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ON THE HARTSHORNE–SPEISER–LYUBEZNIK THEOREM ABOUT ARTINIAN MODULES WITH A FROBENIUS ACTION

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ABSTRACT. Let R be a commutative Noetherian local ring of prime characteristic. The purpose of this paper is to provide a short proof of G. Lyubeznik's extension of a result of R. Hartshorne and R. Speiser about a module over the skew polynomial ring $R[x, f]$ (associated to R and the Frobenius homomorphism f, in the indeterminate x) that is both x-torsion and Artinian over R.

0. INTRODUCTION

In the theory of tight closure of ideals in a d-dimensional commutative (Noetherian) local ring $(R, \mathfrak{m})$ of prime characteristic p, the study of properties of the 'top' local cohomology module $H^d_{\mathfrak{m}}(R)$ related to the Frobenius homomorphism $f : R \longrightarrow R$ has been a very effective tool (see, for example, K. E. Smith [10, 11]). Some of the properties of $H^d_{\mathfrak{m}}(R)$ related to f can be neatly described in terms of a natural structure which $H^d_{\mathfrak{m}}(R)$ possesses as a left module over the skew polynomial ring $R[x, f]$; also, it is well known that $H^d_{\mathfrak{m}}(R)$ is Artinian as an R-module. One can take the view that $H^d_{\mathfrak{m}}(R)$ is an important example of a left $R[x, f]$-module that is Artinian as an R-module.

In 1977, R. Hartshorne and R. Speiser [2, Proposition 1.11] proved, in the case where the local ring R of characteristic p contains its residue field which is perfect, that, given a left $R[x, f]$-module H that is Artinian as an R-module, there exists a non-negative integer e with the following property: whenever $h \in H$ is such that $x^j h = 0$ for some positive integer j, then $x^e h = 0$.

Twenty years later, G. Lyubeznik [7, Proposition 4.4] proved this result without restriction on the local ring R of characteristic p, that is, he was able to drop the hypotheses about the residue field of R. Lyubeznik's proof is an application of his substantial theory of F-modules.

There is some evidence that the Hartshorne–Speiser–Lyubeznik Theorem can be exploited to good effect in tight closure theory. For example, it has recently been used in [9] to prove that, if c is a test element for a reduced excellent equidimensional local ring $(R, \mathfrak{m})$ of characteristic p, then there exists a power of p that is

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a test exponent for $c, \mathfrak{a}$ (see [3, Definition 2.2]) for every parameter ideal $\mathfrak{a}$ of R simultaneously.

It therefore seems desirable to have a short proof of the Hartshorne–Speiser–Lyubeznik Theorem that does not rely on the theory of F-modules. This paper provides one that actually follows the general line of the Hartshorne–Speiser proof.

1. Left modules over the skew polynomial ring $R[x, f]$

1.1. **Notation.** Throughout the paper, A will denote a general commutative Noetherian ring, and R will denote a commutative Noetherian ring of prime characteristic p. In cases where such a ring is assumed to be local, the notation $(A, \mathfrak{m})$ or $(R, \mathfrak{m})$ will indicate that $\mathfrak{m}$ is the maximal ideal.

We shall always denote by $f : R \longrightarrow R$ the Frobenius homomorphism, for which $f(r) = r^p$ for all $r \in R$. We use $\mathbb{N}$ and $\mathbb{N}_0$ to denote the sets of positive integers and non-negative integers, respectively. We shall work with the skew polynomial ring $R[x, f]$ associated to R and f in the indeterminate x over R. Recall that $R[x, f]$ is, as a left R-module, freely generated by $(x^i)_{i \in \mathbb{N}_0}$, and so consists of all polynomials $\sum_{i=0}^{n} r_i x^i$, where $n \in \mathbb{N}_0$ and $r_0, \ldots, r_n \in R$; however, its multiplication is subject to the rule

$$xr = f(r)x = r^p x \quad \text{for all } r \in R.$$

1.2. **Definition and remarks.** We say that the left $R[x, f]$-module H is *x-torsion-free* if $xh = 0$, for $h \in H$, only when $h = 0$. The set $\Gamma_x(H) := \{h \in H : x^j h = 0 \text{ for some } j \in \mathbb{N}\}$ is an $R[x, f]$-submodule of H, called the *x-torsion submodule* of H. In general, the $R[x, f]$-module $H/\Gamma_x(H)$ is x-torsion-free.

