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ARTINIANESS OF GRADED LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let $R = \bigoplus_{n\in\mathbb{N}} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$ and let M be a finitely generated graded R-module. Let a be the largest integer such that $H^a_{R_+}(M)$ is not Artinian. We will prove that $H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M)$ are Artinian for all $i \geq a$ and there exists a polynomial $\widetilde{P} \in \mathbb{Q}[\mathbf{x}]$ of degree less than a such that $\text{length}_{R_0}(H^a_{R_+}(M)_n/\mathfrak{m}_0 H^a_{R_+}(M)_n) = \widetilde{P}(n)$ for all $n \ll 0$. Let s be the first integer such that the local cohomology module $H^s_{R_+}(M)$ is not R_+-cofinite. We will show that for all $i \leq s$ the graded module $\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M))$ is Artinian.

1. Introduction

Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$. So R_0 is a Noetherian ring and there are finitely many elements $l_1, \ldots, l_r \in R_1$ such that $R = R_0[l_1, \ldots, l_r]$. Let $R_+ := \bigoplus_{n\in\mathbb{N}} R_n$ denote the irrelevant ideal of R and let $\mathfrak{m} := \mathfrak{m}_0 \oplus R_+$ denote the graded maximal ideal of R. Moreover let $\mathfrak{q}_0 \subseteq R_0$ be an $\mathfrak{m}_0$-primary ideal. Finally let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a finitely generated graded R-module.

Let ${}^*E := {}^*E_R(R/\mathfrak{m})$ be the *injective envelope of the graded R-module $R/\mathfrak{m}$, and let $E_0 := E_{R_0}(R_0/\mathfrak{m}_0)$ be the injective envelope of the R_0-module $R_0/\mathfrak{m}_0$. Moreover, for a graded R-module $T = \bigoplus_{n\in\mathbb{Z}} T_n$ and an R_0-module U, let ${}^*D(T) := {}^*\text{Hom}_R(T, {}^*E)$ and $D_0(U) := \text{Hom}_{R_0}(U, E_0)$ denote the *Matlis dual of T and the Matlis dual of U respectively (cf. [BS, Exercise 13.4.5], [BH, Theorem 3.6.17]). Let A be a graded Artinian R-module and let $\widehat{R_0}$ be the completion of R_0 with respect to $\mathfrak{m}_0$-adic topology. Then A carries a natural structure as a graded $\widehat{R}$-module where $\widehat{R} = \widehat{R_0} \otimes_{R_0} R$. Clearly, as an $\widehat{R}$-module A is again Artinian with $\text{length}_{R_0}(A_n) = \text{length}_{\widehat{R_0}}(A_n)$ for all $n \in \mathbb{Z}$. In particular by [K] the length of graded components of an $\widehat{R}$-module A has polynomial growth and as an $\widehat{R}$-module A has the same polynomial as it has over R. We now take the *Matlis dual of the $\widehat{R}$-module A and denote it by ${}^*\widehat{D}(A)$.

The modules $H^i_{R_+}(M)$ and their graded components are closely related to sheaf cohomology over projective schemes (cf. [BS, Chap. 20]). Then it is very important

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to study the Artinianess of these graded modules which has a very close relation with the minimal generators of their components.

Brodmann, Fumasoli and Tajarod in [BFT] showed that if the local base ring R_0 is of dimension one, then for all i and for all $\mathfrak{m}_0$-primary ideals $\mathfrak{q}_0$ the graded R-modules $H^i_{R_+}(M)/\mathfrak{q}_0 H^i_{R_+}(M), (0:_{H^i_{R_+}(M)} \mathfrak{q}_0)$ are Artinian and hence the length of the components of these graded modules have polynomial growth. Next, the authors in [BRS] showed that the degrees of these polynomials are independent of the choice of $\mathfrak{q}_0$. In the case $\dim(R_0) = 2$, the situation changes drastically. Here, the graded R-modules $(0:_{H^i_{R_+}(M)} \mathfrak{m}_0)$ and $H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M)$ need not be Artinian in general (cf. [BFT, Examples 4.1, 4.2]). Moreover the above numerical functions need not be polynomial in this case, as shown by examples of Katzman and Sharp.

