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GLOBAL EXISTENCE OF SOLUTIONS TO A HYPERBOLIC-PARABOLIC SYSTEM

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Abstract. In this paper, we investigate the global existence of solutions to a hyperbolic-parabolic model of chemotaxis arising in the theory of reinforced random walks. To get L^2-estimates of solutions, we construct a nonnegative convex entropy of the corresponding hyperbolic system. The higher energy estimates are obtained by the energy method and *a priori* assumptions.

1. Introduction and main result

In this paper, we consider the following system:

$$\begin{cases} u_t - v_x = 0, \\ v_t - (uv)_x = v_{xx}, \end{cases} \tag{1.1}$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \geq 0, \tag{1.2}$$

and initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x) > 0, \quad x \in [0,1]. \tag{1.3}$$

Here the compatibility conditions on u_0, v_0 assume that $(u_0(0), v_0(0)) = (0,0)$, which will be used in Section 3.

Motivated by biological considerations and numerical computations carried out by Othmer and Stevens in [6] and Levine and Sleeman in [3], the system (1.1) comes from:

$$\begin{cases} \dfrac{\partial p}{\partial t} = D\dfrac{\partial}{\partial x}\left(p\dfrac{\partial}{\partial x}\left(\ln\dfrac{p}{\Phi(w)}\right)\right), \qquad x \in (0,l), \quad t > 0, \\ \dfrac{\partial w}{\partial t} = R(p,w), \end{cases} \tag{1.4}$$

where $p(x,t)$ is the particle density of a particular species, $w(x,t)$ is the concentration of the "active agent", and D and B are positive constants.

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In fact, as in [7], let

$$\Phi(w) = w^{-\alpha}, \quad R(p,w) = \lambda pw - \mu w, \tag{1.5}$$

where α and λ, μ are positive constants.

Then the system (1.4) is transformed into the following form:

$$\begin{cases} p_t = Dp_{xx} + D\alpha \left(p\dfrac{w_x}{w}\right)_x, \\ w_t = \lambda pw - \mu w. \end{cases} \tag{1.6}$$

Furthermore, set

$$q = (\ln w)_x = \frac{w_x}{w}. \tag{1.7}$$

Then the system (1.6) can be rewritten as:

$$\begin{cases} p_t = Dp_{xx} + D\alpha(pq)_x, \\ q_t = \lambda p_x. \end{cases} \tag{1.8}$$

Let

$$\tau = At, \quad \xi = lx, \quad p = Bv, \quad q = c_1 u, \tag{1.9}$$

where A, l and c_1 are positive constants to be determined below.

Then the system (1.8) becomes

$$\begin{cases} u_\tau = \dfrac{\lambda l B}{A c_1} v_\xi, \\ v_\tau = \dfrac{Dl^2}{A} v_{\xi\xi} + \dfrac{D\alpha l c_1}{A} (uv)_\xi. \end{cases} \tag{1.10}$$

Choosing

$$\begin{cases} \dfrac{\lambda l B}{A c_1} = 1, \\ \dfrac{Dl^2}{A} = 1, \\ \dfrac{D\alpha l c_1}{A} = 1, \end{cases}$$

i.e.,

$$A = B\alpha\lambda > 0, \quad l = \sqrt{\frac{B\alpha\lambda}{D}} > 0, \quad c_1 = \sqrt{\frac{B\lambda}{\alpha D}} > 0, \tag{1.11}$$

then it is easy to verify that u, v satisfy

$$\begin{cases} u_\tau = v_\xi, \\ v_\tau = v_{\xi\xi} + (uv)_\xi. \end{cases} \tag{1.12}$$

If we replace (ξ, τ) by (x, t), then (1.12) can be rewritten as (1.1).

The system (1.4) describes the model of chemotaxis in biology. Othmer and Stevens [6] have developed a number of mathematical models of chemotaxis to illustrate aggregation leading (numerically) to nonconstant steady-states, blow-up resulting in the formation of singularities and collapse or the formation of a spatially uniform steady state. The models developed in [6] have been studied in depth by Levine and Sleeman [3]. They gave some heuristic understanding of some of these phenomena and investigated the properties of solutions of a system of chemotaxis

equations arising in the theory of reinforced random walks. Y. Yang, H. Chen and W.A. Liu [9] studied the global existence and blow-up in a finite time of solutions for the case considered in [3], respectively, and found that even at the same growth rate the behavior of the biological systems can be very different just because they started their action under different conditions. For the other results on the initial boundary value problem of (1.4) refer to [1, 4].