2. The Hartshorne–Speiser Theorem

As explained in the Introduction, this paper is concerned with the following result of R. Hartshorne and R. Speiser.

2.1. **Theorem** (Hartshorne–Speiser [2, Proposition 1.11])**.** *Suppose that R is local and contains its residue field which is perfect. Let H be a left $R[x, f]$-module which is Artinian as an R-module. Then there exists $e \in \mathbb{N}_0$ such that $x^e \Gamma_x(H) = 0$.*

G. Lyubeznik [7, Proposition 4.4] proved this result without restriction on the local ring R of characteristic p, that is, he was able to drop the hypotheses about the residue field of R; his proof is an application of his theory of F-modules. The main purpose of this section is to show how one can modify the argument of Hartshorne and Speiser to obtain a short and direct proof of the result in the generality achieved by Lyubeznik. To achieve this aim, we shall establish a generalization of Proposition 1.9 of Hartshorne–Speiser [2].

Our first preparatory result concerns an Artinian module of finite injective dimension over a general local ring $(A, \mathfrak{m})$. Let E denote $E_A(A/\mathfrak{m})$, the injective envelope of the simple A-module $A/\mathfrak{m}$. Recall that an A-module is Artinian if and only if it is isomorphic to a submodule of E^t, the direct sum of t copies of E, for some $t \in \mathbb{N}$. It follows that, if G is an Artinian A-module, then, for each $i \in \mathbb{N}_0$, the i-th term $E_A^i(G)$ in the minimal injective resolution of G is isomorphic to a direct sum of finitely many copies of E. When J is an Artinian injective A-module, we shall use the Bass number $\mu^0(\mathfrak{m}, J)$ to denote the number of copies of E that occur in a decomposition of J as a direct sum of indecomposable injective A-modules.

2.2. **Proposition.** *Let G be an Artinian module over the local ring $(A, \mathfrak{m})$ such that* $\operatorname{inj\,dim}_A G < \infty$.

(i) *Let*

$$I^\bullet : 0 \longrightarrow I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots \longrightarrow I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

be a finite injective resolution of G in which each term is isomorphic to a direct sum of copies of $E := E_A(A/\mathfrak{m})$. (It should be noted that the minimal injective resolution of G has this property.) Then the integer $\sum_{i=0}^{\infty}(-1)^i \mu^0(\mathfrak{m}, I^i)$ is independent of the choice of the finite injective resolution $I^\bullet$ of G having the stated property. We call this integer the Euler number *of G, and denote it by $\chi(G)$ (or $\chi_A(G)$ when it is desirable to emphasize the local ring A).*

(ii) *The Euler number $\chi(G)$ of G is non-negative.*

(iii) *Let $G', \overline{G}$ be further Artinian A-modules of finite injective dimension and suppose that there is an exact sequence $0 \longrightarrow G' \longrightarrow G \longrightarrow \overline{G} \longrightarrow 0$ in the category of A-modules and A-homomorphisms. Then*

$$\chi(G) = \chi(G') + \chi(\overline{G}).$$

(iv) *When A is complete, the following three conditions are equivalent:*

(a) $(0 :_A G) \neq 0$;

(b) $\chi(G) = 0$;

(c) $(0 :_A G)$ *contains a non-zerodivisor of A.*

Proof. (i),(ii),(iii) There is an A-homomorphism $\alpha : G \longrightarrow I^0$ such that the sequence

$$0 \longrightarrow G \xrightarrow{\alpha} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots \longrightarrow I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

is exact. Note that E, G, G', $\overline{G}$ and all the I^j ($j \in \mathbb{N}_0$) have natural structures as modules over the completion $(\widehat{A}, \widehat{\mathfrak{m}})$ of A, and that, when they are given these, there is an $\widehat{A}$-isomorphism $E \cong E_{\widehat{A}}(\widehat{A}/\widehat{\mathfrak{m}})$ and the above-displayed exact sequence provides an injective resolution of G as an $\widehat{A}$-module. Furthermore, $0 \longrightarrow G' \longrightarrow G \longrightarrow \overline{G} \longrightarrow 0$ is an exact sequence in the category of $\widehat{A}$-modules and $\widehat{A}$-homomorphisms. It thus follows that it is sufficient to prove parts (i), (ii) and (iii) under the additional assumption that A is complete.