Let $g = g(M)$ (referred to as the *cohomological finite length dimension*) be the least integer i such that the R_0-module $H^i_{R_+}(M)_n$ is of infinite length for infinitely many integers n. Authors in [BRS] showed that if $i \leq g$, then $\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M))$ is Artinian. In this paper we will obtain a parallel conclusion to this result. Let $c = c(M)$ (referred to as the *cohomological dimension of M with respect to R_+*) be the largest integer i such that $H^i_{R_+}(M) \neq 0$. Rotthaus and Şega in [RS] proved that $H^c_{R_+}(M)/\mathfrak{m}_0 H^c_{R_+}(M)$ is Artinian. We will also extend this result for the largest a such that $H^a_{R_+}(M)$ is not Artinian.

Let $a = a_{R_+}(M)$ be the largest integer such that $H^a_{R_+}(M)$ is not Artinian. We will prove that $H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M)$ is Artinian for all $i \geq a$. We will also show that there exists a polynomial $\widetilde{P} \in \mathbb{Q}[\mathbf{x}]$ of degree less than a such that $\mathrm{length}_{R_0}(H^a_{R_+}(M)_n/\mathfrak{m}_0 H^a_{R_+}(M)_n) = \widetilde{P}(n)$ for all $n \ll 0$. Next we deduce that $H^i_{R_+}(M)/\mathfrak{q}_0 H^i_{R_+}(M)$ is Artinian for all $\mathfrak{m}_0$-primary ideals $\mathfrak{q}_0$, the length of the components of this graded module has polynomial growth and the degrees of these polynomials are independent of $\mathfrak{q}_0$.

For any graded ideal $\mathfrak{a}$ of R and any graded R-module N we say that N is $\mathfrak{a}$-*cofinite* if $\mathrm{Supp}(N) \subset V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, N)$ is finitely generated graded for all $i \geq 0$. Also, we define $s = c_{\mathfrak{a}}(N)$ as the first integer such that the local cohomology module $H^s_{\mathfrak{a}}(N)$ is not $\mathfrak{a}-$cofinite. We will show that for all $i \leq s = c_{R_+}(M)$ the graded module $\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M))$ is Artinian and there exists a polynomial $\widetilde{P} \in \mathbb{Q}[\mathbf{x}]$ of degree $\dim_{\widehat{R}}({}^*\widehat{D}(0:_{\Gamma_{\mathfrak{m}_0}(H^s_{R_+}(M))} \mathfrak{m}_0))$ such that $\mathrm{length}_{R_0}(\Gamma_{\mathfrak{m}_0}(H^s_{R_+}(M)_n)) = \widetilde{P}(n)$ for all $n \ll 0$.

2. The results

2.1. **Definition.** For any graded ideal $\mathfrak{a}$ and any finitely generated graded R-module M we define

$$a_{\mathfrak{a}}(M) = \sup\{i | H^i_{\mathfrak{a}}(M) \text{ is not Artinian}\}.$$

In view of this definition we have the following lemma.

2.2. **Lemma.** *Let $x \in \mathfrak{m}$ be a homogeneous non-zero divisor of M. Then we have*

$$a_{R_+}(M/xM) \leq a_{R_+}(M).$$

Proof. Let $\deg x = d$. As x is a non-zero divisor of M, there is an exact sequence

$$0 \to M(-d) \xrightarrow{x.} M \to M/xM \to 0$$

of R-modules. Application of the functor $H^i_{R_+}(-)$ to it induces the following exact sequence $H^i_{R_+}(M) \to H^i_{R_+}(M/xM) \xrightarrow{x.} H^{i+1}_{R_+}(M)(-d) \to H^{i+1}_{R_+}(M)$. If $a_{R_+}(M) = t$, then for all $i > t$ the R-modules $H^i_{R_+}(M)$ are Artinian and so $H^i_{R_+}(M/xM)$ are Artinian. Therefore the result follows. □

