

GEOMETRIC COHOMOLOGY FRAMES ON HAUSMANN–HOLM–PUPPE CONJUGATION SPACES

JOOST VAN HAMEL

(Communicated by Paul Goerss)

ABSTRACT. For certain manifolds with an involution the mod 2 cohomology ring of the set of fixed points is isomorphic to the cohomology ring of the manifold, up to dividing the degrees by two. Examples include complex projective spaces and Grassmannians with the standard antiholomorphic involution (with real projective spaces and Grassmannians as fixed point sets).

Hausmann, Holm and Puppe have put this observation in the framework of equivariant cohomology, and come up with the concept of *conjugation spaces*, where the ring homomorphisms arise naturally from the existence of what they call *cohomology frames*. Much earlier, Borel and Haefliger had studied the degree-halving isomorphism between the cohomology rings of complex and real projective spaces and Grassmannians using the theory of complex and real analytic cycles and cycle maps into cohomology.

The main result in the present note gives a (purely topological) connection between these two results and provides a geometric intuition into the concept of a cohomology frame. In particular, we see that if every cohomology class on a manifold X with involution is the Thom class of an equivariant topological cycle of codimension twice the codimension of its fixed points (inside the fixed point set of X), these topological cycles will give rise to a cohomology frame.

Let X be a topological space with a continuous involution τ . We denote the cyclic group of order two by C_2 , and the field with two elements by \mathbf{F}_2 . Since the cohomology ring of $BC_2 = P^\infty(\mathbf{R})$ over \mathbf{F}_2 is isomorphic to a polynomial ring $\mathbf{F}_2[u]$ in one variable (with u of degree 1), the restriction map from X to the fixed point set X^τ in Borel's version of equivariant cohomology $H_{C_2}^*(-; \mathbf{F}_2) := H^*(EC_2 \times_{C_2} -; \mathbf{F}_2)$ ([B]) gives a homomorphism of $\mathbf{F}_2[u]$ -algebras

$$r: H_{C_2}^*(X; \mathbf{F}_2) \rightarrow H_{C_2}^*(X^\tau; \mathbf{F}_2) = H^*(X^\tau; \mathbf{F}_2) \otimes_{\mathbf{F}_2} \mathbf{F}_2[u].$$

The localisation theorem in equivariant cohomology tells us that if X is finite dimensional, this homomorphism becomes an isomorphism when we invert the variable u . Together with the ring homomorphism

$$\rho: H_{C_2}^*(X; \mathbf{F}_2) \rightarrow H^*(X; \mathbf{F}_2)$$

this provides a close, but somewhat indirect relation between the cohomology of X and the cohomology of X^τ .

In the paper [HHP] J.-C. Hausmann, T. Holm and V. Puppe have exhibited an interesting class of spaces with an involution where this relation can be lifted to

Received by the editors October 7, 2005 and, in revised form, December 23, 2005.
2000 *Mathematics Subject Classification*. Primary 55M35, 55N91, 57S17, 57R91.

a ring homomorphism between the cohomology of X and the cohomology of X^τ which divides the degrees by two.

They call these spaces “conjugation spaces”, and they are defined to be those spaces X with an involution τ for which $H^{\text{odd}}(X; \mathbf{F}_2) = 0$, and which admit what they call a *cohomology frame*: a pair (κ, σ) of additive homomorphisms

$$\begin{aligned}\kappa &: H^{2^*}(X; \mathbf{F}_2) \rightarrow H^*(X^\tau; \mathbf{F}_2), \\ \sigma &: H^{2^*}(X; \mathbf{F}_2) \rightarrow H_{C_2}^{2^*}(X; \mathbf{F}_2),\end{aligned}$$

such that κ is a degree-halving isomorphism, $\rho \circ \sigma = \text{id}$, and for every $m \geq 0$ and every $a \in H^{2^*}(X; \mathbf{F}_2)$ we have the so-called *conjugation relation*

$$(1) \quad r \circ \sigma(a) = \kappa(a)u^m + \omega_{m+1}u^{m-1} + \omega_{m+2}u^{m-2} + \cdots + \omega_{2m}$$

with $\omega_i \in H^i(X^\tau; \mathbf{F}_2)$ for $i = m + 1, \dots, 2m$. What is remarkable about this definition is that the conjugation relation implies that

- such a pair (κ, σ) is unique (if it exists),
- the homomorphisms κ and σ are ring homomorphisms.