In this paper, we will study the global existence of solutions to the initial boundary value problem (1.1)-(1.3) in $L^\infty([0,\infty),\ H^2([0,1]))$. To get L^2-estimates of solutions, we construct a nonnegative convex entropy of the corresponding hyperbolic system. The higher energy estimates are obtained by the energy method and *a priori* assumptions.

The corresponding hyperbolic system of (1.1) is

$$\begin{cases} u_t - v_x = 0, \\ v_t - (uv)_x = 0. \end{cases} \tag{1.13}$$

The eigenvalues of (1.13) are:

$$\lambda_1 = -\frac{1}{2}u - \frac{1}{2}\sqrt{u^2+4v}, \qquad \lambda_2 = -\frac{1}{2}u + \frac{1}{2}\sqrt{u^2+4v}. \tag{1.14}$$

Therefore, the system (1.13) is strictly hyperbolic when $v > 0$.

Remark 1.1. By the boundary conditions (1.2) and $(1.1)_1$, we have

$$u_t(0,t) = u_t(1,t) = v_x(0,t) = v_x(1,t) = 0. \tag{1.15}$$

Notation. Throughout this paper, we denote positive constants by C. Moreover, the character "C" may differ in different places. $L^p = L^p([0,1])$ $(1 \le p \le \infty)$ denotes the usual Lebesgue space with the norm

$$||f||_{L^p} = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$|f|_\infty = \sup_{[0,1]} |f(x)|.$$

$H^l([0,1])$ $(l \ge 0)$ denotes the usual lth-order Sobolev space with the norm

$$||f||_l = \left(\sum_{j=0}^{l} ||\partial_x^j f||^2\right)^{\frac{1}{2}},$$

where $||\cdot|| = ||\cdot||_0 = ||\cdot||_{L^2}$. For simplicity, $||f(\cdot,t)||_{L^p}$ and $||f(\cdot,t)||_l$ are denoted by $||f(t)||_{L^p}$ and $||f(t)||_l$ respectively.

We will prove the following global existence theorem.

Theorem 1.2. *Let $u_0,\ v_0 - 1 \in H^2([0,1])$ and let $||u_0||_2^2 + ||v_0 - 1||_2^2$ be sufficiently small. Then there exists a unique global solution $(u(x,t), v(x,t))$ of (1.1)-(1.3) satisfying*

$$\text{(i)} \quad u,\ v \in L^\infty([0,\infty), H^2([0,1])), \quad v_x \in L^2([0,\infty), H^2([0,1]));$$

$$\text{(ii)} \quad ||u(t)||_2^2 + ||v(t)-1||_2^2 + \int_0^t ||v_x(\tau)||_2^2 d\tau \le C\left(||u_0||_2^2 + ||v_0 - 1||_2^2\right). \tag{1.16}$$

2. L^2-ENERGY ESTIMATES

In this section, we give L^2-energy estimates of the initial boundary value problem (1.1), (1.2) and (1.3) by a nonnegative convex entropy of the system of hyperbolic conservation laws (1.13). To do this, we first give the following relation on the entropy-entropy flux pair $(\eta(u,v), q(u,v))$ of (1.13) (see [8]):

$$\begin{cases} q_u = -v\eta_v, \\ q_v = -\eta_u - u\eta_v. \end{cases} \tag{2.1}$$

Eliminating q from (2.1), we have

$$\eta_{uu} + u\eta_{uv} - v\eta_{vv} = 0. \tag{2.2}$$

Next, we seek the entropy of (1.13) with the following form:

$$\eta(u,v) = \frac{1}{2}u^2 + a(v), \tag{2.3}$$

where $a(v)$ is a nonnegative convex function.

