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STABILIZATION OF TSIRELSON-TYPE NORMS ON ℓ_p SPACES

ANNA MARIA PELCZAR

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Abstract. We consider classical Tsirelson-type norms of $T[\mathcal{A}_n, \theta]$ and their modified versions on ℓ_p spaces, $1 < p < \infty$. We show that the modified Tsirelson-type norms do not distort any of the subspaces of the ℓ_p spaces. We prove that Tsirelson-type norms, being equivalent to their modified versions, may at most 2-distort ℓ_p spaces.

1. Introduction

The Tsirelson space T provided the base for further constructions of Banach spaces, which solved crucial problems in the theory of Banach spaces, such as Schlumprecht space—the first space known to be arbitrarily distortable, and further on the hereditarily indecomposable Gowers-Maurey space. Tsirelson-type norms provide important examples of Banach spaces, as well as a uniform approach to the classical spaces and the new ones, by defining norms implicitly, as solutions to certain equalities. The mixed and modified mixed Tsirelson-type norms were studied in various contexts, with respect to their distortion and asymptotic properties (cf. [1, 2, 3, 5, 4]).

The modified Tsirelson norm on T was introduced in [8] and later proved to be equivalent to the Tsirelson norm in [7] and in [6] with constant 2. Other equivalent norms $\|\cdot\|_n$ on T, of spaces $T[\mathcal{S}_n, 1/2^n]$, were studied in [11] in the context of the question of arbitrary distortability of T. It was shown that there is a universal constant K, such that the norms $\|\cdot\|_n$ do not K-distort any infinite-dimensional subspace of T. Their modified versions considered in [9] appear to be 3-equivalent to the original versions.

The equivalence of certain classical Tsirelson-type norms and their modified versions was shown in [6]. In the same paper the norms of $T[\mathcal{A}_n, \theta]$ isomorphic to ℓ_p were proved to be θ^2-equivalent to the classical $\|\cdot\|_p$ norms on ℓ_p. General Tsirelson-type norms appeared in [2], where equivalence between classical ℓ_p norm and Tsirelson-type norms of $T[\mathcal{A}_n, \theta]$ was shown by means of tree analysis of norming vectors. The fact, applied in our paper, that classical and modified Tsirelson-type norms on ℓ_p spaces are 3-equivalent follows from [12]. Let us recall that, in the mixed case, original and modified versions of Tsirelson-type norms define non-isomorphic spaces, as in the case of Schlumprecht space (cf. [4]).

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E. Odell and T. Schlumprecht solved in [10] the famous distortion problem showing that the spaces ℓ_p, $1 < p < \infty$, are arbitrarily distortable. They have shown in fact that these spaces are biorthogonally distortable, transferring the so-called biorthogonal system from the Schlumprecht space. The question of norms on ℓ_p arbitrarily distorting the original norm, defined only by means of ℓ_p, remains open. The obvious candidates to be studied in this context are the Tsirelson-type norms.

We show in this paper that modified Tsirelson-type norms do not $(1+\varepsilon)$-distort the original norms of ℓ_p spaces for any $\varepsilon > 0$ (Theorem 4.1). We prove also, using the reasoning of [12], the equivalence of Tsirelson-type norms and their modified versions (Corollary 3.2), and show that Tsirelson-type norms may at most 2-distort the original norms of ℓ_p spaces.

We recall first the standard notation. By c_{00} we denote the space of real sequences which are eventually zero, endowed with the supremum norm $\|\cdot\|_\infty$; by ℓ_p, $1 < p < \infty$, we denote the space of p-summable real sequences with the canonical norm $\|\cdot\|_p$ given by the formula

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x(i)|^p\right)^{1/p} \quad \text{for any } x = (x(i))_i \in \ell_p.$$

By (e_n) we denote the unit vectors basis. We put $B_{\ell_p} = \{x \in \ell_p : \|x\|_p \le 1\}$.

For $1 < p < \infty$ we have $(\ell_p)^* \cong \ell_q$, where $1/p + 1/q = 1$. We put

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x(i)y(i) \text{ for any } x = (x(i))_i \in \ell_p,\ y = (y(i))_i \in \ell_q.$$

For any sets $I, J \subseteq \mathbb{N}$ we write $I < J$ if $\max I < \min J$ and for any vectors $x, y \in c_{00}$ we write $x < y$ if $\operatorname{supp} x < \operatorname{supp} y$. A sequence $(x_n) \subseteq c_{00}$ is called a block sequence provided $x_1 < x_2 < \cdots$. Given a block sequence (x_n), by $[x_n]$ we denote the vector space spanned by (x_n). For any $x \in c_{00}$ and $E \subseteq \mathbb{N}$ by Ex or x_E we denote the restriction of x to E, i.e., $Ex(i) = x(i)$ if $i \in E$ and $Ex(i) = 0$ otherwise.

Finally, we say that a set $K \subseteq c_{00}$ is invariant under

(a) *restriction*, if for any $x \in K$ and $E \subset \mathbb{N}$ also $Ex \in K$,
(b) *spreading*, if for any $x = \sum a_n e_n \in K$ and any strictly increasing function $\phi : \mathbb{N} \to \mathbb{N}$ we have $\sum a_n e_{\phi(n)} \in K$,
(c) *permutation*, if for any $x = \sum a_n e_n \in K$ and any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ we have $\sum a_n e_{\sigma(n)} \in K$.