The main example in [HHP] of a conjugation space is a so-called *spherical conjugation complex*, which is constructed by attaching *conjugation cells*: closed unit disks in \mathbf{C}^n with the involution corresponding to complex conjugation. Basic examples of spherical conjugation complexes are complex projective spaces and complex Grassmannians with the involution given by complex conjugation (hence the fixed point sets are real projective spaces and real Grassmannians).

Much earlier, A. Borel and A. Haefliger had studied the degree-halving isomorphism between the cohomology rings of complex and real projective spaces and Grassmannians from a different point of view (and without using equivariant cohomology). Namely, it follows from Proposition 5.15 in their classic paper [BH] that if for a compact complex analytic variety X with an antiholomorphic involution τ for which the τ -invariant analytic cycles generate the cohomology $H^*(X; \mathbf{F}_2)$ and the fixed points of these analytic cycles (which are real analytic cycles) generate the cohomology $H^*(X^\tau; \mathbf{F}_2)$ of the fixed points, then the cycle maps induce the desired isomorphism of rings $H^{2^*}(X; \mathbf{F}_2) \xrightarrow{\sim} H^*(X^\tau; \mathbf{F}_2)$.

The aim of this note is to elaborate on the connection between these two points of view and to provide a geometric insight in the conjugation equation (1). We will see that for a complex analytic variety as above, the τ -equivariant analytic cycles give a geometric construction of a cohomology frame (κ, σ) , where the conjugation relation is satisfied because taking fixed points for the antiholomorphic involution halves the dimension of both the ambient space and the analytic cycles.

We will work in a more general topological framework, where we replace complex analytic manifolds with an antiholomorphic involution and equivariant analytic cycles by topological analogues. Since analytic cycles are sums of analytic subvarieties that can be singular (e.g., Schubert cells), it is appropriate for a topological generalisation to work with singular topological varieties in the sense of [BH]. See Section 1 for basic definitions and a discussion of the cohomology classes $[Z] \subset H^*(X; \mathbf{F}_2)$, $[Z^\tau] \subset H^*(X^\tau; \mathbf{F}_2)$ and $[Z]_{C_2} \subset H_{C_2}^{2^*}(X; \mathbf{F}_2)$ that may be associated to an equivariant singular topological subvariety $Z \subset X$ of a topological manifold X with a locally linear involution.

Theorem. *Let X be a (not necessarily compact) connected topological manifold with a locally linear involution τ . If:*

- (A) $H^{odd}(X; \mathbf{F}_2) = 0$,
- (B) for every $k \geq 0$ we have a set \mathcal{Z}^k of good equivariant singular topological subvarieties $Z \subset X$ of codimension type $(2k, k)$ representing a basis of the cohomology group $H^{2k}(X; \mathbf{F}_2)$,
- (C) either:
 - for every $k \geq 0$ the classes represented by the fixed point sets of the $Z \in \mathcal{Z}^k$ are linearly independent in $H^k(X^\tau; \mathbf{F}_2)$,
 - or:
 - for every $k \geq 0$ the classes represented by the fixed point sets of the $Z \in \mathcal{Z}^k$ generate $H^k(X^\tau; \mathbf{F}_2)$,

then the homomorphisms

$$\kappa: H^{2*}(X; \mathbf{F}_2) \rightarrow H^*(X^\tau; \mathbf{F}_2), \quad \sigma: H^{2*}(X; \mathbf{F}_2) \rightarrow H_{C_2}^{2*}(X; \mathbf{F}_2),$$

defined on the basis elements $\{[Z]: Z \in \mathcal{Z}^*\}$ by

$$\kappa([Z]) = [Z^\tau], \quad \sigma([Z]) = [Z]_{C_2},$$

form a cohomology frame for X .

Moreover, if X is compact, then the first two conditions imply the third.

The terminology of singular topological subvarieties and the cohomology classes they represent is explained in Section 1. This section also contains the key technical results Lemma 1.2 and Corollary 1.3. The Theorem is proved in Section 2.

Note that for a compact complex analytic manifold with an antiholomorphic involution, the theorem in fact gives not only a topological explanation for [BH, Prop. 5.15], but it gives a slight strengthening as well, since we no longer have to check surjectivity of the cycle map onto $H^*(X^\tau; \mathbf{F}_2)$.

