsilon, \phi) > 0$ for every ϕ and W . \square

In [K1, Lemma 1] the following was proved. Let f be an even continuous homogeneous function of degree $-n + 1$ on $\mathbb{R}^n \setminus \{0\}$. Then \hat{f} is a continuous function outside of the origin and for every $\xi \in S^{n-1}$,

$$(5) \quad \hat{f}(\xi) = \pi \int_{S^{n-1} \cap \{(\theta, \xi) = 0\}} f(\theta) d\theta.$$

Using this formula,

$$(6) \quad ((1 - f_\varepsilon(\theta))r^{-n+1})^\wedge(x) = \pi \int_{S^{n-1} \cap \{(x, \theta) = 0\}} (1 - f_\varepsilon(\theta)) d\theta.$$

Lemma 2.6. *For every $(n - 1)$ -dimensional subspace V of \mathbb{R}^n , the body $K \cap V$ is an intersection body.*

Proof. Fix an $(n - 1)$ -dimensional subspace V of \mathbb{R}^n . By the definition (1) of the body K and formula (6)

$$\frac{1}{\pi} \|x\|_K^{-1} = \int_{S^{n-1} \cap \{(x, \theta) = 0\}} (1 - f_\varepsilon(\theta)) d\theta.$$

In particular, for $x \in V$,

$$\frac{1}{\pi} \|x\|_{K \cap V}^{-1} = \int_{S^{n-1} \cap \{(x, \theta) = 0, x \in V\}} (1 - f_\varepsilon(\theta)) d\theta,$$

where $\{(x, \theta) = 0, x \in V\}$ is the hyperplane of all θ that are perpendicular to x for a fixed $x \in V$.

Let us change coordinates from $\theta \in S^{n-2} = S^{n-1} \cap \{(x, \theta) = 0, x \in V\}$ to $\phi \in [0, \pi]$ and $\xi \in S^{n-3} = S^{n-2} \cap \{(x, \theta) = 0, x \in V\}$. The Jacobian is equal to $(\sin \phi)^{n-3}$; cf. [M, p. 1]. So

$$\begin{aligned} \frac{1}{\pi} \|x\|_{K \cap V}^{-1} &= \int_{S^{n-2} \cap \{(x, \theta) = 0, x \in V\}} \left(\int_0^\pi (1 - f_\varepsilon(\xi, \phi)) (\sin \phi)^{n-3} d\phi \right) d\xi \\ &= \int_{S^{n-2} \cap \{(x, \theta) = 0, x \in V\}} \left(\int_0^\pi (\sin \phi)^{n-3} d\phi - \int_0^\pi f_\varepsilon(\xi, \phi) (\sin \phi)^{n-3} d\phi \right) d\xi. \end{aligned}$$

Taking the Fourier transform of both sides, as functions of the variable $x \in V$, and using (5)

$$(\|x\|_{K \cap V}^{-1})^\wedge(\theta) = \pi \int_0^\pi (\sin \phi)^{n-3} d\phi - \pi \int_0^\pi f_\varepsilon(\theta) (\sin \phi)^{n-3} d\phi$$

for $\theta \in S^{n-2}$.

By the definition, f_ε is non-zero only in an ε -neighborhood of x_0 , so there exists a set $R_\varepsilon \subset [0, \pi]$ such that $f_\varepsilon(\xi, \phi) = 0$ for $\phi \in [0, \pi] \setminus R_\varepsilon$. Also, $|f_\varepsilon| \leq 2$ and $|R_\varepsilon|$, the length of the one-dimensional set R_ε , is of the order ε . Therefore,

$$\begin{aligned} \int_0^\pi f_\varepsilon(\xi, \phi) (\sin \phi)^{n-3} d\phi &= \int_{R_\varepsilon} f_\varepsilon(\xi, \phi) (\sin \phi)^{n-3} d\phi \\ &\leq 2|R_\varepsilon| = C\varepsilon, \end{aligned}$$

where C does not depend of the choice of V .

Since $\int_0^\pi (\sin \phi)^{n-3} d\phi$ is equal to some positive constant depending on n only,

$$\int_0^\pi (\sin \phi)^{n-3} d\phi - \int_0^\pi f_\varepsilon(\theta) (\sin \phi)^{n-3} d\phi > 0$$

for a sufficiently small ε , which means we can find ε small enough that $(\|x\|_{K \cap V}^{-1})^\wedge(\theta) > 0$ for all θ . Therefore by [K2, Thm. 1], for small enough ε , for every $(n - 1)$ -dimensional subspace V of \mathbb{R}^n , the body $K \cap V$ is an intersection body. \square

Now Theorem 2.1 follows from Lemmas 2.3, 2.5 and 2.6.

Remark 1. By a result of Neyman [N], for any finite system of equations and inequalities involving the norms, there exists a subspace of L_p , $0 < p < 2$, that does not satisfy this system. Since, by [K2], the unit ball of every finite-dimensional subspace of L_p , $0 < p < 2$, is an intersection body, we conclude that the class of intersection bodies cannot be characterized by a finite number of equations or inequalities.

Remark 2. For $n = 5$ the fact that there exists a non-intersection body whose central sections are intersection bodies follows from the solution of the Busemann-Petty problem, since every four-dimensional symmetric convex body is an intersection body; see [GKS], [Zh2].

Remark 3. It is known that the polar of zonoid is an intersection body (see [K5, p. 127]), but the converse is not true, and the polar of an intersection body may not be a zonoid, as for example a cube in \mathbb{R}^4 . Therefore, the result in this paper does not follow the result of W.Weil on zonoids by applying duality argument, and a new example should be constructed.

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