2. Classical and modified Tsirelson-type norms

We recall briefly the construction of a Tsirelson-type norm, denoted here by $\|\cdot\|_{p,r}$, and a modified Tsirelson-type norm, denoted by $|\cdot|_{p,r}$ (cf. [2, 4]).

Definition 2.1. Fix $1 < p, q < \infty$ with $1/p + 1/q = 1$ and $r \in \mathbb{N}$. Define norms $\|\cdot\|_{p,r}$ and $|\cdot|_{p,r}$ on c_{00} as the unique norms satisfying the following equations:

$$\|x\|_{p,r} = \max\left\{\|x\|_\infty, \frac{1}{\sqrt[q]{r}} \sup \sum_{i=1}^{r} \|E_i x\|_{p,r}\right\}, \quad x \in c_{00},$$

where the supremum is taken over all r-tuples of sets $E_1, \dots, E_r \subseteq \mathbb{N}$ which satisfy $E_1 < \dots < E_r$,

$$|x|_{p,r} = \max\left\{\|x\|_\infty, \frac{1}{\sqrt[q]{r}} \sup \sum_{i=1}^{r} |F_i x|_{p,r}\right\}, \quad x \in c_{00},$$

where the supremum is taken over all r-tuples of sets $F_1, \dots, F_r \subseteq \mathbb{N}$ that are pairwise disjoint.

Remark 2.2.

(a) Basic unit vectors (e_n) form a 1-unconditional and 1-subsymmetric basis of c_{00} endowed with the $\|\cdot\|_{p,r}$ norm and a 1-unconditional and 1-symmetric basis of c_{00} endowed with the $|\cdot|_{p,r}$ norm.

(b) The completion of c_{00} endowed with the norm $\|\cdot\|_{p,r}$, $r > 1$, is isomorphic to ℓ_p [2]. As we have $\|\cdot\|_{p,r} \le |\cdot|_{p,r} \le \|\cdot\|_p$ it follows that the completion of c_{00} endowed with the norm $|\cdot|_{p,r}$, $r > 1$, is also isomorphic to ℓ_p.

The norms introduced above can be defined alternatively by their norming sets presented below.

Definition 2.3. Fix $1 < q < \infty$ and $r \in \mathbb{N}$.

Let $K_{q,r}$ be the smallest set in c_{00} which contains vectors $(\pm e_n)$ and satisfies the following:

$$z_1, \dots, z_l \in K_{q,r}, \;\; l \le r, \;\; z_1 < \dots < z_l \;\; \Rightarrow \;\; \frac{1}{\sqrt[q]{r}}(z_1 + \dots + z_l) \in K_{q,r}.$$

Let $K_{q,r}^{\mathcal{M}}$ be the smallest set in c_{00} which contains vectors $(\pm e_n)$ and satisfies the following:

$$y_1, \dots, y_l \in K_{q,r}^{\mathcal{M}}, \; l \le r, \; \operatorname{supp} y_i \cap \operatorname{supp} y_j = \emptyset, \;\; i \ne j \Rightarrow \frac{1}{\sqrt[q]{r}}(y_1 + \dots + y_l) \in K_{q,r}^{\mathcal{M}}.$$

Remark 2.4.

(a) By the minimality of the considered sets, we have $K_{q,r} \subseteq K_{q,r}^{\mathcal{M}} \subseteq B_{\ell_q}$.

(b) By definition, the sets $K_{q,r}$ and $K_{q,r}^{\mathcal{M}}$ are invariant under restriction and spreading. The set $K_{q,r}^{\mathcal{M}}$ is a "symmetrized" version of $K_{q,r}$ invariant under permutations.

(c) A standard reasoning proves that for any $r \in \mathbb{N}$, $1 < p, q < \infty$ satisfying $1/p + 1/q = 1$, we have

$$\|x\|_{p,r} = \sup\{\langle x, z\rangle \,:\, z \in K_{q,r}\}, \qquad |x|_{p,r} = \sup\{\langle x, y\rangle \,:\, y \in K_{q,r}^{\mathcal{M}}\}.$$

(d) As for $r = 1$, clearly $K_{p,1} = \{\pm e_n\}$ and $\|\cdot\|_{p,1} = |\cdot|_{p,1} = \|\cdot\|_\infty$; we will omit this case in the rest of the paper.

Notation 2.5. Given $\alpha > 0$ and $1 < q < \infty$ put

$$C_\alpha = \{\pm\alpha^j \,:\, j \in \mathbb{Z}\} \cup \{0\},$$
$$\mathcal{N}_\alpha = \{x \in c_{00} \,:\, x(i) \in C_\alpha, i \in \mathbb{N}\}, \qquad \mathcal{N}_\alpha^{(q)} = \mathcal{N}_\alpha \cap B_{\ell_q}.$$

We shall need the following characterization of the set $K_{q,r}^{\mathcal{M}}$:

Lemma 2.6. *For any $r \in \mathbb{N}$, $1 < q < \infty$, $t = r^{1/q}$, we have $K_{q,r}^{\mathcal{M}} = \mathcal{N}_t^{(q)}$.*