Corollary. *Let X be a compact complex analytic manifold with an antiholomorphic involution τ . If every class in $H^*(X; \mathbf{F}_2)$ can be represented by a τ -invariant complex analytic cycle, then:*

- (i) every class in $H^*(X^\tau; \mathbf{F}_2)$ can be represented by the fixed points of a τ -invariant complex analytic cycle,
- (ii) X admits a cohomology frame.

Remark. In the algebro-geometric context (i.e., X projective), a statement equivalent to the corollary was proved by V.A. Krasnov using quite different, not purely topological methods ([K, Th. 0.1]).

1. THOM CLASSES AND SINGULAR SUBVARIETIES

Let X be a (not necessarily compact) topological manifold. In this paper, we adhere to the convention that manifolds are finite-dimensional and do not have boundaries. Let $Z \subset X$ be a closed subspace. We write $H^*(X, X \setminus Z; \mathbf{F}_2)$ for the cohomology relative to the complement of Z . This cohomology is also known as cohomology with supports in Z . Recall that if for each connected component $X_i \subset X$ of dimension d_i the subspace $Z_i := Z \cap X_i$ has cohomological dimension $\leq d_i - k$, then $H^m(X, X \setminus Z; \mathbf{F}_2) = 0$ for $m < k$ by Alexander–Lefschetz duality.

In analogy with [BH, 2.1] we say that a closed subspace $Z \subset X$ is a *singular subvariety* of codimension k if Z contains a nonempty open topological submanifold $U \subset Z$ such that the submanifold $U_i := U \cap X_i$ has codimension k in each connected

component $X_i \subset X$ where U_i is nonempty, and such that each $\Sigma_i := Z_i \setminus U_i$ has cohomological dimension strictly less than $d_i - k$. We call any such U a *fat nonsingular open* of Z . Note that (as in [BH]) the open U is not required to be dense in Z (and for this reason we are using a liberal, rather than literal, translation of the original French term ‘épais’). Also note that we do not require U to be a maximal submanifold, so the corresponding ‘singular subset’ $\Sigma := Z \setminus U$ may include points where Z is locally a manifold.

Recall that the *Thom class* $[U] \in H^*(Y, Y \setminus U; \mathbf{F}_2)$ of a not necessarily connected closed submanifold of pure codimension k in a manifold Y can be characterised by the fact that it is the unique class in $H^k(Y, Y \setminus U; \mathbf{F}_2)$ such that cup-product with $[U]$ induces an isomorphism

$$[U] \cup - : H^*(U; \mathbf{F}_2) \xrightarrow{\sim} H^{*+k}(Y, Y \setminus U; \mathbf{F}_2).$$

We say that a singular subvariety $Z \subset X$ of codimension k *admits a Thom class* if there is a cohomology class

$$[Z] \in H^k(X, X \setminus Z; \mathbf{F}_2)$$

that maps to the Thom class $[U]$ of a fat nonsingular open U under the restriction map

$$H^k(X, X \setminus Z; \mathbf{F}_2) \rightarrow H^k(X \setminus \Sigma, X \setminus Z; \mathbf{F}_2).$$

As in [BH, Prop. 2.3] we see that $[Z]$ is unique if it exists and that the existence of $[Z]$ is guaranteed if Z admits a fat nonsingular open U such that each $\Sigma_i \subset X_i$ has cohomological dimension $< d_i - k - 1$. An example of a singular subvariety without a Thom class is a bounded closed line segment in the plane. A typical unbounded example would be the union of three half-lines meeting in one point.

By abuse of notation, we will also write $[Z]$ for the image of the Thom class in $H^k(X; \mathbf{F}_2)$ under the natural map $H^k(X, X \setminus Z; \mathbf{F}_2) \rightarrow H^k(X; \mathbf{F}_2)$. If we say (for example, in the statement of the main theorem) that Z *represents a cohomology class* $a \in H^k(X; \mathbf{F}_2)$, then this combines two assertions: first, that $Z \subset X$ admits a Thom class, and second, that $a = [Z]$.

Remark 1.1. For the sake of concreteness everything is stated and discussed in terms of *topological* manifolds, singular topological subvarieties and locally linear involutions. However, all results and all proofs in this note are valid for *cohomological* \mathbf{F}_2 -manifolds, arbitrary continuous involutions and the corresponding class of singular subvarieties as well.

