

## LARGE CARDINALS WITH FEW MEASURES

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**ABSTRACT.** We show, assuming the consistency of one measurable cardinal, that it is consistent for there to be exactly  $\kappa^+$  many normal measures on the least measurable cardinal  $\kappa$ . This answers a question of Stewart Baldwin. The methods generalize to higher cardinals, showing that the number of  $\lambda$  strong compactness or  $\lambda$  supercompactness measures on  $P_\kappa(\lambda)$  can be exactly  $\lambda^+$  if  $\lambda > \kappa$  is a regular cardinal. We conclude with a list of open questions. Our proofs use a critical observation due to James Cummings.

### 1. INTRODUCTION AND PRELIMINARIES

Set theorists have long been occupied with the problem of determining the number of normal measures there can be on a measurable cardinal. In [12], Kunen showed that it is consistent, relative to the existence of a measurable cardinal, for there to be exactly one normal measure on the least measurable cardinal  $\kappa$ . In [13], Kunen and Paris showed that it is consistent, relative to the existence of a measurable cardinal, for there to be exactly  $2^{2^\kappa}$  many normal measures on the least measurable cardinal  $\kappa$ , the maximal possible number. In the result of [13], it can be the case that  $2^{2^\kappa} = \kappa^{++}$ . In [16], Mitchell showed that it is consistent, relative to a measurable cardinal  $\kappa$  with  $o(\kappa) = \delta$ , for there to be exactly  $\delta$  many normal measures on  $\kappa$ . In this result,  $\delta \leq \kappa^{++}$  is an arbitrary finite or infinite cardinal. In [3], S. Baldwin generalized Mitchell's results of [16] and showed that it is consistent, relative to measurable cardinals of high Mitchell order, for there to be exactly  $\delta$  many normal measures on the least measurable cardinal  $\kappa$ , where  $\delta < \kappa$  is an arbitrary finite or infinite cardinal. Note that the result of [13] uses forcing, while the results of [12], [16], and [3] use inner model techniques, so that the GCH holds in the models constructed.

Until recently, little had been known concerning generalizations of these results to strongly compact and supercompact cardinals, primarily because of the limited inner model theory available for these cardinals. Some questions remained open even at the level of measurable cardinals. For instance, Baldwin [3] leaves open the questions of whether it is consistent, relative to some large cardinal hypothesis, for

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there to be either exactly  $\kappa$  many or exactly  $\kappa^+$  many normal measures on the least measurable cardinal  $\kappa$ .

In this paper, we rectify the situation described in the preceding paragraph to some extent by proving the following theorem.

**Main Theorem 1.** *If  $\kappa$  is a measurable cardinal, then there is a forcing extension, neither creating nor destroying any measurable cardinals, where there are exactly  $\kappa^+$  many normal measures on  $\kappa$ .*

Consequently, there can be exactly  $\kappa^+$  many normal measures on the least measurable cardinal  $\kappa$ . The argument used in the proof of Theorem 1 easily adapts to show that  $\kappa$  can have a specified Mitchell order, or have a Laver function for measurability, or be  $\mu$ -measurable, while still having exactly  $\kappa^+$  many normal measures.

This argument also extends to the case of supercompactness and strong compactness. For example, Theorem 7 shows that for  $\lambda > \kappa$  a regular cardinal, it is relatively consistent that  $\kappa$  is  $\lambda$  supercompact, but there are fewer than the maximal number of fine, normal  $\kappa$ -additive measures on  $P_\kappa(\lambda)$ . In addition, Theorem 8 shows the same for strong compactness.

All of our results are proved via forcing. The core observation in the proof of the Main Theorem was made by James Cummings and subsequently adapted to the various large cardinal contexts.

We take this opportunity to mention some preliminary information. For  $\kappa < \lambda$  cardinals,  $\kappa$  regular,  $\text{Coll}(\kappa, \lambda)$  is the standard Lévy collapse of  $\lambda$  to  $\kappa$ . For  $\kappa$  a regular cardinal and  $\gamma$  an ordinal,  $\text{Add}(\kappa, \gamma)$  is the standard partial ordering for adding  $\gamma$  Cohen subsets of  $\kappa$ . The partial ordering  $\mathbb{P}$  is  $\kappa$ -*distributive* if the intersection of  $\kappa$  many dense open subsets of  $\mathbb{P}$  is dense open. The partial ordering  $\mathbb{P}$  is  $\kappa$ -*strategically closed* if player II has a winning strategy, ensuring that play can continue to the  $\kappa$ th step, in the two person game in which the players construct a decreasing sequence of conditions  $\langle p_\alpha \mid \alpha \leq \kappa \rangle$ , where player I plays at all odd stages and player II plays at all even stages (including all limit stages), with the trivial condition played at stage 0. The partial ordering  $\mathbb{P}$  is  $\kappa^+$ -*directed closed* if any directed collection of conditions of cardinality at most  $\kappa$  has a common extension. Such partial orderings are necessarily  $\kappa$ -strategically closed.