2.3. *Remark.* a) Any local flat morphism of local Noetherian rings is faithfully flat. So, if R'_0 is flat over R_0 and $\mathfrak{m}'_0 = \mathfrak{m}_0 R'_0$, then R'_0 is faithfully flat over R_0. Moreover, it follows from [K, Theorem 1] that if $(R'_0, \mathfrak{m}'_0)$ be a faithfully flat local R_0-algebra, then A is a graded Artinian R-module if and only if $A' := R'_0 \otimes_{R_0} A$ is a graded Artinian module over $R' := R'_0 \otimes_{R_0} R$.

b) Let $(R'_0, \mathfrak{m}'_0)$ be a faithfully flat local R_0-algebra. One can easily show that $a_{R_+}(M) = a_{(R'_0 \otimes_{R_0} R)_+}(R'_0 \otimes_{R_0} M)$.

2.4. **Theorem.** *Let $a = a_{R_+}(M)$. Then $H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M)$ is Artinian for all $i \geq a$.*

Proof. We proceed by induction on $d = \dim R_0$. At first if $i > a$, then $H^i_{R_+}(M)$ is Artinian, and so it follows from [BFT, Lemma 2.2] that $H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M)$ is Artinian. So we assume that $i = a$. If $d = 1$, then the result follows by [BFT, Theorem 2.5]. Suppose inductively that the result has been proved for all values smaller than d and so we prove it for d. Let $\mathbf{x}$ be an indeterminate and let $R'_0 := R_0[\mathbf{x}]_{\mathfrak{m}_0 R_0[\mathbf{x}]}$, $\mathfrak{m}'_0 := \mathfrak{m}_0 R'_0, R' = R'_0 \otimes_{R_0} R$ and $M' := R'_0 \otimes_{R_0} M$. Then by the flat base change property of local cohomology $R'_0 \otimes_{R_0} H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M) \cong H^a_{R'_+}(M)/\mathfrak{m}'_0 H^a_{R'_+}(M')$. As R'_0 is a faithfully flat local R_0-algebra, in view of the above remark, it suffices show that $H^a_{R'_+}(M)/\mathfrak{m}'_0 H^a_{R'_+}(M')$ is Artinian. Therefore, we may replace R and M by R' and M', respectively and hence we assume that $R_0/\mathfrak{m}_0$ is infinite residue field. Consider the exact sequence $0 \to \Gamma_{m_0}(M) \to M \to M/\Gamma_{m_0}(M) \to 0$. Application of the functor $H^i_{R_+}(-)$ induces the following exact sequence:

$$H^i_{R_+}(\Gamma_{m_0}(M)) \xrightarrow{\alpha} H^i_{R_+}(M) \xrightarrow{\beta} H^i_{R_+}(M/\Gamma_{m_0}(M)) \xrightarrow{\gamma} H^{i+1}_{R_+}(\Gamma_{m_0}(M)).$$

It should be noted that $H^i_{R_+}(\Gamma_{m_0}(M))$ is Artinian for each i by [BFT, Lemma 2.3]. Now consider $U = \mathrm{Im}\alpha, \mathrm{V} = \mathrm{Im}\beta$ and $W = \mathrm{Im}\gamma$. It follows from [BFT, Lemma 2.2] that both $\mathrm{Tor}^{R_0}_i(R_0/\mathfrak{m}_0, U)$ and $\mathrm{Tor}^{R_0}_i(R_0/\mathfrak{m}_0, W)$ are Artinian for all i. Now, let $H^i_{R_+}(M/\Gamma_{m_0}(M))/\mathfrak{m}_0 H^i_{R_+}(M/\Gamma_{m_0}(M))$ be Artinian. It implies that $V/m_0 V$ is Artinian and then we can conclude that $H^i_{R_+}(M)/\mathfrak{m}_0 H^i_{R_+}(M)$ is Artinian. One can also easily show that $a_{R_+}(M) = a_{R_+}(M/\Gamma_{\mathfrak{m}_0}(M))$. So we may assume that $\Gamma_{\mathfrak{m}_0}(M) = \Gamma_{\mathfrak{m}_0 R}(M) = 0$. Now, let x_0 be a non-zero divisor of M and a part of a system of parameters of $\mathfrak{m}_0$, and consider the exact sequence $0 \to M \xrightarrow{x_0.} M \to M/x_0 M \to 0$ of R-modules. Application of the functor $H^a_{R_+}(-)$ induces the exact sequence