Substituting (2.3) into (2.2), we have

$$1 - va''(v) = 0,$$

which implies

$$a(v) = v\ln v - v + k_1 v + k_2, \tag{2.4}$$

where $k_1,\ k_2$ are arbitrary constants.

It is easy to get the flux corresponding to the entropy $\eta(u,v)$ defined by (2.3) and (2.4), namely

$$q(u,v) = -uv\ln v - k_1 uv + k_3, \tag{2.5}$$

where k_3 is an arbitrary constant.

In particular, if we take $k_1 = k_3 = 0,\ k_2 = 1$, we will get an entropy-entropy flux pair of (1.13):

$$\begin{cases} \eta(u,v) = \frac{1}{2}u^2 + v\ln v - v + 1, \\ q(u,v) = -uv\ln v. \end{cases} \tag{2.6}$$

In the next analysis, we devote ourselves to the estimates of the solution $(u(x,t), v(x,t))$ of (1.1), (1.2) and (1.3) under the *a priori* assumptions:

$$|u| \le \varepsilon, \quad |v-1| \le \frac{1}{2}, \quad |u_x| \le \varepsilon, \quad |v_x| \le \varepsilon, \tag{2.7}$$

where $0 < \varepsilon << 1$.

Lemma 2.1. *The entropy $\eta(u,v)$ defined by $(2.6)_1$ satisfies for $|v-1| \le \frac{1}{2}$,*

$$\frac{1}{2}u^2 + \frac{1}{3}(v-1)^2 \le \eta(u,v) \le \frac{1}{2}u^2 + (v-1)^2. \tag{2.8}$$

Proof. Let

$$a_0(v) = v\ln v - v + 1. \tag{2.9}$$

Then

$$a_0(1) = a_0'(1) = 0, \quad \frac{2}{3} \le a_0''(v) = \frac{1}{v} \le 2,$$

which implies

$$\frac{1}{3}(v-1)^2 \le a_0(v) = \frac{1}{2}a_0''(\xi)(v-1)^2 \le (v-1)^2, \tag{2.10}$$

where ξ is between 1 and v. □

From $(2.6)_1$, (2.9) and (2.10), (2.8) follows.

This proves Lemma 2.1.

Lemma 2.2 (L^2-energy estimates)**.** *Under the assumptions of Theorem* 1.2 *and the a priori assumptions* (2.7)*, we have*

$$\int_0^1 \left(\frac{1}{2}u^2 + \frac{1}{3}(v-1)^2\right) dx + \frac{2}{3}\int_0^t \int_0^1 v_x^2 dx d\tau \le \int_0^1 \left(\frac{1}{2}u_0^2(x) + (v_0(x)-1)^2\right) dx. \tag{2.11}$$

Proof. Multiplying (1.1) by $\nabla\eta$ and integrating it, we have by the boundary conditions (1.2) and (1.15),

$$\int_0^1 \eta(x,t)dx + \int_0^t \int_0^1 \frac{v_x^2}{v} dx ds = \int_0^1 \eta(x,0)dx, \tag{2.12}$$

i.e.,

$$\int_0^1 \left(\frac{1}{2}u^2 + v\ln v - v + 1\right) dx + \int_0^t \int_0^1 \frac{v_x^2}{v} dx ds = \int_0^1 \left(\frac{1}{2}u_0^2 + v_0 \ln v_0 - v_0 + 1\right) dx. \tag{2.13}$$

This proves Lemma 2.2 by Lemma 2.1 and the *a priori* assumptions (2.7). □

3. Higher energy estimates

In this section, we will establish higher energy estimates.

Lemma 3.1. *Under the assumptions of Theorem* 1.2 *and the a priori assumptions* (2.7)*, we have*

$$\int_0^1 \left(u_x^2 + \frac{2}{3}v_x^2\right) dx + \frac{4}{3}\int_0^t \int_0^1 v_{xx}^2 dx ds \le C\left(||u_0||_1^2 + ||v_0 - 1||_1^2\right), \tag{3.1}$$

where C is a positive constant.

