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EMBEDDINGS OF n-DIMENSIONAL SEPARABLE METRIC SPACES INTO THE PRODUCT OF SIERPIŃSKI CURVES

DARIA MICHALIK

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Abstract. We give a short proof of the following fact: the set of embeddings of any n-dimensional separable metric space X into a certain n-dimensional subset of the $(n+1)$-product of Sierpiński curves Σ is residual in $C(X, \Sigma^{n+1})$.

Introduction and notation

In a Sierpiński curve we can specify the 0-dimensional subset of a "rational" points. In [5], Ivanšić and Milutinović proved that the $(n+1)$-product of Sierpiński curves, with points whose coordinates are all rational removed, is a universal space for n-dimensional metric separable spaces.

In this paper we present a new short proof of the last result. The proof is similar in spirit to proof of Theorem 1.1 in [6]. Sternfeld proved in his paper that any n-dimensional compact metric space may be embedded in the $(n+1)$-product of dendrites D. He also showed that the set of the basic embeddings is dense in $C(X, D^{n+1})$. However, it is worth pointing out that concerned in [6] is the fact that dendrites are ARs. Since a Sierpiński curve has not a good extension property, we use Lemma 3 to approximate it by ANRs. We also use in our proof the idea of the disjoint disk property that has been used in the topology of infinite- or finite-dimensional manifolds (cf. [7]).

All maps in this paper are continuous. Maps $f, g : X \to Y$ are said to be ε-near if $\sup_{x\in X} \operatorname{dist}(f(x), g(x)) < \varepsilon$. A map $f : X \to Y$ is an ε-map if each point $y \in Y$ has an open neighbourhood V_y such that $\operatorname{diam}(f^{-1}(V_y)) < \varepsilon$. We denote by $B(x, r)$ the open ball with centre x and radius r.

We shall use the following proposition (it is an easy application of Eilenberg's Theorem; see [1])

Proposition 1. *Let X be a compact metric space and Y be a metric ANR. Then for each map $f : X \to Y$ and each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each surjective δ-map $p : X \to X'$ there exists a map $q : X' \to Y$ such that $dist(f, q \circ p) < \varepsilon$.*

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Main theorem

Let us recall the construction of the triangular Sierpiński curve. Consider the homotheties $\varphi_i : \mathbb{R}^3 \to \mathbb{R}^3$ with scale $\frac{1}{2}$ and centres e_i, where $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$. With $\Sigma = \operatorname{conv}(e_1, e_2, e_3)$ we let

$$\Sigma_m := \bigcup_{\lambda_1,\ldots,\lambda_m \in \Lambda} \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_m}(\Sigma), \text{ where } \Lambda = \{1,2,3\}, \quad \text{and} \quad \Sigma(3) := \bigcap_{m \in \mathbb{N}} \Sigma_m.$$

We also let $\Sigma_n(3) := \Sigma \setminus \operatorname{Int} \Sigma_n$. Clearly, $\Sigma_n(3)$ is a graph and hence an ANR, for each $n \in \mathbb{N}$.

We call the vertices of triangles obtained in the construction of $\Sigma(3)$ *rational points*, and other points of $\Sigma(3)$ *irrational points*. Let

$$L_n(3) := \{x \in \Sigma(3)^{n+1} : \text{at least one coordinate of } x \text{ is irrational}\}.$$

Theorem 2. *Suppose that X is a metric separable space. If* $\dim(X) \leq n$, *then the set of embeddings of X in $L_n(3)$ is dense in the space $C(X, \Sigma(3)^{n+1})$.*

The original proof [5] of this result depended on complex arguments involving brick decompositions. Instead, here we use a simple property of compact metric spaces and the fact that a Sierpiński curve can be approximated by the graphs $\Sigma_n(3)$:

Lemma 3. *For each $\varepsilon > 0$ there exists a natural number k and a retraction r_k of $\Sigma(3)$ onto $\Sigma_k(3)$ such that $d(r_k, id_{\Sigma(3)}) < \varepsilon$.*