$$H^a_{R_+}(M) \xrightarrow{x_0.} H^a_{R_+}(M) \to H^a_{R_+}(M/x_0 M) \xrightarrow{\delta} H^{a+1}_{R_+}(M).$$

Consider $\ker\delta = X$ and $\mathrm{Im}\delta = Y$. As $H^{a+1}_{R_+}(M)$ is Artinian, the graded R-module Y is Artinian, and then in view of [BFT, Lemma 2.2], $H^a_{R_+}(M/x_0 M)/\mathfrak{m}_0 H^a_{R_+}(M/x_0 M)$ is Artinian if and only if $X/\mathfrak{m}_0 X$ is Artinian. On the other hand, application of the functor $R_0/\mathfrak{m}_0 \otimes_{R_0}$ induces the exact sequence

$$R_0/\mathfrak{m}_0 \otimes_{R_0} H^a_{R_+}(M) \overset{id_{R_0/\mathfrak{m}_0} \otimes_{R_0} x_0.}{\longrightarrow} R_0/\mathfrak{m}_0 \otimes_{R_0} H^a_{R_+}(M) \longrightarrow X/\mathfrak{m}_0 X \longrightarrow 0.$$

As $x_0 \in \mathfrak{m}_0$, the map $id_{R_0/\mathfrak{m}_0} \otimes_{R_0} x_0$ is zero and then $H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M) \cong X/\mathfrak{m}_0 X$. Now, in view of the above arguments one can conclude that $H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M)$ is artinian if and only if $H^a_{R_+}(M/x_0M)/\mathfrak{m}_0 H^a_{R_+}(M/x_0M)$ is Artinian. Set $\bar{R_0} = R_0/x_0R_0$ and $\bar{\mathfrak{m}_0} = \mathfrak{m}_0/x_0R_0$. Using the Independence Theorem for graded local cohomology we have $H^a_{R_+}(M/x_0M) \cong H^a_{(\bar{R_0}\otimes_{R_0} R)_+}(M/x_0M)$. We note that the local base ring of the graded ring $\bar{R_0} \otimes_{R_0} R$ is $(\bar{R_0}, \bar{\mathfrak{m}_0})$ and $\dim \bar{R_0} = d-1$. It is easy to see that

$$\begin{aligned} &H^a_{R_+}(M/x_0M)/\mathfrak{m}_0 H^a_{R_+}(M/x_0M) \\ &\qquad \cong H^a_{(\bar{R_0}\otimes_{R_0} R)_+}(M/x_0M)/\bar{\mathfrak{m}_0} H^a_{(\bar{R_0}\otimes_{R_0} R)_+}(M/x_0M). \end{aligned}$$

Now, in view of Lemma 2.2 and using induction hypotheses this module is Artinian. $\square$

2.5. **Proposition.** *Let $a_{R_+}(M) = a$ and $\mathfrak{m} \notin \operatorname{Att}_R(H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M))$; then there exists an element $x \in R_1$ such that $a_{R_+}(M/xM) = a-1$.*

Proof. As for any finitely generated graded R-module M and any non-negative integer i there is an isomorphism $H^i_{R_+}(M) \cong H^i_{R_+}(M/\Gamma_{R_+}(M))$; we may assume that $\Gamma_{R_+}(M) = 0$. Since, by Theorem 2.4, $H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M)$ is Artinian, the set of its attached prime ideal is finite, and so we set

$$\mathcal{P} = \operatorname{Att}_R(H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M)) \bigcup \operatorname{Ass}_R(M) \setminus \operatorname{Var}(R_+).$$