Proof. Differentiating (1.1) with respect to x, we have

$$\begin{cases} u_{xt} - v_{xx} = 0, \\ v_{xt} - (uv)_{xx} = v_{xxx}. \end{cases} \tag{3.2}$$

For any smooth function $g_1(v)$, take $(3.2)_1 \times u_x + (3.2)_2 \times g_1(v)v_x$, and integrate it to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 u_x^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 g_1(v)v_x^2 dx \\ = &\int_0^1 u_x v_{xx} dx + \frac{1}{2}\int_0^1 g_1'(v)v_x^2 v_t dx + \int_0^1 g_1(v)uv_x v_{xx} dx \\ &+2\int_0^1 g_1(v)u_x v_x^2 dx + \int_0^1 g_1(v)u_{xx}vv_x dx + \int_0^1 g_1(v)v_x v_{xxx} dx. \end{aligned} \tag{3.3}$$

Next, we estimate the terms in the right side of (3.3) as follows:

$$\begin{aligned}\frac{1}{2}\int_0^1 g_1'(v)v_x^2v_tdx = &\ \frac{1}{2}\int_0^1 g_1'(v)v_x^2(uv)_xdx+\frac{1}{2}\int_0^1 g_1'(v)v_x^2v_{xx}dx\\ = &\ -\frac{1}{6}\int_0^1 g_1''(v)v_x^4dx+\frac{1}{6}\left(g_1'(v)v_x^3\right)\Big|_0^1\\ &+\frac{1}{2}\int_0^1 g_1'(v)uv_x^3dx+\frac{1}{2}\int_0^1 g_1'(v)u_xvv_x^2dx,\end{aligned}\tag{3.4}$$

$$\begin{aligned}\int_0^1 g_1(v)uv_xv_{xx}dx &= -\frac{1}{2}\int_0^1 (g_1(v)u)_xv_x^2dx+\frac{1}{2}\left(g_1(v)uv_x^2\right)\Big|_0^1\\ &= -\frac{1}{2}\int_0^1 g_1(v)u_xv_x^2dx-\frac{1}{2}\int_0^1 g_1'(v)uv_x^3dx+\frac{1}{2}\left(g_1(v)uv_x^2\right)\Big|_0^1,\end{aligned}\tag{3.5}$$

$$\begin{aligned}\int_0^1 g_1(v)vv_xu_{xx}dx = &\ -\int_0^1 (g_1(v)vv_x)_xu_xdx+(g_1(v)vv_xu_x)|_0^1\\ = &\ -\int_0^1 g_1'(v)vv_x^2u_xdx-\int_0^1 g_1(v)v_x^2u_xdx\\ &-\int_0^1 g_1(v)vv_{xx}u_xdx+(g_1(v)vv_xu_x)|_0^1,\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}\int_0^1 g_1(v)v_xv_{xxx}dx = &\ -\int_0^1 (g_1(v)v_x)_xv_{xx}dx+(g_1(v)v_xv_{xx})|_0^1\\ = &\ \frac{1}{3}\int_0^1 g_1''(v)v_x^4dx-\int_0^1 g_1(v)v_{xx}^2dx\\ &-\frac{1}{3}\left(g_1'(v)v_x^3\right)\Big|_0^1+(g_1(v)v_xv_{xx})|_0^1.\end{aligned}\tag{3.7}$$

Substituting (3.4)-(3.7) into (3.3) and using the boundary conditions (1.2) and (1.15), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\left(u_x^2+g_1(v)v_x^2\right)dx+\int_0^1 g_1(v)v_{xx}^2dx\\ &\quad=\frac{1}{6}\int_0^1 g_1''(v)v_x^4dx-\int_0^1(g_1(v)v-1)u_xv_{xx}dx-\frac{1}{2}\int_0^1(g_1'(v)v-g_1(v))u_xv_x^2dx.\end{aligned}\tag{3.8}$$

Choosing

$$g_1(v)=\frac{1}{v}>0,\tag{3.9}$$

we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1\left(u_x^2+\frac{v_x^2}{v}\right)dx+\int_0^1\frac{v_{xx}^2}{v}dx=\frac{1}{3}\int_0^1\frac{v_x^4}{v^3}dx+\int_0^1\frac{u_xv_x^2}{v}dx.\tag{3.10}$$