A forcing notion  $\mathbb{P}$  (and the forcing extensions to which it gives rise) admits a *closure point* at  $\delta$  if it factors as  $\mathbb{Q} * \dot{\mathbb{R}}$ , where  $\mathbb{Q}$  is nontrivial,  $|\mathbb{Q}| \leq \delta$ , and  $\Vdash_{\mathbb{Q}} \dot{\mathbb{R}}$  is  $\delta$ -strategically closed". Our arguments will rely on the following consequence of the main result of [6] (which generalizes results of [7]).

**Theorem 2** ([6]). *If  $V \subseteq V[G]$  admits a closure point at  $\delta$  and  $j : V[G] \rightarrow M[j(G)]$  is an ultrapower embedding in  $V[G]$  with  $\delta < \text{cp}(j)$ , then  $j \restriction V : V \rightarrow M$  is a definable class in  $V$ .*

This theorem follows from [6, Theorem 3, Corollary 14]. If  $j : V[G] \rightarrow M[j(G)]$  witnesses the  $\lambda$  supercompactness of  $\kappa$  in  $V[G]$ , then by [6, Corollary 4], the restriction  $j \restriction V : V \rightarrow M$  witnesses the  $\lambda$  supercompactness of  $\kappa$  in  $V$ . In the case of strong compactness, a slight additional hypothesis is used. Specifically, a forcing extension  $V \subseteq V[G]$  exhibits  $\kappa$ -*covering* if every set of ordinals  $x$  of size less than  $\kappa$  in  $V[G]$  is covered by a set of ordinals  $y$  of size less than  $\kappa$  in  $V$  (this captures the power of *mildness* in [7]). The fact proved in [6, Theorem 31] is that if  $V \subseteq V[G]$  has  $\kappa$ -covering and admits a closure point at  $\delta < \kappa$ , and  $j : V[G] \rightarrow M[j(G)]$  is the

ultrapower by a fine,  $\kappa$ -additive measure on  $P_\kappa(\lambda)$  in  $V[G]$ , then  $j \restriction V : V \rightarrow M$  is a definable class in  $V$  and witnesses the  $\lambda$  strong compactness of  $\kappa$  in  $V$ .

Finally, let us mention that we assume familiarity with the large cardinal properties of measurability, strong compactness, and supercompactness, along with some other related notions. Interested readers may consult [11] for further details.

## 2. LIMITING THE NUMBER OF MEASURES ON A MEASURABLE CARDINAL

We now prove the Main Theorem.

**Main Theorem 1.** *If  $\kappa$  is a measurable cardinal, then there is a forcing extension, neither creating nor destroying any measurable cardinals, where there are exactly  $\kappa^+$  many normal measures on  $\kappa$ .*

*Proof.* Suppose that  $\kappa$  is a measurable cardinal in  $V$ . By forcing if necessary, we may assume that there are at least  $\kappa^+$  many normal measures on  $\kappa$ . One way to accomplish this, for example, is via a reverse Easton iteration using  $\text{Add}(\delta^+, 1)$  at every inaccessible cardinal  $\delta \leq \kappa$ , and trivial forcing at all other stages. Note that this forcing ensures the GCH will hold in the extension at all such nontrivial stages of forcing  $\gamma \leq \kappa$ . Standard arguments (see Lemma 1.1 of [1] or Lemma 6 of [4]) then show that every measurable cardinal  $\gamma \leq \kappa$  in  $V$  remains measurable after the forcing, with  $2^{\gamma^+}$  many normal measures, computed in the extension. The results of [14] show that all measurable cardinals greater than  $\kappa$  are also preserved. Further, an easy application of Theorem 2 shows that this iteration creates no new measurable cardinals. So we assume without loss of generality that there are at least  $\kappa^+$  many normal measures on  $\kappa$  in  $V$ .