Proof. Obviously $K_{q,r}^{\mathcal{M}} \subseteq \mathcal{N}_t^{(q)}$. Suppose now that $x \in \mathcal{N}_t^{(q)}$. It is easy to see that there is some $y \in \mathcal{N}_t^{(q)}$ such that $x < y$ and $\|x+y\|_q = 1$. Since $K_{q,r}^{\mathcal{M}}$ is invariant under restrictions, we may assume that $\|x\|_q = 1$. Since both $K_{q,r}^{\mathcal{M}}$ and $\mathcal{N}_t^{(q)}$ are invariant under permutations, we may assume that $|x(1)| \geq |x(2)| \geq \cdots$. The proof goes by induction on

$$n(x) = \min\{n \,:\, \text{there is some } j \in \operatorname{supp} x \text{ with } |x(j)| = t^{-n}\}.$$

For $n(x) = 0$ the result is clear. Suppose now that $n(x) > 0$. □

Claim. *There is a block sequence $(x_i)_{i=1}^r \subseteq \mathcal{N}_t^{(q)}$ such that $x = x_1 + \cdots + x_r$ and $\|x_i\|_q = t^{-1}$, for any $1 \leq i \leq r$.*

Proof of Claim. The proof goes by induction on

$$m(x) = \max\{n \,:\, \text{there is some } j \in \operatorname{supp} x \text{ with } |x(j)| = t^{-n}\}.$$

For $m(x) = 1$ the result is obvious. Let $I = \{j \in \operatorname{supp} x \,:\, |x(j)| > t^{-m(x)}\}$ and $J = \operatorname{supp} x \setminus I$. Notice that I is an initial part of $\operatorname{supp} x$. Denote by x_I and x_J the projections of x on I and J respectively. Then we have that

$$1 = \|x\|_q^q = \|x_I\|_q^q + \|x_J\|_q^q = \|x_I\|_q^q + \frac{|J|}{r^{m(x)}} = \frac{l}{r^{m(x)-1}} + \frac{|J|}{r^{m(x)}}$$

for some integer $l \in \mathbb{N}$. Since $m(x) \geq n(x) \geq 1$ it follows that $|J| = kr$ for some integer $k \in \mathbb{N}$. Divide J into k disjoint pieces $(J_i)_{i=1}^k$, with $J_1 < \cdots < J_k$ and $|J_i| = r$ $(1 \leq i \leq k)$. Now pick $n_i \in J_i$ $(1 \leq i \leq k)$ and set

$$y = x_I + \frac{1}{t^{m(x)-1}} \sum_{i=1}^k e_{n_i}.$$

It is clear that $m(y) = m(x) - 1$, $\|y\|_q = \|x\|_q = 1$, and $|y(1)| \geq |y(2)| \geq \cdots$; hence by the induction hypothesis there exists a decomposition $y = y_1 + \cdots + y_r$ into a sum of a block sequence with $\|y_i\|_q = t^{-1}$ for any $1 \leq i \leq r$.

Define $F : \operatorname{supp} y \to K_{q,r}^{\mathcal{M}}$ by $F(j) = e_j$ if $j \in I$, and $F(n_i) = t^{-1} \sum_{n \in J_i} e_n$, for $1 \leq i \leq k$. It is clear that $\|F(j)\|_q = 1$ for any $j \in \operatorname{supp} y$ and that $F(j) < F(j')$ for any $j < j'$. For any $1 \leq i \leq r$ define

$$x_i = \sum_{j \in \operatorname{supp} y_i} y_i(j) F(j).$$

By the previous observations, we obtain that $\|x_i\|_q = \|y_i\|_q = t^{-1}$ for any $1 \leq i \leq r$ and (x_i) is a block sequence; therefore we have the decomposition $x = x_1 + \cdots + x_r$.

Now we continue the proof of Lemma 2.6. Consider the decomposition of the vector $x = x_1 + \cdots + x_r$ as in the Claim. Then $\|tx_i\|_q = 1$, and $n(tx_i) \leq n(x) - 1$ for any $1 \leq i \leq r$; hence by the induction hypothesis we have that $(tx_i)_i \subseteq K_{q,r}^{\mathcal{M}}$. Therefore $x = t^{-1}(tx_1 + \cdots + tx_r) \in K_{q,r}^{\mathcal{M}}$. □

3. Equivalence of $|\cdot|_{p,r}$ and $\|\cdot\|_{p,r}$ norms

The fact that $\|\cdot\|_{p,r}$ and $|\cdot|_{p,r}$ are equivalent follows immediately from results in [12]. We recall the reasoning from this preprint for the sake of completeness.

First we introduce some notation. Let $\mathbb{N}^{<\infty}$ denote the set of finite sequences of $\mathbb{N}$. For any $m = (m(1), \ldots, m(n)) \in \mathbb{N}^{<\infty}$ and $k \in \mathbb{Z}$ put

$$m + k\mathbf{1} = (m(1) + k, \ldots, m(n) + k).$$

Given any sequences $m, l \in \mathbb{N}^{<\infty}$, $m = (m(1), \dots, m(n))$, $l = (l(1), \dots, l(j))$ put

$$m^\frown l = (m(1), \dots, m(n), l(1), \dots, l(j)).$$

Fix $1 < q < \infty$ and $r \in \mathbb{N}$. Define the function $\Phi : \mathbb{N}^{<\infty} \to (0, \infty)$ as follows:

$$\Phi(m(1)) = r^{-m(1)},$$
$$\Phi(m(1), m(2)) = r^{-m(1)} + r^{-m(2)},$$
$$\Phi(m(1), \dots, m(n)) = r^{-m(1)} + 2\sum_{i=2}^{n-1} r^{-m(i)} + r^{-m(n)}, \quad n > 2.$$