From the *a priori* assumptions (2.7), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\left(u_x^2+\frac{v_x^2}{v}\right)dx+\int_0^1\frac{v_{xx}^2}{v}dx &\le \frac{4}{3}\varepsilon^2\int_0^1\frac{v_x^2}{v}dx+\varepsilon\int_0^1\frac{v_x^2}{v}dx\\ &=\left(\frac{4}{3}\varepsilon^2+\varepsilon\right)\int_0^1\frac{v_x^2}{v}dx.\end{aligned}\tag{3.11}$$

Integrating (3.11) in t over $[0,t]$, we can obtain

$$\begin{aligned}&\int_0^1\left(u_x^2+\frac{v_x^2}{v}\right)dx+2\int_0^t\int_0^1\frac{v_{xx}^2}{v}dxds\\ &\le\int_0^1\left(u_{0x}^2+\frac{v_{0x}^2}{v_0}\right)dx+2\left(\frac{4}{3}\varepsilon^2+\varepsilon\right)\int_0^t\int_0^1\frac{v_x^2}{v}dxds,\end{aligned}\tag{3.12}$$

which implies (3.1) by (2.11) and the *a priori* assumptions (2.7). □

The proof of Lemma 3.1 is completed.

Lemma 3.2. *Under the assumptions of Theorem* 1.2 *and the a priori assumptions* (2.7)*, we have*

$$\int_0^1\left(u_{xx}^2+\frac{2}{3}v_{xx}^2\right)dx+\frac{1}{3}\int_0^t\int_0^1 v_{xxx}^2dxds\le C\left(||u_0||_2^2+||v_0-1||_2^2\right),\tag{3.13}$$

where C is a positive constant.

Before proving Lemma 3.2, we give the following result.

Proposition 3.3. *The smooth function $v(x,t)$ obtained by Theorem* 1.2 *satisfies the following properties:*

$$\left|\int_0^1\frac{1}{v^2}v_{xx}^3dx\right|\le\int_0^1\frac{1}{v}v_{xxx}^2dx+C\int_0^1\frac{1}{v}v_x^2dx,\tag{3.14}$$

where C is a positive constant.

Proof. From the Gagliardo-Nirenberg-Moser inequality, we have

$$||v_{xx}(t)||_{L^3}^3\le C||v_x(t)||_{L^6}^{\frac{3}{2}}||v_{xxx}(t)||_{L^2}^{\frac{3}{2}},\tag{3.15}$$

where C is a positive constant.

(3.15) and Young's inequality show that

$$||v_{xx}(t)||_{L^3}^3\le C||v_x(t)||_{L^6}^6+\frac{2}{3}||v_{xxx}(t)||_{L^2}^2.\tag{3.16}$$

Therefore, we have from the *a priori* assumptions (2.7),

$$\begin{aligned}\left|\int_0^1\frac{1}{v^2}v_{xx}^3dx\right| &\le C||v_x(t)||_{L^6}^6+\frac{2}{3}||v_{xxx}(t)||_{L^2}^2\\ &\le C|v_x|_{L^\infty}^4\int_0^1\frac{v_x^2}{v}dx+\int_0^1\frac{1}{v}v_{xxx}^2dx\\ &\le C\int_0^1\frac{v_x^2}{v}dx+\int_0^1\frac{1}{v}v_{xxx}^2dx.\end{aligned}\tag{3.17}$$

This proves Proposition 3.3. □

Proof of Lemma 3.2. Differentiating (3.2) with respect to x, we have

$$\begin{cases} u_{xxt} - v_{xxx} = 0, \\ v_{xxt} - (uv)_{xxx} = v_{xxxx}. \end{cases} \tag{3.18}$$

For any smooth function $g_2(v)$, taking $(3.18)_1 \times u_{xx} + (3.18)_2 \times g_2(v)v_{xx}$, and integrating it, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 \left(u_{xx}^2 + g_2(v)v_{xx}^2\right) dx = & \int_0^1 u_{xx}v_{xxx}dx + \frac{1}{2}\int_0^1 g_2'(v)v_t v_{xx}^2 dx \\ & + \int_0^1 g_2(v)v_{xx}(uv)_{xxx}dx + \int_0^1 g_2(v)v_{xx}v_{xxxx}dx. \end{aligned} \tag{3.19}$$