1.1. Equivariant singular subvarieties and equivariant Thom classes. Assume that the ambient manifold X admits an involution τ . Since for concreteness we work with topological manifolds, we will assume this involution to be locally linear, so that $X^\tau \subset X$ is a submanifold. For a singular topological subvariety $Z \subset X$ of codimension k to be an *equivariant singular topological subvariety*, we require not only that $\tau(Z) = Z$, but also that Z admits a fat nonsingular open $U \subset Z$ such that τ acts locally linear on the pair $U \subset X$. This ensures that $U^\tau \subset X^\tau$ is a submanifold, so we call U a *good equivariant submanifold*. Of course, this condition on U does not ensure that Z^τ is a singular subvariety with fat nonsingular open U^τ , since the dimension of Σ^τ may be too big. We say that $Z \subset X$ is a *good equivariant singular topological subvariety of codimension type* (k, k') if Z admits a fat nonsingular open $U \subset Z$ such that Z^τ is a singular topological subvariety of codimension k' in X^τ with fat nonsingular open U^τ .

Writing

$$H_{C_2}^*(X, X \setminus Z; \mathbf{F}_2) := H^*(EC_2 \times_{C_2} X, EC_2 \times_{C_2} (X \setminus Z); \mathbf{F}_2)$$

for the equivariant cohomology with supports in Z , the Borel–Serre spectral sequence

$$E_2^{pq} = H^p(BC_2, \mathcal{H}^q(X, X \setminus Z; \mathbf{F}_2)) \Rightarrow H_{C_2}^{p+q}(X, X \setminus Z; \mathbf{F}_2)$$

(i.e., the Leray spectral sequence for the fibration $EC_2 \times_{C_2} X \rightarrow BC_2$) tells us that

$$H_{C_2}^j(X, X \setminus Z; \mathbf{F}_2) = 0$$

when $d - j$ is greater than the cohomological dimension of Z and that

$$(2) \quad \rho: H_{C_2}^j(X, X \setminus Z; \mathbf{F}_2) \rightarrow H^j(X, X \setminus Z; \mathbf{F}_2)$$

is injective when $d - j$ is equal to the cohomological dimension of Z , with the image equal to the C_2 -invariant cohomology classes. It follows that if Z admits a nonequivariant Thom class $[Z] \in H^k(X, X \setminus Z; \mathbf{F}_2)$, then $[Z]$ lifts to a unique class

$$[Z]_{C_2} \in H_{C_2}^k(X, X \setminus Z; \mathbf{F}_2)$$

which we call the *equivariant Thom class*.

The equivariant Thom class $[U]_{C_2} \in H_{C_2}^*(Y, Y \setminus U; \mathbf{F}_2)$ of a closed equivariant submanifold U of a manifold with involution Y can be characterised by the fact that it is the unique homogeneous class in $H_{C_2}^*(Y, Y \setminus U; \mathbf{F}_2)$ such that the cup-product with $[U]_{C_2}$ induces an isomorphism

$$(3) \quad [U]_{C_2} \cup -: H_{C_2}^*(U; \mathbf{F}_2) \xrightarrow{\sim} H_{C_2}^*(Y, Y \setminus U; \mathbf{F}_2).$$

The equivariant Thom class $[Z]_{C_2} \in H_{C_2}^*(X, X \setminus Z; \mathbf{F}_2)$ of a good equivariant singular topological subvariety $Z \subset X$ can be characterised (if it exists) as the unique homogeneous class in $H_{C_2}^*(X, X \setminus Z; \mathbf{F}_2)$ that maps by restriction to the equivariant Thom class of any fat equivariant nonsingular open U of Z .

The following results links the equivariant Thom class to the Thom class of the set of fixed points.

Lemma 1.2. *Let $U \subset Y$ be a good equivariant closed submanifold of codimension k of a not necessarily compact topological manifold Y with a locally linear involution τ . Assume that $U^\tau \subset Y^\tau$ has codimension k' . Then the restriction map*

$$r: H_{C_2}^*(Y, Y \setminus U; \mathbf{F}_2) \rightarrow H_{C_2}^*(Y^\tau, Y^\tau \setminus U^\tau; \mathbf{F}_2) = H^*(Y^\tau, Y^\tau \setminus U^\tau; \mathbf{F}_2)[u]$$

maps the equivariant Thom class $[U]_{C_2} \in H_{C_2}^(Y, Y \setminus U; \mathbf{F}_2)$ to the class*

$$r([U]_{C_2}) = [U^\tau]u^{k-k'} + \eta_{k'+1}u^{k-k'-1} + \dots + \eta_k$$

with $\eta_i \in H^i(Y^\tau, Y^\tau \setminus U^\tau; \mathbf{F}_2)$, and $[U^\tau] \in H^{k'}(Y^\tau, Y^\tau \setminus U^\tau; \mathbf{F}_2)$ the Thom class of $U^\tau \subset X^\tau$.