Notice that the function Φ has the following property (an addition rule):

$$\Phi(m(1), \dots, m(n)) = \Phi(m(1), \dots, m(i)) + \Phi(m(i), \dots, m(n)) \quad \text{for} \quad 1 < i < n.$$

Put $t = r^{1/q}$ and define the function

$$V : \mathbb{N}^{<\infty} \ni (m(1), \dots, m(n)) \mapsto (t^{-m(1)}, t^{-m(2)}, \dots, t^{-m(n)}, 0, \dots) \in c_{00}.$$

Theorem 3.1 ([12]). *With the above notation, let the sequence $m \in \mathbb{N}^{<\infty}$ satisfy $\Phi(m) \le 1$. Then $V(m) \in K_{q,r}$.*

Proof. We show the theorem by induction on the length n of the sequence. For $n = 1$ the assertion holds true since $(t^{-m(1)}, 0, \dots) \in K_{q,r}$ for any $m(1) \in \mathbb{N}$.

Fix $n \in \mathbb{N}$ and assume that the Theorem holds true for any sequence of integers of length not greater than n and pick some sequence $m = (m(1), \dots, m(n+1))$ of integers.

Let us first notice that we can consider only the case $r^{-1} < \Phi(m) \le 1$. Indeed, for any $m \in \mathbb{N}^{<\infty}$ pick $k \in \mathbb{N}$ such that $r^{-k-1} < \Phi(m) \le r^{-k}$. Notice that $r^{-1} < \Phi(m - k\mathbf{1}) \le 1$. If $V(m - k\mathbf{1}) \in K_{q,r}$, then also

$$V(m) = t^{-k} V(m - k\mathbf{1}) \in K_{q,r}.$$

Let $r^{-1} < \Phi(m) \le 1$. Put $k_0 = 1$ and define inductively $k_1 < \dots < k_l = n + 2$ as

$$k_{i+1} = \max\{k \in \{k_i, \dots, n+2\} : \ \Phi(m(k_i), \dots, m(k-1)) \le r^{-1}\}, \quad i \ge 1.$$

Since $r^{-1} < \Phi(m)$ we have $k_1 \le n + 1$.

We will show that $l \le r$. Assume that $l > 1$. By the definition of k_{i+1} we have

$$\Phi(m(k_i), \dots, m(k_{i+1})) > r^{-1} \quad \text{for} \quad 0 \le i \le l - 2.$$

By the addition rule for Φ we have

$$1 \ge \Phi(m) = \sum_{i=0}^{l-1} \Phi(m(k_i), \dots, m(k_{i+1})) > \sum_{i=0}^{l-2} \frac{1}{r} = \frac{l-1}{r},$$

which implies that $l - 1 < r$; hence $l \le r$.

Define sequences $m_1, \dots, m_l$ by

$$m_i = (m(k_i), \dots, m(k_{i+1} - 1)) \quad \text{for} \quad 0 \le i \le l - 1.$$

Since $k_l = n + 2$ we have $m = m_1{}^\frown \cdots {}^\frown m_l$. By construction the length of each m_i is not greater than n and $\Phi(m_i - \mathbf{1}) \le 1$ for any $1 \le i \le l$. By the induction hypothesis $V(m_i - \mathbf{1}) \in K_{q,r}$ for any $1 \le i \le l$. Notice that

$$V(m) = V(m_1{}^\frown \cdots {}^\frown m_l) = t^{-1} V((m_1 - \mathbf{1})^\frown \cdots {}^\frown (m_l - \mathbf{1})) = t^{-1}(v_1 + \dots + v_l),$$

where $(v_1, \dots, v_l)$ is a block sequence of suitably shifted vectors $V(m_1 - \mathbf{1}), \dots, V(m_l - \mathbf{1})$. By the definition of the set $K_{q,r}$ and its invariance under spreading it follows that $V(m) \in K_{q,r}$. □

Corollary 3.2. *Fix $1 < p < \infty$ and $r \in \mathbb{N}$. Then*

$$3^{-1}|x|_{p,r} \le \|x\|_{p,r} \le |x|_{p,r}, \quad x \in \ell_p.$$

Moreover, for any infinite-dimensional subspace X of ℓ_p and any $\varepsilon > 0$ there is an infinite-dimensional subspace Y of X such that

$$(2+\varepsilon)^{-1}|x|_{p,r} \le \|x\|_{p,r} \le |x|_{p,r}, \quad x \in Y.$$

Proof. Take $1 < q < \infty$ with $1/p + 1/q = 1$. The sets $K_{q,r}$ and $K_{q,r}^{\mathcal{M}}$ are invariant under spreading and $\Phi(m) \le 2\|V(m)\|_q^q$, $m \in \mathbb{N}^{<\infty}$; hence by Theorem 3.1 we have