Let D be the functor $\operatorname{Hom}_A(\,\bullet\,, E)$ on the category of A-modules. We use Matlis duality. Since $D(E) \cong A$, application of the functor D to $I^\bullet$ yields an exact sequence

$$\cdots \longrightarrow D(I^{i+1}) \longrightarrow D(I^i) \longrightarrow \cdots \longrightarrow D(I^0) \longrightarrow D(G) \longrightarrow 0,$$

and this provides a finite free resolution of the finitely generated A-module $D(G)$. Moreover, for each $i \in \mathbb{N}_0$, the free A-module $D(I^i)$ is finitely generated of rank $\mu^0(\mathfrak{m}, I^i)$. Thus

$$\sum_{i=0}^{\infty}(-1)^i \mu^0(\mathfrak{m}, I^i) = \sum_{i=0}^{\infty}(-1)^i \operatorname{rank} D(I^i),$$

which is just the Euler number $\chi(D(G))$, and so is independent of the choice of finite injective resolution $I^\bullet$ of G of the type under consideration (see [8, p. 159], for example). Likewise, the claim in part (ii) now follows from the corresponding statement (see [8, Theorem 19.7], for example) about modules with finite free resolutions, and the claim in part (iii) follows from the well-known fact that χ is additive on short exact sequences of modules with finite free resolutions.

(iv) Since the annihilators of G and $D(G)$ are equal, the equivalence of (a), (b) and (c) is now immediate from a theorem of M. Auslander and D. A. Buchsbaum [1] (see [8, Theorem 19.8], for example). $\square$

2.3. *Remark.* It is a consequence of Proposition 2.2 that, with the notation of that result, $\chi_A(G) = \sum_{i=0}^{\infty}(-1)^i\mu^i(\mathfrak{m}, G) = \sum_{i=0}^{\operatorname{inj}\dim G}(-1)^i\mu^i(\mathfrak{m}, G)$, because, for each $i \in \mathbb{N}_0$, the i-th term in the minimal injective resolution of G is isomorphic to the direct sum of $\mu^i(\mathfrak{m}, G)$ copies of E, and $\mu^j(\mathfrak{m}, G) = 0$ for all $j > \operatorname{inj}\dim_A G$.

We can now establish the promised generalization of Proposition 1.9 of Hartshorne–Speiser [2].

2.4. **Proposition** (Compare Hartshorne–Speiser [2, Proposition 1.9])**.** *Assume that $(R, \mathfrak{m})$ is a complete regular local ring, and that H is a left $R[x, f]$-module which is Artinian as an R-module and such that $RxH = H$. Let $K = \{h \in H : xh = 0\}$, an $R[x, f]$-submodule of H. Then $(0 :_R K) \neq 0$.*

Proof. Here, we shall use R' to denote R considered as an R-module by means of f (at points where care is needed). Also F will denote the *Frobenius functor* $R' \otimes_R (\,\bullet\,)$ from the category of all R-modules and R-homomorphisms to the category of all R'-modules and R'-homomorphisms.

Since $axrh = ar^pxh$ for $a \in R'$, $r \in R$ and $h \in H$, there is an R-homomorphism $\phi : F(H) \longrightarrow H$ for which $\phi(a \otimes h) = axh$ for all $h \in H$ and $a \in R'$. Note that ϕ is surjective because $RxH = H$. Note also that, if $h \in K$, then the element $1 \otimes h$ of $F(H)$ lies in $\operatorname{Ker}\phi$. Since R is regular, $f : R \longrightarrow R$ is flat (by E. Kunz [6]), and therefore faithfully flat. The $\mathbb{Z}$-homomorphism $\gamma : K \longrightarrow \operatorname{Ker}\phi$ for which $\gamma(h) = 1 \otimes h$ for all $h \in K$ is therefore injective. It is therefore enough for us to show that $(0 :_R \operatorname{Ker}\phi) \neq 0$, for if $0 \neq a \in R$ annihilates $\operatorname{Ker}\phi$, then, for each $h \in K$, we have $\gamma(ah) = 1 \otimes ah = a^p \otimes h = a^p(1 \otimes h) = 0$.