Proof. Since $H^i(Y^\tau, Y^\tau \setminus U^\tau; \mathbf{F}_2) = 0$ for $i < k'$ we have that

$$r([U]_{C_2}) = \eta_{k'}u^{k-k'} + \eta_{k'+1}u^{k-k'-1} + \dots + \eta_k.$$

The isomorphism (3) together with the localisation theorem implies that the cup-product with the class $r([Z]_{C_2}) \in H_{C_2}^k(Y^\tau, Y^\tau \setminus U^\tau; \mathbf{F}_2)$ induces an isomorphism

$$(4) \quad r([U]_{C_2}) \cup -: H^*(U^\tau; \mathbf{F}_2)[u, u^{-1}] \xrightarrow{\sim} H^*(Y^\tau, Y^\tau \setminus U^\tau; \mathbf{F}_2)[u, u^{-1}].$$

It follows that the cup-product with $\eta_{k'} \in H^{k'}(Y^\tau, Y^\tau \setminus U^\tau; \mathbf{F}_2)$ induces an isomorphism

$$\eta_{k'} \cup - : H^*(U^\tau; \mathbf{F}_2) \xrightarrow{\sim} H^{*+k'}(Y^\tau, Y^\tau \setminus U^\tau; \mathbf{F}_2),$$

hence $\eta_{k'}$ is the Thom class of U^τ . □

Corollary 1.3. *Let $Z \subset X$ be a good equivariant singular topological subvariety of codimension type (k, k') of a not necessarily compact topological manifold X with a locally linear involution τ .*

If $Z \subset X$ admits a Thom class $[Z] \in H^k(X, X \setminus Z; \mathbf{F}_2)$, then

(i) *$Z \subset X$ admits an equivariant Thom class $[Z]_{C_2} \in H_{C_2}^k(X, X \setminus Z; \mathbf{F}_2)$, and the natural map*

$$\rho : H_{C_2}^k(X, X \setminus Z; \mathbf{F}_2) \rightarrow H^k(X, X \setminus Z; \mathbf{F}_2)$$

sends $[Z]_{C_2}$ to $[Z]$,

(ii) *$Z^\tau \subset X^\tau$ admits a Thom class $[Z^\tau] \in H^{k'}(X^\tau, X^\tau \setminus Z^\tau; \mathbf{F}_2)$, and the restriction map*

$$r : H_{C_2}^*(X, X \setminus Z; \mathbf{F}_2) \rightarrow H_{C_2}^*(X^\tau, X^\tau \setminus Z^\tau; \mathbf{F}_2) = H^*(X^\tau, X^\tau \setminus Z^\tau; \mathbf{F}_2)[u]$$

sends the equivariant Thom class $[Z]_{C_2} \in H_{C_2}^(X, X \setminus Z; \mathbf{F}_2)$ to the class*

$$r([Z]_{C_2}) = [Z^\tau]u^{k-k'} + \omega_{k'+1}u^{k-k'-1} + \dots + \omega_k$$

with $\omega_i \in H^i(X^\tau, X^\tau \setminus Z^\tau; \mathbf{F}_2)$, and $[Z^\tau] \in H^{k'}(X^\tau, X^\tau \setminus Z^\tau; \mathbf{F}_2)$ the Thom class of $Z^\tau \subset X^\tau$.