$$K_{q,r}^{\mathcal{M}} \cap 2^{-1/q} B_{\ell_q} \subseteq K_{q,r}. \tag{1}$$

Take arbitrary $y \in K_{q,r}^{\mathcal{M}}$. If for some $i \in \mathbb{N}$ we have $|y(i)| = 1$, then $y \in K_{q,r}$. Assume $|y(i)| < 1$ for all $i \in \mathbb{N}$ and put $i_0 = \max\{i \in \operatorname{supp} y : \|y_{[1,i]}\|_q^q \le 1/2\} + 1$, $y_1 = y_{[1,i_0)}$, $y_2 = y_{\{i_0\}}$, $y_3 = y_{(i_0,\infty)}$. Then $y = y_1 + y_2 + y_3$ and $\|y_j\|_q^q \le 1/2$, $j = 1, 2, 3$. By (1) we have $y_1, y_2, y_3 \in K_{q,r}$. Now for any $x \in \ell_p$ compute

$$|x|_{p,r} = \sup_{y \in K_{q,r}^{\mathcal{M}}} |\langle x, y\rangle| \le \sup_{y_1,y_2,y_3 \in K_{q,r}} |\langle x, y_1 + y_2 + y_3\rangle| \le 3 \sup_{z \in K_{q,r}} |\langle x, z\rangle| = 3\|x\|_{p,r}.$$

By definition we have $\|\cdot\|_{p,r} \le |\cdot|_{p,r}$, which proves the first part of the Corollary.

For the second part fix $\varepsilon > 0$. By the equivalence of $\|\cdot\|_{p,r}$ and $\|\cdot\|_p$ pick $\delta > 0$ such that $\delta \le \varepsilon\|x\|_{p,r}$ for any $x \in \ell_p$ with $\|x\|_p = 1$. Given any infinite-dimensional subspace X of ℓ_p take an infinite-dimensional subspace $Y \subseteq X$ such that $\|x\|_\infty \le \delta$ for any $x \in B_{\ell_p} \cap Y$. Using the decomposition of vectors from $K_{q,r}^{\mathcal{M}}$ described above compute for any $x \in B_{\ell_p} \cap Y$,

$$|x|_{p,r} = \sup_{y \in K_{q,r}^{\mathcal{M}}} |\langle x, y\rangle| \le \sup_{y_1,y_3 \in K_{q,r}} |\langle x, y_1 + y_3\rangle| + \|x\|_\infty \le (2+\varepsilon)\|x\|_{p,r},$$

which ends the proof. □

4. Stabilization of $|\cdot|_{p,r}$ and $\|\cdot\|_{p,r}$ norms on ℓ_p

Now we present the main theorem of this paper, claiming that $|\cdot|_{p,r}$ norms do not distort $\|\cdot\|_p$ norms and $\|\cdot\|_{p,r}$ norms may at most 2-distort $\|\cdot\|_p$ norms.

Theorem 4.1. *Fix $1 < p < \infty$ and $r \in \mathbb{N}$, $r > 1$. For any $\varepsilon > 0$ every infinite-dimensional subspace $X \subseteq \ell_p$ has an infinite-dimensional subspace $Y \subseteq X$ satisfying*

$$(1-\varepsilon)\|x\|_p \le C|x|_{p,r} \le (1+\varepsilon)\|x\|_p, \quad x \in Y,$$
$$(1/2-\varepsilon)\|x\|_p \le C\|x\|_{p,r} \le (1+\varepsilon)\|x\|_p, \quad x \in Y,$$

for some constant $C = C(p, r, \varepsilon)$.

Notation 4.2. Fix $1 < p, q < \infty$ with $1/p + 1/q = 1$ and $r \in \mathbb{N}$, $r > 1$. Put $t = r^{1/q}$ and $s = r^{1/p}$.

Given $M \in \mathbb{N}$, $M \ge 1$ let $\alpha = s^{1/M} \in (1, s]$. For any vector $x \in \mathcal{N}_\alpha$ and any $m \in \mathbb{N}$ put

$$J_{m,x} = \{i \in \operatorname{supp} x : \alpha^{-m} x(i) \in C_s\}, \quad J_m x = J_{m,x} x.$$

Notice that $J_{kM+m}x = J_m x$ for any $k, m \in \mathbb{N}$. Using this notation we can write any vector $x \in \mathcal{N}_\alpha$ as a sum $x = J_0 x + \cdots + J_{M-1}x$ of vectors with disjoint supports such that $\alpha^{-m} J_m x \in \mathcal{N}_s$ for any $0 \le m \le M-1$.

The proof of the main theorem uses two lemmas: Lemma 4.4 showing stabilization of $|\cdot|_{p,r}$ norms on subspaces of a certain form, and Lemma 4.5 implying the saturation of ℓ_p by subspaces close to those used in Lemma 4.4. In both lemmas we will work only on vectors from $\mathcal{N}_\alpha$. It is sufficient, as for any seminormalized block sequence (v_n) in ℓ_p, that there is a block sequence $(x_n) \subseteq \mathcal{N}_\alpha$ which is α-equivalent to (v_n). First we deal with a technical Lemma 4.3 estimating relation between $\|\cdot\|_p$ and $|\cdot|_{p,r}$ norms on vectors of a specific form.