There is a short exact sequence

$$0 \longrightarrow \operatorname{Ker}\phi \hookrightarrow F(H) = R' \otimes_R H \xrightarrow{\phi} H \longrightarrow 0$$

of R-modules and R-homomorphisms. By Huneke–Sharp [4, Proposition 1.5], for each injective R-module I, we have $F(I) \cong I$. Observe that every R-module has finite injective dimension because R has finite global dimension. If one applies the exact functor F to the minimal injective resolution for H, one can deduce, with the aid of Proposition 2.2, that $F(H)$ is isomorphic to a submodule of the direct sum of finitely many copies of $F(E) \cong E$ and so is Artinian, and that $\chi(F(H)) = \chi(H)$. Hence $\operatorname{Ker}\phi$ is an Artinian R-module, and it follows from Proposition 2.2(iii) that $\chi(\operatorname{Ker}\phi) = \chi(F(H)) - \chi(H) = 0$. Hence $(0 :_R \operatorname{Ker}\phi) \neq 0$ by Proposition 2.2(iv). $\square$

We shall need the following lemma of Hartshorne and Speiser.

2.5. **Lemma** (Hartshorne–Speiser [2, Lemma 1.10])**.** *Let H be a left $R[x, f]$-module, and set*

$$K := \{h \in H : xh = 0\},$$

an $R[x, f]$-submodule of H. Suppose that $a \in R$ is such that $aK = 0$. Then $a^2\Gamma_x(H) = 0$.

The short proof, presented in the next theorem, of Lyubeznik's extension of the Hartshorne–Speiser Theorem follows the general line of argument of Hartshorne and Speiser.

2.6. **Theorem** (G. Lyubeznik [7, Proposition 4.4]; compare Hartshorne–Speiser [2, Proposition 1.11])**.** *Suppose that* $(R, \mathfrak{m})$ *is local, and let* H *be a left* $R[x, f]$*-module which is Artinian as an* R*-module. Then there exists* $e \in \mathbb{N}_0$ *such that* $x^e\Gamma_x(H) = 0$.

Proof. Recall the natural $\widehat{R}$-module structure on the Artinian R-module H: given $h \in H$, there exists $t \in \mathbb{N}$ such that $\mathfrak{m}^t h = 0$; for an $\widehat{r} \in \widehat{R}$, choose any $r \in R$ such that $\widehat{r} - r \in \mathfrak{m}^t\widehat{R}$; then $\widehat{r}h = rh$. It is easy to see from this that $x\widehat{r}h = \widehat{r}^p xh$ for all $h \in H$ and $\widehat{r} \in \widehat{R}$, and we can then use [5, Lemma 1.3] to see that one can assume that R is complete.

Argue by induction on $n := \dim R$; note that, when $n = 0$, the Artinian R-module H has finite length and then the claim follows easily. Suppose that $n > 0$ and assume inductively that the result has been proved when the underlying complete local ring R has dimension smaller than n.

Let k be a coefficient field for R, and let $r_1, \dots, r_n$ be a system of parameters for R. Then R is module-finite over the complete regular local ring $k[[r_1, \dots, r_n]]$, which we denote by $(S, \mathfrak{n})$. Since $\mathfrak{m} \cap S = \mathfrak{n}$, it is clear that each element of H is annihilated by some power of $\mathfrak{n}$. Since $\mathfrak{n}R$ is $\mathfrak{m}$-primary, it contains $\mathfrak{m}^t$ for some $t \in \mathbb{N}$, and so $(0 :_H \mathfrak{n}) = (0 :_H \mathfrak{n}R) \subseteq (0 :_H \mathfrak{m}^t)$, which is finitely generated over R and therefore over S. Thus H is Artinian as an S-module.