We note that $R_1 \nsubseteq \bigcup_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}$, otherwise for some $\mathfrak{p} \in \operatorname{Att}_R(H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M))$ we should have $R_1 \subseteq \mathfrak{p}$ and hence $R_+ \subseteq \mathfrak{p}$. On the other hand, since

$$\operatorname{Att}_R(H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M)) \subseteq \operatorname{Var}(\mathfrak{m}_0 R),$$

one can conclude that $\mathfrak{p} = \mathfrak{m}$ and this is a contradiction. Now, consider $x \in R_1 \setminus \bigcup_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}$ and the exact sequence $0 \to M(-1) \to M \to M/xM \to 0$. Application of the functor $H^i_{R_+}(-)$ induces the following exact sequence:

$$H^i_{R_+}(M)(-1) \xrightarrow{x.} H^i_{R_+}(M) \to H^i_{R_+}(M/xM) \to H^{i+1}_{R_+}(M)(-1).$$

Set $i = a$. This fact that $\mathfrak{m} \notin \operatorname{Att}_R(H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M))$ implies that any prime ideal in $\operatorname{Att}_R(H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M))$ belongs to $\mathcal{P}$. So using the same proof mentioned in [BFT, Lemma 3.2], we deduce that $\operatorname{Coker} x. = 0$, and hence $H^a_{R_+}(M/xM)$ is embedded in the Artinian module $H^{a+1}_{R_+}(M)$. Thus $a_{R_+}(M/xM) \leq a_{R_+}(M) - 1$. Conversely, for $i > a_{R_+}(M/xM)$ the graded module $H^i_{R_+}(M/xM)$ is Artinian and so $(0 :_{H^{i+1}_{R_+}(M)} x)$ is Artinian. Now, since $H^{i+1}_{R_+}(M)$ is x-torsion, using the Melkersson Lemma the result follows. $\square$

2.6. **Theorem.** *Let $a = a_{R_+}(M)$. Then there exists a polynomial $\widetilde{P} \in \mathbb{Q}[\mathbf{x}]$ of degree less than a such that* $\operatorname{length}_{R_0}(H^a_{R_+}(M)_n/\mathfrak{m}_0 H^a_{R_+}(M)_n) = \widetilde{P}(n)$ *for all $n \ll 0$.*

Proof. By the fact that for each R_0-module T and any Noetherian local flat R_0-algebra $(R'_0, \mathfrak{m}'_0)$ we have $\text{length}_{R_0}(M) = \text{length}_{R'_0}(R'_0 \otimes_{R_0} M)$, we may assume that $R_0/\mathfrak{m}_0$ is infinite and $\Gamma_{R_+}(M) = 0$. As $H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M)$ is Artinian, by using [K], there exists a polynomial $\widetilde{P} \in \mathbb{Q}[\mathbf{x}]$ such that $\text{length}_{R_0}(H^a_{R_+}(M)_n/\mathfrak{m}_0 H^a_{R_+}(M)_n) = \widetilde{P}(n)$ for all $n \ll 0$. Now, we prove $\deg\widetilde{P} < a$. Since $H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M)$ is Artinian, it has a graded secondary representation. Let $H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M) = S^1 + \cdots + S^t$ be a minimal graded secondary representation with $\mathfrak{p}_j = \sqrt{(0 :_R S^j)}$ for all $1 \leq j \leq t$. Let $\mathfrak{p}_t = \mathfrak{m}$. So in this case S^t is a graded R-module of finite length and hence it is concentrated in finitely many degrees. If we take $d = \text{beg}S^t - 1$ where $\text{beg}S^t$ is the beginning degree of S^t, then $H^a_{R_+}(M)_n/\mathfrak{m}_0 H^a_{R_+}(M)_n = (S^1)(n) + \cdots + (S^{t-1})(n)$ for all sufficiently small n such that $n < d$. So

$$\text{length}_{R_0}(H^a_{R_+}(M)_n/\mathfrak{m}_0 H^a_{R_+}(M)_n) = \widetilde{P}(n) = \text{length}_{R_0}((S^1 + \cdots + S^{t-1})_n)$$