Proof. The existence of a unique lift $[Z]_{C_2}$ of $[Z]$ was already proved above, so we only have to prove the second part. Since $Z^\tau \subset X^\tau$ is a singular topological subvariety of dimension k' , we have that

$$r([Z]_{C_2}) = \omega_{k'}u^{k-k'} + \omega_{k'+1}u^{k-k'-1} + \dots + \omega_k$$

with $\omega_i \in H^i(X^\tau, X^\tau \setminus Z^\tau; \mathbf{F}_2)$. Let $j : U \hookrightarrow Z$ be a nice fat equivariant nonsingular open. By definition, $j^*([Z]_{C_2}) = [U]_{C_2}$. Since $r \circ j^* = j^* \circ r$, Lemma 1.2 implies that

$$j^*\omega_{k'}u^{k-k'} + j^*\omega_{k'+1}u^{k-k'-1} + \dots + j^*\omega_k = [U^\tau]u^{k-k'} + \eta_{k'+1}u^{k-k'-1} + \dots + \eta_k.$$

Hence $\omega_{k'}$ is the Thom class of $Z^\tau \subset X^\tau$. □

Remarks 1.4.

(i) A treatment more in the spirit of [BH] would be in terms of equivariant Borel–Moore homology and fundamental classes (compare [vH, III.7.1, IV.1]), which is linked to the present treatment via equivariant Poincaré duality (*loc. cit.*). The above treatment was chosen to remain closer to the language and approach in [HHP].

(ii) Observe that *a priori* it is not obvious at all that the existence of a Thom class for Z implies the existence of a Thom class for Z^τ . For example, in [BH], complexifications of real analytic sets play an important role, but the existence of a fundamental class for a real analytic set is proved using the highly nontopological operation of *normalisation*, whereas for a complex analytic space the existence of a fundamental class is deduced from the simple topological observation that the set of singular points is of topological codimension ≥ 2 .

(iii) The proof of Lemma 1.2 is analogous to the proof of [vH, Th. III.7.4]. This proof was inspired by the proof of [AP, Prop. 5.3.7], which proves the case $U = Y^\tau$ of Lemma 1.2 in the case where Y is a Poincaré duality space, rather than a manifold.

2. PROOF OF THE THEOREM

Let X be a (not necessarily compact) connected topological manifold with a locally linear involution τ . Assume that

- (A) $H^{\text{odd}}(X; \mathbf{F}_2) = 0$,
- (B) for every $k \geq 0$ we have a set \mathcal{Z}^k of good equivariant singular topological subvarieties $Z \subset X$ of codimension type $(2k, k)$ representing a basis of the cohomology group $H^{2k}(X; \mathbf{F}_2)$.

We define homomorphisms

$$\kappa: H^{2*}(X; \mathbf{F}_2) \rightarrow H^*(X^\tau; \mathbf{F}_2), \quad \sigma: H^{2*}(X; \mathbf{F}_2) \rightarrow H_{C_2}^{2*}(X; \mathbf{F}_2),$$

on the basis elements $\{[Z]: Z \in \mathcal{Z}^*\}$ by

$$\kappa([Z]) = [Z^\tau], \quad \sigma([Z]) = [Z]_{C_2},$$

where the classes $[Z]_{C_2} \in H_{C_2}^*(X; \mathbf{F}_2)$ and $[Z^\tau] \in H^*(X^\tau; \mathbf{F}_2)$ exist by Corollary 1.3.

By construction, κ is a degree-halving homomorphism, $\rho \circ \sigma = \text{id}$, and the conjugation relation (1) holds by Corollary 1.3. In order to prove that κ is an isomorphism, we observe that the existence of σ implies that X is *equivariantly formal*, i.e., we have an isomorphism of $\mathbf{F}_2[u]$ -modules

$$H^*(X; \mathbf{F}_2)[u] \xrightarrow{\sim} H_{C_2}^*(X; \mathbf{F}_2).$$

It follows from the localisation theorem that this isomorphism induces an isomorphism

$$H^*(X; \mathbf{F}_2)[u, u^{-1}] \xrightarrow{\sim} H^*(X^\tau; \mathbf{F}_2)[u, u^{-1}].$$

Putting $u = 1$, we get an isomorphism of (possibly infinite-dimensional) \mathbf{F}_2 -vector spaces

$$(5) \quad H^*(X; \mathbf{F}_2) \xrightarrow{\sim} H^*(X^\tau; \mathbf{F}_2),$$

which does not necessarily halve the grading, but which, by the above hypotheses and Corollary 1.3, does map $H^{\geq 2k}(X; \mathbf{F}_2)$ to $H^{\geq k}(X^\tau; \mathbf{F}_2)$ for every $k \geq 0$. In other words, with the appropriate filtrations on source and target, the isomorphism (5) is an isomorphism of filtered vector spaces that preserves the filtrations, with κ the corresponding homomorphism of the graded quotients associated to the filtrations. Since the filtrations are of finite length, κ is an isomorphism if it is either injective or surjective, which is the case by the third hypothesis:

- (C) either:
 - for every $k \geq 0$ the classes represented by the fixed point sets of the $Z \in \mathcal{Z}^k$ are linearly independent in $H^k(X^\tau; \mathbf{F}_2)$,
 - or:
 - for every $k \geq 0$ the classes represented by the fixed point sets of the $Z \in \mathcal{Z}^k$ generate $H^k(X^\tau; \mathbf{F}_2)$.