Lemma 4.3. *Fix p, r, M as in Notation 4.2. Then there exists a constant $D = D(p, r, M)$ such that for any $\mu > 0$ there is $\delta = \delta(p, r, M, \mu)$ satisfying for any $x \in \mathcal{N}_\alpha$ the following:*

$$\|x\|_\infty \le \delta, \quad \mu \le \|J_m x\|_p^p \le 1, \quad m = 0, \dots, M-1 \implies \mu D/\alpha^2 \le |x|_{p,r} \le D.$$

Proof. Let

$$D = D(p, r, M) = \sup \left\{ \sum_{m=0}^{M-1} t^{-m/M} \sum_{k \in K_m} t^k \gamma_k^{(m)} \right\},$$

where the supremum is taken over all finite families

$$K_0, \dots, K_{M-1} \subseteq \mathbb{Z} \quad \text{and} \quad (\gamma_k^{(0)})_{k \in K_0}, \dots, (\gamma_k^{(M-1)})_{k \in K_{M-1}} \subseteq (0, 1]$$

such that

$$\sum_{m=0}^{M-1} r^{-m/M} \sum_{k \in K_m} r^k \gamma_k^{(m)} \le 1, \ \sum_{k \in K_m} \gamma_k^{(m)} \le 1 \quad \text{for} \quad m = 0, \dots, M-1. \tag{2}$$

Consider families almost realizing the supremum: sets $L_0, \dots, L_{M-1} \subseteq \mathbb{Z}$ and sets $(\xi_k^{(0)})_{k \in L_0}, \dots, (\xi_k^{(M-1)})_{k \in L_{M-1}} \subseteq (0, 1]$ satisfying conditions (2) such that

$$d = \sum_{m=0}^{M-1} t^{-m/M} \sum_{k \in L_m} t^k \xi_k^{(m)}$$

satisfies $D/\alpha \le d$. Take $\delta > 0$ satisfying

$$\delta^{1/p} < \min\{\mu \xi_k^{(m)} (1 - 1/\alpha) \,:\, k \in L_m, \ m = 0, \dots, M-1\}.$$

Take $x \in \mathcal{N}_\alpha$ as in Lemma 4.3 and $y \in K_{q,r}^{\mathcal{M}}$. It is enough to consider vectors $y \in \mathcal{N}_t^{(q)}$ with $\operatorname{supp} y \subseteq \operatorname{supp} x$ and $\operatorname{sign} y(i) = \operatorname{sign} x(i)$, for any $i \in \operatorname{supp} y$. For any such y there is a finite sequence $(k_i)_i \subseteq \mathbb{Z}$ so that

$$|y(i)| = \frac{t^{k_i}}{\alpha^{mp/q}} |x(i)|^{p/q} = \frac{t^{k_i}}{t^{m/M}} |x(i)|^{p/q}, \ \ i \in J_{m,x} \cap \operatorname{supp} y, \ \ m = 0, \dots, M-1.$$

Then

$$\|y\|_q^q = \sum_{m=0}^{M-1} r^{-m/M} \sum_{i \in J_{m,x} \cap \operatorname{supp} y} r^{k_i} |x(i)|^p$$

and

$$\langle x, y \rangle = \sum_{m=0}^{M-1} t^{-m/M} \sum_{i \in J_{m,x} \cap \operatorname{supp} y} t^{k_i} |x(i)|^p.$$

Hence we can compute the norm $|x|_{p,r}$ in the following way:

$$|x|_{p,r} = \sup\left\{\sum x(i)y(i) : \ y \in \mathcal{N}_t^{(q)}\right\} = \sup\left\{\sum_{m=0}^{M-1} t^{-m/M} \sum_{k\in K_m} t^k \|x_{I_k^{(m)}}\|_p^p\right\},$$

where the last supremum is taken over finite families $K_0, \dots, K_{M-1} \subseteq \mathbb{Z}$ and pairwise disjoint sets $I_k^{(m)} \subseteq J_{m,x}$, $k \in K_m$, $m = 0, \dots, M-1$, such that

$$\sum_{m=0}^{M-1} r^{-m/M} \sum_{k\in K_m} r^k \|x_{I_k^{(m)}}\|_p^p \le 1.$$

By the assumption on $\|J_m x\|_p$ norms it follows immediately that $|x|_{p,r} \le D$.

Fix $m \in \{0, \dots, M-1\}$. Apply the notation: $z = J_m x$, $L = L_m = \{k_1, \dots, k_l\}$, $\xi_1 = \xi_{k_1}^{(m)}, \dots, \xi_l = \xi_{k_l}^{(m)}$. Define $i_1, \dots, i_l \in \mathbb{N}$ inductively as follows:

$$\begin{aligned} i_1 &= \min\{i \in \operatorname{supp} z \ : \ \mu\xi_1/\alpha \le \|z_{[1,i]}\|_p^p\}, \\ i_{k+1} &= \min\{i \in \operatorname{supp} z \ : \ i > i_k, \ \mu\xi_{k+1}/\alpha \le \|z_{(i_k,i]}\|_p^p\}, \quad k = 1, \dots, l-1. \end{aligned}$$

Such an i_1 exists by the assumptions on the norms $\|J_m x\|_p$. The inductive construction is possible, since for any $1 < n < l$, by the choice of δ and the definition of $i_2, \dots, i_n$ we have $\|z_{(i_k,i_{k+1}]}\|_p^p \le \mu\xi_{k+1}$, $k = 1, \dots, n-1$. Hence $\|z_{[1,i_n]}\|_p^p \le \mu(\xi_1 + \dots + \xi_n)$ and so $\|z_{(i_n,\infty)}\|_p^p \ge \mu\xi_{n+1}$.