Let  $\mathbb{P} = \text{Add}(\omega, 1) * \text{Coll}(\kappa^+, 2^{2^\kappa})$  be the forcing to add a Cohen real and then collapse  $2^{2^\kappa}$  to  $\kappa^+$ , and suppose that  $V[c][G]$  is the resulting forcing extension. Every normal measure on  $\kappa$  in  $V$  generates a unique normal measure in  $V[c]$ , since this is small forcing (see [14]). These remain normal measures in  $V[c][G]$ , since no additional subsets of  $\kappa$  are added by the collapse forcing. Thus, there are at least  $\kappa^+$  many normal measures on  $\kappa$  in  $V[c][G]$ . Conversely, suppose that  $\mathcal{U}$  is a normal measure on  $\kappa$  in  $V[c][G]$ , with the associated ultrapower embedding  $j : V[c][G] \rightarrow M[c][j(G)]$ . In particular,  $X \in \mathcal{U}$  if and only if  $\kappa \in j(X)$  for all  $X \subseteq \kappa$  in  $V[c][G]$ . By Theorem 2, it follows that the restriction  $j \restriction V : V \rightarrow M$  is a definable class in  $V$ . Since the forcing to add  $c$  is small with respect to  $\kappa$ , it follows (by [14]) that  $j \restriction V$  lifts uniquely to  $V[c]$ , and so  $j \restriction V[c] : V[c] \rightarrow M[c]$  is a definable class in  $V[c]$ . The key observation is now that because  $V[c]$  and  $V[c][G]$  have the same subsets of  $\kappa$ , one can reconstruct  $\mathcal{U}$  inside  $V[c]$  by observing  $X \in \mathcal{U}$  if and only if  $\kappa \in j(X)$ , using only  $j \restriction V[c]$ . Thus,  $\mathcal{U} \in V[c]$ . So every normal measure on  $\kappa$  in  $V[c][G]$  is actually in  $V[c]$ . The number of such normal measures, therefore, is at most  $(2^{2^\kappa})^{V[c]}$ , which is  $\kappa^+$  in  $V[c][G]$ , because  $(2^{2^\kappa})^{V[c]}$  was collapsed by  $G$ . So in  $V[c][G]$ , there are exactly  $\kappa^+$  many normal measures on  $\kappa$ , as desired. By the results of [14] and the closure properties of the Lévy collapse, forcing with  $\mathbb{P}$  does not affect measurable cardinals either above or below  $\kappa$ , so this completes the proof of Theorem 1.  $\square$

The proof of the Main Theorem is easily adapted, as we now illustrate. In particular, our first corollary answers one of Baldwin's questions from [3], by showing that it is consistent for there to be exactly  $\kappa^+$  many normal measures on the least measurable cardinal  $\kappa$ .

**Corollary 3.** *If  $\kappa$  is measurable, then there is a forcing extension in which  $\kappa$  is the least measurable cardinal and there are exactly  $\kappa^+$  many normal measures on  $\kappa$ .*

*Proof.* Suppose that  $\kappa$  is measurable. By either iterating Prikry forcing (see [15]) or iterating nonreflecting stationary set forcing (see [2]) if necessary, we may arrange that  $\kappa$  becomes the least measurable cardinal. By Theorem 1, there is a further extension in which  $\kappa$  remains the least measurable cardinal and there are exactly  $\kappa^+$  many normal measures on  $\kappa$ .  $\square$

Our next corollary shows that a measurable cardinal  $\kappa$  of nontrivial Mitchell rank can have exactly  $\kappa^+$  many normal measures.

**Corollary 4.** *Suppose  $\kappa$  is measurable and  $o(\kappa) = \delta$ . There is then a class model in which  $\kappa$  is measurable,  $o(\kappa) = \delta$ , and there are exactly  $\kappa^+$  many normal measures on  $\kappa$ .*

*Proof.* By the main result of [4], we may assume without loss of generality (by first passing to a Mitchell inner model and then doing the relevant forcing if necessary) that in our ground model  $V$ ,  $o(\kappa) = \delta$  and there are exactly  $\kappa^{++}$  many normal measures on  $\kappa$ . Alternatively, one can accomplish this purely by forcing: first increase the number of normal measures on  $\kappa$  with a reverse Easton iteration, as in the Main Theorem, and then observe that because all the ground model normal ultrapower embeddings lift to the extension, we preserve  $o(\kappa) \geq \delta$ ; if it happens that  $o(\kappa) > \delta$  in the extension, then  $o(\kappa) = \delta$  in the ultrapower by a normal measure of rank  $\delta$ , where there are still sufficient normal measures.

We now force over  $V$  with the partial ordering  $\mathbb{P}$  of Theorem 1. If  $j : V \rightarrow M$  is any normal ultrapower embedding for  $\kappa$  in  $V$ , then it lifts uniquely through the small forcing  $\text{Add}(\omega, 1)$  to  $j : V[c] \rightarrow M[c]$ , and then uniquely again through the directed closed forcing  $\text{Coll}(\kappa^+, 2^{2^\kappa})$  to  $j : V[c][G] \rightarrow M[c][j(G)]$ , since  $j(G)$  can (and must) be taken to be the filter generated by  $j \restriction G$ . Conversely, we have already argued above that if  $j : V[c][G] \rightarrow M[c][j(G)]$  is the ultrapower by a normal measure  $\mathcal{U}$  in  $V[c][G]$ , then  $j : V[c] \rightarrow M[c]$  is definable in  $V[c]$ . In fact,  $j \restriction V[c]$  is the ultrapower by  $\mathcal{U}$  in  $V[c]$ , since the collapse forcing  $G$  adds no new functions from  $\kappa$  to  $V[c]$ . Since  $V[c]$  is a small forcing extension, it follows that  $j \restriction V$  is the ultrapower in  $V$  by  $\mathcal{U} \cap V \in V$ . So we have established that every normal ultrapower embedding in  $V$  lifts to  $V[c][G]$ , and all ultrapower embeddings in  $V[c][G]$  arise in this way. It follows that  $o(\kappa)$  is preserved, since by induction the Mitchell rank of every measure is preserved to its unique extension in  $V[c]$ .  $\square$