This finishes the proof of the first part of the Theorem.

For the final assertion of the Theorem we let X be compact, satisfying the first two hypotheses. These hypotheses imply that X is even dimensional and that the dimension d of X^τ is half the dimension of X . We will establish the injectivity of κ by a weak form of equivariant intersection theory. Assume that we have a $k \geq 0$ and an $a \in H^{2k}(X; \mathbf{F}_2)$ such that $\kappa(a) = 0$. The conjugation equation implies that

$$r \circ \sigma(a) = \omega_{k+1}u^{k-1} + \omega_{k+2}u^{k-2} + \cdots + \omega_{2k},$$

with $\omega_i \in H^i(X^\tau; \mathbf{F}_2)$. By Poincaré duality we have a cohomology class $b \in H^{2d-2k}(X; \mathbf{F}_2)$ with $a \cup b \neq 0$, hence

$$\sigma(a) \cup \sigma(b) \neq 0.$$

On the other hand, the conjugation equation implies

$$r \circ \sigma(b) = \eta_{d-k}u^{d-k} + \eta_{d-k+1}u^{d-k-1} + \cdots + \eta_{2d-2k}$$

with $\eta_i \in H^i(X^\tau; \mathbf{F}_2)$ (in fact, $\eta_{d-k} = \kappa(b)$, but we will not need that here). Since $\eta_i \cup \eta_j = 0$ for $i + j > d$, we see that

$$r \circ \sigma(a) \cup r \circ \sigma(b) = 0,$$

which implies that $r \circ \sigma(a \cup b) = 0$. Since $\sigma(a \cup b) = \sigma(a) \cup \sigma(b) \neq 0$, this contradicts the injectivity of r (which follows from the localisation theorem). Hence we have shown that κ is injective, which finishes the proof of the theorem. \square

ACKNOWLEDGEMENTS

This work has its origins in discussions with Jean-Claude Hausmann and Tara Holm at MSRI, Berkeley in February 2004 about their work in progress, and it has benefited from later discussions with Jean-Claude Hausmann during a visit to Geneva. The author would like to thank them both for the inspiring discussions, and to acknowledge the financial support received from MSRI and from the Swiss National Science Foundation for visits to Berkeley and Geneva.

The author would also like to thank Volker Puppe for his useful comments on a preliminary version of this note, and the referee for the careful reading.

REFERENCES

- [AP] C. Allday and V. Puppe, *Cohomological methods in transformation groups*, Cambridge Stud. Adv. Math., vol. 32, Cambridge University Press, 1993. MR1236839 (94g:55009)
- [B] A. Borel, *Seminar on transformation groups*, Ann. of Math. Studies, vol. 46, Princeton University Press, 1960. MR0116341 (22:7129)
- [BH] A. Borel and A. Haefliger, *La classe d'homologie fondamentale d'un espace analytique*, Bull. Soc. Math. France **89** (1961), 461–513. MR0149503 (26:6990)
- [vH] J. van Hamel, *Algebraic cycles and topology of real algebraic varieties*, CWI Tract, vol. 129, 2000. MR1824786 (2002c:14018)
- [HHP] J.-C. Hausmann, T. Holm, and V. Puppe, *Conjugation spaces*, Algebr. Geom. Topol. **5** (2005), 923–964. MR2171799 (2006e:55008)
- [K] V.A. Krasnov, *Real algebraically maximal varieties*, Mat. Zametki **73** (2003), 853–860 (Russian, with Russian summary). English translation in: *Math. Notes* **73** (2003), 806–812. MR2010655 (2004k:14111)

KATHOLIEKE UNIVERSITEIT LEUVEN, DEPARTEMENT WISKUNDE, CELESTIJNENLAAN 200B,
B-3001 LEUVEN (HEVERLEE), BELGIUM

E-mail address: vanhamel@member.ams.org