Next, we estimate the terms in the right side of (3.19) as follows:

$$\begin{aligned} \frac{1}{2}\int_0^1 g_2'(v)v_t v_{xx}^2 dx = & \frac{1}{2}\int_0^1 g_2'(v)[(uv)_x + v_{xx}]v_{xx}^2 dx \\ = & \frac{1}{2}\int_0^1 g_2'(v)uv_x v_{xx}^2 dx + \frac{1}{2}\int_0^1 g_2'(v)u_x v v_{xx}^2 dx \\ & + \frac{1}{2}\int_0^1 g_2'(v)v_{xx}^3 dx, \end{aligned} \tag{3.20}$$

$$\begin{aligned} \int_0^1 g_2(v)v_{xx}(uv)_{xxx}dx = & -\int_0^1 (uv)_{xx}(g_2(v)v_{xx})_x dx + ((uv)_{xx}g_2(v)v_{xx})\,|_0^1 \\ = & -\int_0^1 ug_2(v)v_{xx}v_{xxx}dx - 2\int_0^1 g_2(v)u_x v_x v_{xxx}dx \\ & -\int_0^1 vg_2(v)u_{xx}v_{xxx}dx - \int_0^1 ug_2'(v)v_x v_{xx}^2 dx \\ & -2\int_0^1 g_2'(v)u_x v_x^2 v_{xx}dx + \int_0^1 (vg_2'(v)v_x v_{xx})_x u_x dx \\ & -(vg_2'(v)v_x u_x v_{xx})\,|_0^1 + \left(ug_2(v)v_{xx}^2\right)|_0^1 \\ & +2\,(g_2(v)u_x v_x v_{xx})\,|_0^1 + (vg_2(v)u_{xx}v_{xx})\,|_0^1, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \int_0^1 g_2(v)v_{xx}v_{xxxx}dx = & -\int_0^1 (g_2(v)v_{xx})_x v_{xxx}dx + (g_2(v)v_{xx}v_{xxx})|_0^1 \\ = & -\int_0^1 g_2(v)v_{xxx}^2 dx - \int_0^1 g_2'(v)v_x v_{xx}v_{xxx}dx \\ & +(g_2(v)v_{xx}v_{xxx})|_0^1. \end{aligned} \tag{3.22}$$

Substituting (3.20)-(3.22) into (3.19), we obtain by the boundary conditions (1.2) and (1.15),

(3.23)
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left(u_{xx}^2+g_2(v)v_{xx}^2\right)dx+\int_0^1 g_2(v)v_{xxx}^2dx\\
=\ &\int_0^1 u_{xx}v_{xxx}dx-\int_0^1 vg_2(v)u_{xx}v_{xxx}dx+\frac{1}{2}\int_0^1 g_2'(v)uv_xv_{xx}^2dx\\
&+\frac{1}{2}\int_0^1 g_2'(v)u_xvv_{xx}^2dx+\frac{1}{2}\int_0^1 g_2'(v)v_{xx}^3dx-\int_0^1 ug_2(v)v_{xx}v_{xxx}dx\\
&-2\int_0^1 g_2(v)u_xv_xv_{xxx}dx-\int_0^1 ug_2'(v)v_xv_{xx}^2dx-2\int_0^1 g_2'(v)u_xv_x^2v_{xx}dx\\
&+\int_0^1 g_2'(v)v_x^2v_{xx}u_xdx+\int_0^1 vg_2''(v)v_x^2v_{xx}u_xdx+\int_0^1 vg_2'(v)v_{xx}^2u_xdx\\
&+\int_0^1 vg_2'(v)v_xv_{xxx}u_xdx-\int_0^1 g_2'(v)v_xv_{xx}v_{xxx}dx\\
&+(g_2(v)v_{xx}(vu_{xx}+v_{xxx}))|_0^1.
\end{aligned}$$