By the definition of  $\mathbb{P}$ , the cardinal  $(\kappa^{++})^V$  is of course collapsed. Thus, if  $o(\kappa) = \kappa^{++}$  in  $V$ , then in our forcing extension, the Mitchell order of  $\kappa$  remains  $(\kappa^{++})^V$ , which is now an ordinal between  $\kappa^+$  and  $\kappa^{++}$  of the extension.

Next, we adapt the argument to allow for Laver functions. As in [5], define that  $\ell : \kappa \rightarrow V_\kappa$  is a *Laver function for measurability* if for every  $x \in H_{\kappa^+}$ , there is a normal ultrapower embedding  $j : V \rightarrow M$  with  $\text{cp}(j) = \kappa$  and  $j(\ell)(\kappa) = x$ . Since different values for  $x$  give rise to different induced normal measures, the existence of a Laver function for measurability implies that there are at least  $2^\kappa = |H_{\kappa^+}|$  many normal measures on  $\kappa$ . Hamkins asked in [5] whether or not the existence of such a Laver function implies that  $\kappa$  must have  $2^{2^\kappa}$  many normal measures. This is answered in the negative by the following result.

**Corollary 5.** *If  $\kappa$  is a measurable cardinal, then there is a forcing extension in which there are exactly  $\kappa^+$  many normal measures on  $\kappa$ , yet  $\kappa$  has a Laver function for measurability.*

*Proof.* Suppose that  $\kappa$  is measurable. We may assume, by preliminary forcing as in [9, Theorem 2.3] if necessary, that  $\kappa$  already has in  $V$  a Laver function  $\ell : \kappa \rightarrow V_\kappa$ . Thus, in  $V$ , there are at least  $\kappa^+$  many normal measures on  $\kappa$ .

Let  $V[c][G]$  be the forcing extension of Theorem 1, where there are exactly  $\kappa^+$  many normal measures on  $\kappa$ . Define  $\ell^*(\alpha) = \ell(\alpha)_c$ , provided  $\ell(\alpha)$  is an  $\text{Add}(\omega, 1)$ -name (choose anything otherwise). For any  $x \in H_{\kappa^+}^{V[c][G]} = H_{\kappa^+}^{V[c]}$ , there is a name  $\dot{x} \in H_{\kappa^+}^V$  such that  $x = \dot{x}_c$ . Since  $\ell$  is a Laver function in  $V$ , there is a normal ultrapower embedding  $j : V \rightarrow M$  with  $j(\ell)(\kappa) = \dot{x}$ . This embedding lifts (uniquely) to  $j : V[c][G] \rightarrow M[c][j(G)]$ . To conclude the argument, observe that  $j(\ell^*)(\kappa) = j(\ell)(\kappa)_c = \dot{x}_c = x$ , so  $\ell^*$  is a Laver function in  $V[c][G]$ .  $\square$

A cardinal  $\kappa$  is  $\mu$ -measurable if there is an embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that the induced normal measure  $\mathcal{U} = \{X \subseteq \kappa \mid \kappa \in j(X)\}$  is in  $M$ .

**Corollary 6.** *If  $\kappa$  is  $\mu$ -measurable, then there is a forcing extension preserving this in which there are exactly  $\kappa^+$  many normal measures on  $\kappa$ .*

*Proof.* Suppose that  $\kappa$  is  $\mu$ -measurable. From this, it follows that there are at least  $\kappa^+$  many normal measures (of varying Mitchell rank) on  $\kappa$ . Thus, in the extension  $V[c][G]$  of Theorem 1, there are exactly  $\kappa^+$  many normal measures on  $\kappa$ .