Proof. Let k be such that $\frac{1}{2^k} < \varepsilon$. For each $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Lambda^k$ the Sierpiński curve $\Sigma(3)$ intersects the simplex $S_\lambda = \varphi_{\lambda_1} \circ \cdots \circ \varphi_{\lambda_k}(\Sigma)$ along a set containing the boundary of $\operatorname{Bd}(S_\lambda)$ but different from S_λ. Therefore $\Sigma(3) \cap S_\lambda$ may be retracted onto $\operatorname{Bd}(S_\lambda)$. The union of these retractions, taken over all $\lambda \in \Lambda^k$, does the job. $\square$

Lemma 4. *Let X be a compact metric space and $f : X \to \Sigma(3)^{n+1}$ be a map. Then for each $\varepsilon > 0$ and for each pair of disjoint compact sets $A, B \subset X$ such that* $\dim A \leq n$, *there exists a map $f_\varepsilon : X \to \Sigma(3)^{n+1}$ such that* $\operatorname{dist}(f, f_\varepsilon) < \varepsilon$, *$f_\varepsilon(A) \cap f_\varepsilon(B) = \emptyset$ and $f_\varepsilon(A) \subset L_n(3)$.*

Proof. The proof is by induction on n. First let $n = 0$. We fix $\varepsilon > 0$ and take k so large that the retraction $r_k : \Sigma(3) \to \Sigma_k(3)$ satisfies $\operatorname{dist}(\mathrm{id}_{\Sigma(3)}, r_k) < \varepsilon/3$. We also cover A by finitely many disjoint δ-small compact sets, where $\delta < \operatorname{dist}(A, B)$, and denote by X' the space obtained from X by squeezing each of them to a point. Let us observe that by [3] (Theorem 4.4.15) X' is a metrizable space. By Proposition 1, for δ small enough the map $r_k \circ f$ is $\varepsilon/3$-close to the composition of the projection $p : X \to X'$ and of a map $f' : X' \to \Sigma_k(3) \subset \Sigma(3)$. Thus, by replacing f by f', and A and B by $p(A)$ and $p(B)$, respectively, we may assume that the set A is finite, and by treating each of its points individually - that A has only one point, which we denote a. We assume these arrangements have been made; in particular, $\operatorname{Im}(f) \subset \Sigma_k(3)$.

Let $B(a, \eta)$ be a ball in X centred at a, $W = X/\operatorname{Bd}B(a, \eta)$ be the decomposition space, obtained by squeezing $\operatorname{Bd}B(a, \eta)$ to a point, and let $p : X \to W$ be the projection. By taking η small enough we ensure that $\operatorname{diam} f(B(a, \eta)) < \varepsilon/3$ and there exists a map $q : W \to \Sigma_k(3) \subset \Sigma(3)$ such that $q \circ p$ is $\varepsilon/3$-near to f. (We use Proposition 1.)

Let us fix $b \in \mathrm{Bd}(B(a,\eta))$. There exists an $\varepsilon/3-$short arc J in $\Sigma(3)$ joining $q(p(b))$ with an irrational point $c \in \Sigma(3) \setminus \Sigma_k(3)$. Let $h : W \to \Sigma(3)$ be a map such that $h(x) = q(x)$ for $x \in W \setminus p(B(a,\eta))$ and $h(p(a)) = c$. The arc J is an AR so we can extend the map h to the whole space W so that $h(p(B(a,\eta)) \subset J$.

Let us define $f_\varepsilon = h \circ p$. If $x \in X \setminus B(a,\eta)$, then $f_\varepsilon(x) = q \circ p(x)$ and hence $d(f_\varepsilon(x), f(x)) < \varepsilon/3$, while for $x \in B(a,\eta)$ we have $f(x), f_\varepsilon(x) \in J$ and $f(b), f(x) \in f(B(a,\eta))$, whence

$$d(f_\varepsilon(x), f(x)) \leq d(f_\varepsilon(x), f_\varepsilon(b)) + d(f_\varepsilon(b), f(b)) + d(f(b), f(x)) < 3 \cdot \varepsilon/3$$

because $\mathrm{diam} J, \mathrm{diam} f(B(a,\eta)) < \varepsilon/3$. Thus $d(f, f_\varepsilon) < \varepsilon$ and since $f_\varepsilon(a) \in \Sigma(3) \setminus \Sigma_k(3)$ and $f_\varepsilon(B) \subset \Sigma_k(3)$ the assertion of the lemma is true if $n = 0$.