If choosing $g_2(v)=\frac{1}{v}$, then we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left(u_{xx}^2+\frac{1}{v}v_{xx}^2\right)dx+\int_0^1 \frac{1}{v}v_{xxx}^2dx\\
=\ &\frac{1}{2}\int_0^1 \frac{u}{v^2}v_xv_{xx}^2dx-\frac{1}{2}\int_0^1 \frac{1}{v}u_xv_{xx}^2dx-\frac{1}{2}\int_0^1 \frac{1}{v^2}v_{xx}^3dx\\
&-\int_0^1 \frac{u}{v}v_{xx}v_{xxx}dx-3\int_0^1 \frac{1}{v}u_xv_xv_{xxx}dx+3\int_0^1 \frac{1}{v^2}v_x^2u_xv_{xx}dx\\
&-\int_0^1 \frac{1}{v}u_xv_{xx}^2dx+\int_0^1 \frac{1}{v^2}v_xv_{xx}v_{xxx}dx+\left(\frac{v_{xx}}{v}(vu_{xx}+v_{xxx})\right)\Big|_0^1. \qquad (3.24)
\end{aligned}$$

From $(3.2)_2$ and the boundary conditions (1.2) and (1.15), we have

$$v_{xxx}|_0^1=-((uv)_{xx})|_0^1=-(uv_{xx})|_0^1-2(u_xv_x)|_0^1-(u_{xx}v)|_0^1=-(u_{xx}v)|_0^1.$$

Thus

$$\left(\frac{v_{xx}}{v}(vu_{xx}+v_{xxx})\right)\Big|_0^1=0. \qquad (3.25)$$

By (3.24), (3.25), (3.14) and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left(u_{xx}^2+\frac{1}{v}v_{xx}^2\right)dx+\int_0^1 \frac{1}{v}v_{xxx}^2dx\\
\le\ &\frac{3}{4}\int_0^1 \frac{1}{v}v_{xxx}^2dx+C\int_0^1 \frac{1}{v}(v_x^2+v_{xx}^2)dx. \qquad (3.26)
\end{aligned}$$

Integrating (3.26) in t over $[0,t]$ and using Lemmas 2.2 and 3.1, and the *a priori* assumptions (2.7), we get (3.13).

This proves Lemma 3.2. □

4. The proof of Theorem 1.2

The global existence in Theorem 1.2 follows from a local existence theorem (see [2, 5]) and the *a priori* estimate (1.16) obtained by (2.11), (3.1) and (3.13).

Now, we have to show that the *a priori* assumptions (2.7) can be closed since, under the *a priori* assumptions (2.7), we have proved that (1.16) holds.

In fact, by Sobolev's embedding theorem $W^{1,1}([0,1]) \hookrightarrow L^\infty([0,1])$ and Hölder's inequality, we have

$$\begin{aligned} |v(x,t)-1| \leq & \; C\int_0^1 |v(x,t)-1|dx + C\int_0^1 \left|\frac{\partial}{\partial x}(v(x,t)-1)\right| dx \\ \leq & \; C\left(\int_0^1 (v(x,t)-1)^2 dx\right)^{\frac{1}{2}} + C\left(\int_0^1 v_x^2(x,t)dx\right)^{\frac{1}{2}} \\ \leq & \; C\left(||u_0||_1^2 + ||v_0-1||_1^2\right)^{\frac{1}{2}}, \end{aligned}$$

which implies

$$|v(x,t)-1| \leq C\left(||u_0||_2^2 + ||v_0-1||_2^2\right)^{\frac{1}{2}}. \tag{4.1}$$

Similarly, we have

$$|u(x,t)|, \quad |u_x(x,t)|, \quad |v_x(x,t)| \leq C\left(||u_0||_2^2 + ||v_0-1||_2^2\right)^{\frac{1}{2}}. \tag{4.2}$$

By (4.1) and (4.2), it is easy to see that the *a priori* assumptions (2.7) hold provided $||u_0||_2^2 + ||v_0-1||_2^2$ is sufficiently small. Therefore, the *a priori* assumptions (2.7) are always true provided $||u_0||_2^2 + ||v_0-1||_2^2$ is sufficiently small.

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