The  $\mu$ -measurability of  $\kappa$  in  $V$  is witnessed by an ultrapower embedding  $j : V \rightarrow M$  by a (nonnormal) measure  $\nu$  on  $\kappa$ , with the induced normal measure  $\mathcal{U}$  in  $M$ . Just as in Theorem 1, this embedding lifts uniquely to an embedding  $j^* : V[c][G] \rightarrow M[c][j(G)]$  in  $V[c][G]$ . This embedding witnesses  $\mu$ -measurability in  $V[c][G]$ , since the induced normal measure  $\mathcal{U}^*$  is precisely the measure generated as a filter by  $\mathcal{U} \in M$ .  $\square$

One can easily generalize the Main Theorem to cardinals other than  $\kappa^+$ . For example, if  $\kappa$  is measurable,  $\delta$  is a regular cardinal in the interval  $(\kappa^+, 2^{2^\kappa}]$  and there are at least  $\delta$  many normal measures on  $\kappa$ , then in the forcing extension  $V[c][G]$ , obtained by forcing with  $\text{Add}(\omega, 1) * \text{Coll}(\delta, 2^{2^\kappa})$ , there will be exactly  $\delta$  many normal measures on  $\kappa$ . Also, as above, one can similarly preserve various properties of  $\kappa$ , such as  $\mu$ -measurability, the existence of a Laver function, or any particular value of  $o(\kappa)$ .

### 3. LIMITING THE NUMBER OF MEASURES ON HIGHER CARDINALS

We now extend the method to the case of supercompactness and strong compactness.

**Theorem 7.** *Suppose that  $\kappa$  is  $\lambda$  supercompact,  $\lambda > \kappa$  is regular, and the GCH holds. Then there is a forcing extension preserving this in which there are exactly  $\lambda^+$  many fine, normal,  $\kappa$ -additive measures on  $P_\kappa(\lambda)$ .*

*Proof.* By forcing if necessary, we may assume without loss of generality (see, for example, [9, Theorem 1.15]) that in  $V$ , there are at least  $\lambda^+$  many fine, normal,  $\kappa$ -additive measures on  $P_\kappa(\lambda)$ . Next, we force as in Theorem 1 with  $\mathbb{P} = \text{Add}(\omega, 1) * \text{Coll}(\lambda^+, 2^{2^\lambda})$ , giving rise to the forcing extension  $V[c][G]$ . Standard arguments show that the GCH holds in  $V[c][G]$ .

As before, every  $\lambda$  supercompactness embedding  $j : V \rightarrow M$  in  $V$  lifts uniquely through the small forcing to  $j : V[c] \rightarrow M[c]$ , and then uniquely to the full extension  $j : V[c][G] \rightarrow M[c][j(G)]$ . This is because the collapse forcing is  $\lambda$ -distributive, which implies that the filter generated by  $j \restriction G$  is already  $M[c]$ -generic, and so one can (and must) take  $j(G)$  to be this filter. Conversely, if  $\mathcal{U}$  is a fine, normal,  $\kappa$ -additive measure on  $P_\kappa(\lambda)$  in  $V[c][G]$ , with the associated ultrapower embedding  $j : V[c][G] \rightarrow M[c][j(G)]$ , then by Theorem 2, the restricted embedding  $j \restriction V : V \rightarrow M$  is a definable class in  $V$ . This embedding lifts uniquely to  $j \restriction V[c] : V[c] \rightarrow M[c]$  in  $V[c]$ .

The point now is just as in Theorem 1, namely that  $V[c]$  and  $V[c][G]$  have the same subsets of  $P_\kappa(\lambda)$ . Therefore,  $\mathcal{U}$  is constructible from  $j \restriction V[c]$  in  $V[c]$ , because  $X \in \mathcal{U}$  if and only if  $j \restriction \lambda \in j(X)$ . Hence, every  $\lambda$  supercompactness measure in  $V[c][G]$  is actually in  $V[c]$ . Consequently, the number of such measures is at most  $(2^{2^\lambda})^{V[c]}$ , which has size  $\lambda^+$  in  $V[c][G]$ .  $\square$

**Theorem 8.** *Suppose that  $\kappa$  is  $\lambda$  supercompact,  $\lambda > \kappa$  is regular, and the GCH holds. Then there is a forcing extension in which  $\kappa$  is  $\lambda$  strongly compact but not  $\lambda$  supercompact, the GCH holds, and there are exactly  $\lambda^+$  many fine,  $\kappa$ -additive measures on  $P_\kappa(\lambda)$ .*