Assume now that the lemma holds true if $\dim A \leq n-1$. From the Baire Theorem and from the fact that $L_0(3)$ is of G_δ type we conclude that

$$(*) \qquad \text{the lemma is true if } A \text{ is } \sigma-\text{compact and } \dim A \leq n-1.$$

Now, given $f = (f_1, f_2) : X \to \Sigma(3)^n \times \Sigma(3)$ and a compact set $A \subset X$ with $\dim A \leq n$, we can represent A as $A = Y \cup Z$, where $\dim Y \leq n-1$, $\dim Z \leq 0$ and Z is σ-compact. (See [2], §1.5.) By $(*)$ we can assume that $f_2(Z) \cap f_2(B) = \emptyset$ and $f_2(Z) \subset L_0(3)$. The set $Y' = (f_2 \mid_A)^{-1}(f_2(B) \cup (\Sigma(3) \setminus L_0(3)))$ is disjoint from Z, $n-1$-dimensional, and σ-compact (for $L_0(3)$ is of type G_δ). Thus by $(*)$ there exists a map $f_1^\varepsilon : X \to \Sigma(3)^n$ ε-near to f_1 such that $f_1^\varepsilon(Y') \subset L_{n-1}(3)$ and $f_1^\varepsilon(Y') \cap f_1^\varepsilon(B) = \emptyset$. We take $f_\varepsilon(x) = (f_1^\varepsilon(x), f_2(x))$ to complete the proof. $\square$

Proof of Theorem 2. Let $f \in C(X, \Sigma(3)^{n+1})$. By [4] there exists a dimension preserving compactification X^* of the space X such that the map $f : X \to \Sigma(3)^{n+1}$ can be extended to $f^* : X^* \to \Sigma(3)^{n+1}$. Hence, without loss of generality, we may assume that X is compact.

Let $\mathcal{B}$ be a countable family of closed subsets of X whose interiors are a base of the topology of X. Let us observe that

$$\begin{aligned}&\{h \in C(X, \Sigma(3)^{n+1}) : h \text{ is an embedding}\} \\ &\quad = \bigcap_{\substack{A,B \in \mathcal{B} \\ A \cap B = \emptyset}} \{f \in C(X, \Sigma(3)^{n+1}) : f(A) \cap f(B) = \emptyset\}.\end{aligned}$$

Fix $A, B \in \mathcal{B}$ such that $A \cap B = \emptyset$. By Lemma 4, the set $\{f \in C(X, \Sigma(3)^{n+1}) : f(A) \cap f(B) = \emptyset\}$ is dense in $C(X, \Sigma(3)^{n+1})$. Also by Lemma 4 (with $A = X$ and $B = \emptyset$) the set $\{f \in C(X, \Sigma(3)^{n+1}) : f(X) \subset L_n(3)\}$ is dense in $C(X, \Sigma(3)^{n+1})$. The sets $\{f \in C(X, \Sigma(3)^{n+1}) : f(X) \subset L_n(3)\}$ and $\bigcap_{\substack{A,B \in \mathcal{B} \\ A \cap B = \emptyset}} \{f \in C(X, \Sigma(3)^{n+1}) : f(A) \cap f(B) = \emptyset\}$ are obviously of type G_δ. Now the proof is completed by an application of Baire's Theorem. $\square$

Remark 5. The following property of a Sierpiński curve can be proved analogously to Lemma 4:

Let F be a σ-closed 0*-dimensional subset of a compact metric space X. Then* $\{f \in C(X, \Sigma(3)) : f^{-1}f(x) = \{x\}$ for all $x \in F$ and $f(F)$ contains no rational points$\}$ *is a dense G_δ-set in $C(X, \Sigma(3))$.*

This fact is similar to Theorem 1.1 in [6] that is a key result of Sternfeld's paper. Using this property of a Sierpiński curve and the argument that the set of trivial fibers of map f is G_δ-set we can construct an embedding into $L_n(3)$. This way of construction is analogous to Sternfeld's methods in [6].

Remark 6. In [6], Sternfeld noted that in the case of embeddings into $(n+1)$-product of dendrites the last space in that product can be replaced by an interval. The same can be done also for the product of Sierpiński curves.

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Institute of Mathematics, Polish Academy of Sciences, P.O. Box 21 Śniadeckich 8 00-956 Warszawa, Poland

E-mail address: daria@impan.gov.pl