Repeating the above reasoning for $m = 0, \dots, M-1$ pick pairwise disjoint sets $(I_k^{(m)})_{k\in L_m}$ such that

$$I_k^{(m)} \subseteq J_{m,x}, \quad \mu\xi_k^{(m)}/\alpha \le \|x_{I_k^{(m)}}\|_p^p \le \xi_k^{(m)} \quad \text{for} \quad k \in L_m, \ \ m = 0, \dots, M-1. \tag{3}$$

Let $y \in \mathcal{N}_t$ be the vector defined by

$$\begin{aligned} \operatorname{supp} y &= \bigcup\{I_k^{(m)} : \ k \in L_m, m = 0, \dots, M-1\}, \\ \operatorname{sign} y(i) &= \operatorname{sign} x(i), \quad i \in \operatorname{supp} y, \\ |y(i)| &= \frac{t^k}{t^{m/M}} |x(i)|^{p/q}, \ \ i \in I_k^{(m)}, \ \ k \in L_m, \ \ m = 0, \dots, M-1. \end{aligned}$$

By (2) and (3) clearly $\|y\|_q^q \le 1$; therefore by Lemma 2.6 we have $y \in K_{q,r}^{\mathcal{M}}$. Moreover, by construction we have $\mu d/\alpha \le \langle x, y\rangle \le |x|_{p,r}$, which ends the proof. □

An analogous result can be shown for any $x \in \mathcal{N}_\alpha$, but having J_m parts of almost equal $\|\cdot\|_p$ norm is hereditary with respect to taking suitable blocks. We will profit from this fact in the next "stabilization" lemma.

Lemma 4.4. *Apply the notation from Lemma* 4.3 *with* $\mu = \alpha^{-4p}$. *Let* $(x_n)_n \subseteq \mathcal{N}_\alpha$ *be a block sequence such that* $\|x_n\|_\infty \le \delta$ *and* $\alpha^{-3} \le \|J_m x_n\|_p \le 1$, *for* $n, m \in \mathbb{N}$. *Then for any finite sequence* $(a_n) \subseteq \mathbb{R}$ *we have*

$$\alpha^{-4p-2}\|\sum a_n x_n\|_p \le (M^{1/p}/D)|\sum a_n x_n|_{p,r} \le \alpha^4 \|\sum a_n x_n\|_p.$$

Proof. Take (x_n) as in Lemma 4.4. Clearly $\alpha^{-3}M^{1/p} \le \|x_n\|_p \le M^{1/p}$ for any $n \in \mathbb{N}$. It is enough to consider scalars $a_1, \dots, a_N \in \mathbb{R}$ satisfying $a_1, \cdots, a_N > 0$ and $a_1^p + \dots + a_N^p = 1$. Then

$$\alpha^{-3}M^{1/p} \le \|\sum a_n x_n\|_p \le M^{1/p}. \tag{4}$$

We approximate $\sum a_n x_n$ by vectors from $\mathcal{N}_\alpha$. For any $1 \le n \le N$ pick $k_n \in \mathbb{N}$ such that $\alpha^{-k_n} \le a_n < \alpha^{-k_n+1}$. Put $x = \sum_n \alpha^{-k_n} x_n \in \mathcal{N}_\alpha$. Notice that, for $\|\cdot\|$ denoting either $\|\cdot\|_p$, $\|\cdot\|_\infty$ or $|\cdot|_{p,r}$, we have

$$\alpha^{-1}\|\sum a_n x_n\| \le \|x\| \le \|\sum a_n x_n\|; \tag{5}$$

in particular $\|x\|_\infty \le \delta$. Observe that for any $m \in \mathbb{N}$ we have

$$J_m x = \sum_{n=1}^{N} \alpha^{-k_n} J_{m+k_n} x_n.$$

As $\|J_m x\|_p^p = \sum \alpha^{-pk_n} \|J_{m+k_n} x_n\|_p^p$, $m \in \mathbb{N}$, by the assumptions on (x_n) and the choice of (k_n) we have $\alpha^{-4} \le \|J_m x\|_p \le 1$, $m \in \mathbb{N}$. By Lemma 4.3 we have $\alpha^{-4p-2} D \le |x|_{p,r} \le D$ and hence, applying (5), we obtain

$$\alpha^{-4p-2} D \le |\sum a_n x_n|_{p,r} \le \alpha D,$$

which by (4) ends the proof. □

Lemma 4.5. *Fix p, r, M as in Notation 4.2. Let $(x_n) \subseteq \mathcal{N}_\alpha$ be a block sequence with $\alpha^{-2} \le \|x_n\|_p \le \alpha^{-1}$ for $n \in \mathbb{N}$. Then there is a block sequence $(y_n) \subseteq \mathcal{N}_\alpha$ of (x_n) with $\alpha^{-3} \le \|J_m y_n\|_p \le 1$ for any $n, m \in \mathbb{N}$.*

Proof. In order to prove the lemma it will be sufficient to find one vector y with the property described above. Without loss of generality, passing to a subsequence if necessary, we may assume that for a fixed $\varepsilon > 0$ there are scalars $b_0, \dots, b_{M-1}$ such that

$$0 \le b_m - \|J_m x_n\|_p^p < \varepsilon, \quad 0 \le m < M, \quad n \in \mathbb{N}.$$