*Proof.* As in the proof of Theorem 7, we may assume without loss of generality that there are at least  $\lambda^+$  many fine, normal,  $\kappa$ -additive measures on  $P_\kappa(\lambda)$  in  $V$ . If we force with Magidor's iteration of Prikry forcing found in [15] which turns  $\kappa$  into the least measurable cardinal, then by the work of [15],  $\kappa$  remains  $\lambda$  strongly compact. Further, each fine, normal,  $\kappa$ -additive measure  $\mathcal{U}$  on  $P_\kappa(\lambda)$  extends to a fine,  $\kappa$ -additive measure  $\mathcal{U}^* \supseteq \mathcal{U}$  on  $P_\kappa(\lambda)$  in the generic extension  $\overline{V}$  resulting from the iterated Prikry forcing. Thus, if we now force over  $\overline{V}$  with  $\mathbb{P} = \text{Add}(\omega, 1) * \text{Coll}(\lambda^+, \lambda^{++})$ , giving rise to the forcing extension  $\overline{V}[c][G]$ , then the argument given in the proof of Theorem 7 (using strong compactness embeddings rather than supercompactness embeddings) goes through and shows there are  $\lambda^+$  many fine,  $\kappa$ -additive measures on  $P_\kappa(\lambda)$  in  $\overline{V}[c][G]$ . Since all partial orderings used preserve the GCH and the fact  $\kappa$  is the least measurable cardinal,  $\kappa$  is  $\lambda$  strongly compact but is not  $\lambda$  supercompact in  $\overline{V}[c][G]$ .  $\square$

We note that since forcing with  $\text{Coll}(\lambda^+, \lambda^{++})$  in both Theorems 7 and 8 adds a new subset of  $\lambda^+$  to a model obtained by small forcing, it follows by the main result of [10, p. 552] that  $\kappa$  is not  $\lambda^+ = 2^\lambda$  strongly compact in  $\overline{V}[c][G]$ .

To this point, in Section 3, we have constructed generic extensions in which we force over a ground model satisfying the GCH and obtain another universe which also satisfies the GCH. Our methods, however, also work in situations where the GCH fails. For instance, we have the following theorem.

**Theorem 9.** *Suppose that  $\kappa$  is  $\kappa^{++}$  supercompact and the GCH holds in  $V$ . Then there is a forcing extension in which:*

- (1)  $2^\delta = 2^{\delta^+} = \delta^{++}$  for every inaccessible cardinal  $\delta \leq \kappa$ .
- (2)  $\kappa$  is  $\kappa^+$  supercompact, but  $\kappa$  is not  $2^\kappa$  supercompact.
- (3) There are exactly  $\kappa^{++}$  many normal measures on  $\kappa$ .
- (4) There are exactly  $\kappa^{++}$  many fine, normal,  $\kappa$ -additive measures on  $P_\kappa(\kappa^+)$ .

The point of Theorem 9 is that the GCH fails at  $\kappa$ ,  $\kappa$  is  $\kappa^+$  supercompact, yet both  $\kappa$  and  $P_\kappa(\kappa^+)$  have fewer than the maximal number of normal measures.

*Proof.* To prove Theorem 9, suppose we start with a ground model  $V$  in which the GCH holds and  $\kappa$  is  $\kappa^{++}$  supercompact. Force over  $V$  with the reverse Easton iteration of length  $\kappa + 1$  which begins by adding a Cohen subset of  $\omega$  and then does nontrivial forcing only at those stages which are inaccessible cardinals in  $V$ . At such a stage  $\delta \leq \kappa$ , we force with  $\text{Add}(\delta, \delta^{++})$ . Standard arguments then show that in the resulting model  $\bar{V}$ , for every inaccessible cardinal  $\delta \leq \kappa$ ,  $2^\delta = 2^{\delta^+} = \delta^{++}$ , and  $\kappa$  is  $2^\kappa = 2^{[\kappa^+]^{<\kappa}} = 2^{\kappa^+} = \kappa^{++}$  supercompact. Furthermore, there are exactly  $2^{2^\kappa} = 2^{\kappa^{++}} = \kappa^{+++}$  many normal measures on  $\kappa$ , and there are exactly  $2^{2^{[\kappa^+]^{<\kappa}}} = 2^{\kappa^{++}} = \kappa^{+++}$  many fine, normal,  $\kappa$ -additive measures on  $P_\kappa(\kappa^+)$ .

If we now force over  $\bar{V}$  with  $\mathbb{P} = \text{Add}(\omega, 1) * \text{Coll}(\kappa^{++}, \kappa^{+++})$ , giving rise to the forcing extension  $\bar{V}[c][G]$ , then the proofs of Theorems 1 and 7 remain valid and show that in  $\bar{V}[c][G]$ , there are exactly  $\kappa^{++}$  many normal measures on  $\kappa$ , and there are exactly  $\kappa^{++}$  many fine, normal,  $\kappa$ -additive measures on  $P_\kappa(\kappa^+)$ . Since forcing with  $\mathbb{P}$  preserves the  $\kappa^+$  supercompactness of  $\kappa$ , and since for the same reasons as mentioned after the proof of Theorem 8,  $\kappa$  is not  $\kappa^{++}$  strongly compact in  $\bar{V}[c][G]$ , the proof of Theorem 9 is now complete.  $\square$