We can replace H by its $R[x, f]$- and $S[x, f]$-submodule $\Gamma_x(H)$; thus we can assume that H is x-torsion. Of course, we can assume that $H \neq 0$.

The descending chain of $S[x, f]$-submodules

$$H \supseteq SxH \supseteq Sx^2H \supseteq \cdots \supseteq Sx^iH \supseteq Sx^{i+1}H \supseteq \cdots$$

of H must eventually stabilize: let $t \in \mathbb{N}_0$ be such that $Sx^tH = Sx^{t+j}H$ for all $j \in \mathbb{N}$. Observe that $Sx^tH = Sx(Sx^tH)$, and that it is enough to prove the claim for Sx^tH rather than H.

It thus follows that, in order to complete the inductive step, we can (replace R by S and) assume that R is regular (and complete), that H is x-torsion and that $H = RxH$.

Let $K = \{h \in H : xh = 0\}$. By Proposition 2.4, there exists $0 \neq a \in R$ such that $aK = 0$. Therefore $a^2\Gamma_x(H) = a^2H = 0$, by Lemma 2.5.

Thus H has a natural structure as a module over the complete local ring R/Ra^2, which has dimension $n - 1$. Use $\overline{r}$ to denote the natural image in R/Ra^2 of an element $r \in R$. Then $x\overline{r}h = \overline{r}^p xh$ for all $r \in R$ and $h \in H$; it thus follows from [5, Lemma 1.3] that H inherits a structure as left $(R/Ra^2)[x, f]$-module, compatible with its $R[x, f]$-module structure; note that H is still Artinian over R/Ra^2 and x-torsion, and satisfies $H = (R/Ra^2)xH$. Application of the inductive hypothesis therefore completes the proof. $\square$

I am grateful to Craig Huneke for pointing out that the argument used in the above proof can be modified to prove the next result.

2.7. **Theorem.** *Suppose that* $(R, \mathfrak{m})$ *is local, and let* H *be an* x*-torsion left* $R[x, f]$*-module which is Artinian as an* R*-module. If* $H = RxH$*, then* H *has finite length as an* R*-module.*

Proof. Only a sketch is presented here, as the strategy used is very similar to that used in the above proof of Theorem 2.6.

One can assume that R is complete; then argue by induction on $n := \dim R$, the result being easy when $n = 0$. For the inductive step, in the situation where $n > 0$, again let k be a coefficient field for R, let $r_1, \ldots, r_n$ be a system of parameters for R, and let S be the complete regular local ring $k[[r_1, \ldots, r_n]]$. We can again use the fact that R is module-finite over S to see that H is Artinian as an S-module, so that there exists $t \in \mathbb{N}_0$ such that $Sx^tH = Sx^{t+j}H$ for all $j \in \mathbb{N}$.

Thus the left $S[x, f]$-module Sx^tH is x-torsion, Artinian as an S-module, and such that $Sx^tH = Sx(Sx^tH)$. Let $e_1 = 1, e_2, \ldots, e_h$ generate R as an S-module. If we could prove that Sx^tH is of finite length as an S-module, then, since $H = RxH$, it would follow that

$$H = Rx^tH = \sum_{i=1}^{h} Se_ix^tH = \sum_{i=1}^{h} e_iSx^tH$$

is finitely generated as an S-module, and therefore as an R-module. It follows that, in order to complete the inductive step, we can (replace R by S and) assume that R is regular (and complete). The proof can now be completed by an argument almost identical to that in the last two paragraphs of the above proof of Theorem 2.6. □

2.8. *Remark.* Note that an extension of the Hartshorne–Speiser–Lyubeznik Theorem to non-local situations is proved in [9, Corollary 1.8]. There it is proved that, if R is merely a commutative Noetherian ring (of characteristic p), and H is a left $R[x, f]$-module which is Artinian as an R-module, then there exists $e \in \mathbb{N}_0$ such that $x^e\Gamma_x(H) = 0$.

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