Observe that $\alpha^{-2p} \le \sum_{m=0}^{M-1} b_m < \alpha^{-p} + M\varepsilon$. For technical reasons, we define also $b_{M+m} = b_m$ for any $0 \le m < M$. Take $a \in \mathbb{R}$ large enough so that $[b]/b \ge 1 - \varepsilon$ for any $b \ge a$. Fix $l \in \mathbb{N}$ such that $\alpha^{Ml} \ge a$. Now for $0 \le k, m < M$ consider the following averages:

$$y_k = \frac{1}{\alpha^{Ml+k}}(x_{r^{l+1}k} + \dots + x_{r^{l+1}k+[\alpha^{(Ml+k)p}]-1}), \quad c_k^m = \frac{[\alpha^{(Ml+k)p}]}{\alpha^{(Ml+k)p}} b_{m+k}.$$

It is straightforward to check that (y_k) is a block sequence. The sequences defined above have the following properties:

(a) $J_m y_k = \alpha^{-(Ml+k)} J_{m+k}(x_{r^{l+1}k} + \dots + x_{r^{l+1}k+[\alpha^{(Ml+k)p}]-1})$ for $0 \le k, m < M$,
(b) $0 \le c_k^m - \|J_m y_k\|_p^p < \varepsilon$ for any $0 \le k, m < M$,
(c) $(1-\varepsilon)\alpha^{-2p} \le \sum_{k=0}^{M-1} c_k^m < \alpha^{-p} + M\varepsilon$ for any $0 \le m < M$.

Set $y = y_0 + \dots + y_{M-1}$. Then

(d) $(1-\varepsilon)\alpha^{-2p} - M\varepsilon < \|J_m y\|_p^p < \alpha^{-p} + M\varepsilon$ for any $0 \le m < M$.

Choosing sufficiently small ε we obtain the desired result. □

Now we are ready to prove Theorem 4.1.

Proof. The part concerning $\|\cdot\|_{p,r}$ norms follows from the result for $|\cdot|_{p,r}$ norms and Corollary 3.2; hence we consider only the modified Tsirelson-type norms case.

Take $1 < p < \infty$, $r \in \mathbb{N}$, $r > 1$. Fix $\varepsilon > 0$ and take $M \in \mathbb{N}$ big enough to assure $\alpha^8 \le 1 + \varepsilon$ and $\alpha^{-4p-6} \ge 1 - \varepsilon$. Take constants D and δ as in Lemma 4.4.

Take an infinite-dimensional subspace X of ℓ_p. Take any sequence $(v_n)_n \subseteq X$ converging weakly to zero with $\|v_n\|_p = \alpha^{-1}$, $\|v_n\|_\infty \leq \delta/2$, $n \in \mathbb{N}$. By a well-known procedure applied simultaneously to $\|\cdot\|_p$ and $|\cdot|_{p,r}$ norms pick a block sequence (u_n) with $\|u_n\|_p = \alpha^{-1}$, $\|u_n\|_\infty \leq \delta$, $n \in \mathbb{N}$, which is α-equivalent to some subsequence $(v_{l_n})_n$ in both the $\|\cdot\|_p$ and $|\cdot|_{p,r}$ norms. Approximate vectors $(u_n)_n$ by vectors from $\mathcal{N}_\alpha$: for any $n \in \mathbb{N}$ and $i \in \operatorname{supp} u_n$ pick $k_n(i) \in \mathbb{N}$ such that

$$\alpha^{-k_n(i)} \leq |u_n(i)| \leq \alpha^{-k_n(i)+1} \tag{6}$$

and define (x_n) by $\operatorname{supp} x_n = \operatorname{supp} u_n$, $|x_n(i)| = \alpha^{-k_n(i)}$, sign $x_n(i)$= sign $u_n(i)$, $i \in \operatorname{supp} x_n$, $n \in \mathbb{N}$. Notice that by (6) we have $\|x_n\|_\infty \leq \delta$, $n \in \mathbb{N}$. The sequence $(x_n) \subseteq \mathcal{N}_\alpha$ is a block sequence with $\alpha^{-2} \leq \|x_n\|_p \leq \alpha^{-1}$ for any $n \in \mathbb{N}$. By Lemma 4.5 there is a block sequence (y_n) of (x_n) satisfying the assumptions of Lemma 4.4.

Notice that by (6) the sequence (x_n) is α-equivalent to (u_n), thus α^2-equivalent to (v_{l_n}), with respect to the $\|\cdot\|_p$ and $|\cdot|_{p,r}$ norms. Let $W : [x_n] \to [v_{l_n}]$ be the isomorphism defined by $Wx_n = v_{l_n}$, $n \in \mathbb{N}$. Put $z_n = Wy_n$, $n \in \mathbb{N}$. By Lemma 4.4 and α^2-equivalence of (y_n) and (z_n), for any finite sequence $(a_n) \subseteq \mathbb{R}$ we obtain

$$\alpha^{-4p-6} D \|\sum a_n z_n\|_p \leq M^{1/p} |\sum a_n z_n|_{p,r} \leq \alpha^8 D \|\sum a_n z_n\|_p.$$

By the choice of α the subspace $Y = [z_n]$ of X satisfies the desired condition with the constant $C = M^{1/p}/D$. □

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Institute of Mathematics, Jagiellonian University, Kraków, Poland
E-mail address: anna.pelczar@im.uj.edu.pl