With a little more work, it is possible to obtain the conclusions of Theorem 9 with  $\kappa$  in addition being the least measurable cardinal. A brief outline of the argument is as follows. Start with a ground model  $V$  containing a cardinal  $\kappa$  for which  $2^\kappa = 2^{\kappa^+} = \kappa^{++}$ ,  $2^{\kappa^{++}} = \kappa^{+++}$ ,  $\kappa$  is  $\kappa^+$  supercompact, and  $\kappa$  is the least measurable cardinal. (Note that the consistency of a cardinal with these properties is originally due to Woodin. A construction of a model containing such a cardinal may be found, for example, in [2].) Since  $2^\kappa = \kappa^{++}$  holds in  $V$ , there are  $\kappa^{++}$  many permutations  $\pi : \kappa \rightarrow \kappa$  in  $V$ . Let  $j : V \rightarrow M$  be an elementary embedding witnessing the  $\kappa^+$  supercompactness of  $\kappa$  such that  $\kappa$  is not measurable in  $M$ . Force over  $V$  with the reverse Easton iteration  $\mathbb{P} = \mathbb{P}_\kappa * \dot{\mathbb{Q}}$  of length  $\kappa + 1$  which begins by adding a Cohen subset of  $\omega$ , adds a Cohen subset to each nonmeasurable inaccessible cardinal  $\delta < \kappa$ , does trivial forcing at all other stages  $\delta < \kappa$ , and ends by adding a Cohen subset of  $\kappa$ . Let  $G = G_\kappa * g$  be  $V$ -generic over  $\mathbb{P}$ . By a standard argument (see, for example, the proofs of Lemma 1.1 of [1] or Lemma 6 of [4]), in  $V[G_\kappa][g]$  there are exactly  $2^{2^\kappa} = \kappa^{+++}$  many fine, normal,  $\kappa$ -additive measures on  $P_\kappa(\kappa^+)$ . Also, for each permutation  $\pi : \kappa \rightarrow \kappa$ , a standard argument allows us to lift  $j$  to  $j_\pi^+ : V[G_\kappa][g] \rightarrow M[G_\kappa][g_\pi][H][g^+]$ , where  $g_\pi = \pi''g$ . If  $i_\pi : V[G_\kappa][g] \rightarrow N$  is the induced normal ultrapower embedding, so that  $j_\pi^+$  factors as  $k_\pi \circ i_\pi$ , then it follows that  $i_\pi(G_\kappa)(\kappa) = g_\pi$ . Since there are  $\kappa^{++}$  many permutations  $\pi : \kappa \rightarrow \kappa$  in  $V$ , this means that in  $V[G_\kappa][g]$  there are (at least)  $\kappa^{++}$  many normal measures on  $\kappa$ . If we now force with  $\text{Add}(\omega, 1) * \text{Coll}(\kappa^{++}, \kappa^{+++})$ , the same argument given in the proof of Theorem 9 then shows that in the resulting generic extension, there are  $\kappa^{++}$  many normal measures on  $\kappa$  and  $\kappa^{++}$  many fine, normal,  $\kappa$ -additive measures on  $P_\kappa(\kappa^+)$ .

The arguments we have presented often allow us to conclude that a large cardinal has strictly fewer than the maximal number of measures of the desired kind, even when we are unable to calculate the exact number of measures. For instance, if we start with a ground model  $V$  where  $\kappa$  is the least measurable cardinal,  $\kappa$  is  $\kappa^+$  supercompact,  $2^\kappa = 2^{\kappa^+} = \kappa^{++}$ , and  $2^{\kappa^{++}} = \kappa^{+++}$ , then after forcing with  $\text{Add}(\omega, 1) * \text{Coll}(\kappa^{++}, \kappa^{+++})$ , we may conclude that there are at most  $\kappa^{++}$  many normal measures on  $\kappa$  and at most  $\kappa^{++}$  many fine, normal,  $\kappa$ -additive measures on

$P_\kappa(\kappa^+)$ . Thus, even if we do not know either the exact number of normal measures on  $\kappa$ , or fine,  $\kappa$ -additive measures on  $P_\kappa(\kappa^+)$ , or fine, normal,  $\kappa$ -additive measures on  $P_\kappa(\kappa^+)$  in our ground model, we still know that there are fewer than the maximal number of such measures in the forcing extension.

The method is quite malleable and allows for diverse similar results. For instance, by starting with a model in which the GCH holds,  $\kappa$  is measurable,  $\lambda > \kappa$  is regular, and for  $\delta \leq \kappa^{++}$  any finite or infinite cardinal, there are exactly  $\delta$  many normal measures on  $\kappa$ , by forcing with  $\text{Add}(\omega, 1) * \text{Add}(\kappa^+, \lambda)$ , we have constructed a model in which  $\kappa$  is measurable,  $2^\kappa = \kappa^+$ ,  $2^{\kappa^+} = \lambda$ , and there are still  $\delta$  many normal measures on  $\kappa$ . In addition, we may force the restriction of the number of extenders witnessing the  $\lambda$  strongness of  $\kappa$ .

One limitation of the method is that after the forcing constructions of Theorems 7 and 8, as we have mentioned,  $\kappa$  is no longer strongly compact. Thus, it is not possible to use the techniques of this paper to construct a strongly compact cardinal  $\kappa$  such that there are fewer than the maximal number of fine,  $\kappa$ -additive measures on  $P_\kappa(\lambda)$ , for  $\lambda > \kappa$  regular. In addition, the arguments of this paper do not seem to allow us to construct a model in which  $\kappa$  is the least measurable cardinal and there are exactly  $\kappa$  many normal measures on  $\kappa$ . If one were to use  $\text{Add}(\omega, 1) * \text{Coll}(\kappa, 2^{2^\kappa})$  in the arguments of Theorem 1 above, or any other small forcing followed by the collapse of an ordinal to  $\kappa$ , then the main theorem of [8] shows that the measurability of  $\kappa$  would be destroyed.

We conclude with a list of open questions that the methods of this paper seem not to resolve (although for some, we now have partial answers). They are as follows:

- (1) Is it consistent, relative to anything, for the least measurable cardinal  $\kappa$  to have exactly  $\kappa$  many normal measures?<sup>1</sup>
- (2) How many measures can the least measurable cardinal have, when there is a strongly compact or supercompact cardinal above it? Results here show that any regular cardinal above  $\kappa$  is possible; for smaller values, it seems to be completely open.
- (3) For which values of  $\lambda$  does  $\text{Con}(\text{ZFC} + \text{There is one measurable cardinal } \kappa)$  imply  $\text{Con}(\text{ZFC} + \text{There is a measurable cardinal } \kappa \text{ with exactly } \lambda \text{ many normal measures})$ ? The case  $\lambda = 1$  is provided by  $L[\mu]$ , and the case  $\lambda = 2^{2^\kappa}$  is provided by the usual lifting argument techniques. The results of Mitchell [16] and Baldwin [3], however, use measurable cardinals of high Mitchell order.
- (4) How many normal measures can a measurable cardinal of nontrivial Mitchell rank have? In the canonical Mitchell inner model with  $o(\kappa) = \delta$ , there are exactly  $|\delta|$  many normal measures on  $\kappa$ . The usual lifting arguments show that  $2^{2^\kappa}$  is also always possible, with any value of  $o(\kappa)$ . Our results here show that any cardinal  $\delta \in [\kappa^+, 2^{2^\kappa}]$  with  $\text{cof}(\delta) > \kappa$  is also possible with any value of  $o(\kappa) < \kappa^{++}$ .
- (5) How many normal measures can  $\kappa$  have if  $\kappa$  is measurable and  $2^\kappa > \kappa^+$ ? The work of this paper shows that such a  $\kappa$  can have fewer than the maximal number of normal measures, but does not provide a fully general answer. Our results show that if  $\kappa$  is measurable and  $2^\kappa > \kappa^+$ , then for any regular

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<sup>1</sup>An affirmative answer to the question has been announced by Leaning and, independently, by Gitik.



- cardinal  $\delta$  in the interval  $(\kappa^+, 2^{2^\kappa}]$ , there is a forcing extension preserving  $2^\kappa > \kappa^+$ , where  $\kappa$  carries exactly  $\delta$  many normal measures.
- (6) If  $\kappa$  is  $\lambda$  supercompact, how many fine, normal  $\kappa$ -additive measures can there be on  $P_\kappa(\lambda)$ ? If  $\kappa$  is  $\lambda$ -supercompact, then our results here show that for any  $\delta \in [\lambda^+, 2^{2^\lambda}]$  with  $\text{cof}(\delta) \geq \lambda^+$ , there is a forcing extension where  $\kappa$  has exactly  $\delta$  many such  $\lambda$  supercompactness measures.
  - (7) If  $\kappa$  is  $\lambda$  strongly compact but is not  $\lambda$  supercompact, how many fine,  $\kappa$ -additive measures can  $P_\kappa(\lambda)$  have? Our results show that any  $\delta \in [\lambda^+, 2^{2^\lambda}]$  with  $\text{cof}(\delta) \geq \lambda^+$  is possible.
  - (8) For what values of  $\lambda$  is it consistent for  $\kappa$  to be fully supercompact and for  $\kappa$  to have exactly  $\lambda$  normal measures not concentrating on measurable cardinals? The usual lifting arguments show that  $\lambda = 2^{2^\kappa}$  is always possible, and so any  $\lambda > \kappa^+$  with  $\text{cof}(\lambda) > \kappa^+$  is possible. For smaller values of  $\lambda$ , it seems to be completely open.

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