for all $n \ll 0$. Therefore we may assume that $\mathfrak{m} \notin \text{Att}_R(H^a_{R_+}(M)/\mathfrak{m}_0 H^a_{R_+}(M))$. Now, we prove the assertion by induction on a. Using Proposition 2.5, there exists an element $x \in R_1$ such that $a_{R_+}(M/xM) = a - 1$ and x is a non-zero divisor of M. By the same proof mentioned in [BFT, Lemma 3.2], for all $n \ll 0$ there exists an exact sequence of R_0-modules

$$H^{a-1}_{R_+}(M/xM)_{n+1} \longrightarrow H^a_{R_+}(M)_n \xrightarrow{x.} H^a_{R_+}(M)_{n+1} \longrightarrow 0.$$

Application of the functor $R_0/\mathfrak{m}_0 \otimes_{R_0}$ induces the following exact sequence:

$$\begin{aligned} &H^{a-1}_{R_+}(M/xM)_{n+1}/\mathfrak{m}_0 H^{a-1}_{R_+}(M/xM)_{n+1} \longrightarrow H^a_{R_+}(M)_n/\mathfrak{m}_0 H^a_{R_+}(M)_n \\ &\xrightarrow{x.} H^a_{R_+}(M)_{n+1}/\mathfrak{m}_0 H^a_{R_+}(M)_{n+1} \longrightarrow 0. \end{aligned} \tag{‡}$$

If $a = 1$, then we have the following exact sequence:

$$\begin{aligned} &\Gamma_{R_+}(M/xM)_{n+1}/\mathfrak{m}_0 \Gamma_{R_+}(M/xM)_{n+1} \longrightarrow H^1_{R_+}(M)_n/\mathfrak{m}_0 H^1_{R_+}(M)_n \\ &\xrightarrow{x.} H^1_{R_+}(M)_{n+1}/\mathfrak{m}_0 H^1_{R_+}(M)_{n+1} \longrightarrow 0. \end{aligned}$$

We note that $\Gamma_{R_+}(M/xM)/\mathfrak{m}_0 \Gamma_{R_+}(M/xM)$ is Artinian and finitely generated, and hence $\text{length}_{R_0}(\Gamma_{R_+}(M/xM)_{n+1}/\mathfrak{m}_0 \Gamma_{R_+}(M/xM)_{n+1}) = 0$ for all $n \ll 0$. Therefore $\widetilde{P}(n+1) \leq \widetilde{P}(n) \leq \widetilde{P}(n+1) + \text{length}_{R_0}(\Gamma_{R_+}(M/xM)_{n+1}/\mathfrak{m}_0 \Gamma_{R_+}(M/xM)_{n+1}) = \widetilde{P}(n+1) \leq \widetilde{P}(n+1)$ for all $n \ll 0$. It implies that $\deg\widetilde{P} < 1$. Let $a > 1$ and the result be true for all values smaller than a. In view of the exact sequence (‡), since $H^{a-1}_{R_+}(M/xM)/\mathfrak{m}_0 H^{a-1}_{R_+}(M/xM)$ is Artinian, using induction there is a polynomial $Q \in \mathbb{Q}[\mathbf{x}]$ of degree less than $a - 1$ such that

$$\text{length}_{R_0}(H^{a-1}_{R_+}(M/xM)_n/\mathfrak{m}_0 H^{a-1}_{R_+}(M/xM)_n) = Q(n)$$

for all $n \ll 0$. So we have $\widetilde{P}(n) \leq \widetilde{P}(n+1) + Q(n+1)$ for all $n \ll 0$, and this implies that $\deg\widetilde{P} < a$. □

2.7. **Corollary.** *Let $a = a_{R_+}(M)$ and let $\mathfrak{q}_0$ be a an $\mathfrak{m}_0$-primary ideal of R_0. Then $H^a_{R_+}(M)/\mathfrak{q}_0 H^a_{R_+}(M)$ is Artinian and there is a polynomial $\overline{P} \in \mathbb{Q}[\mathbf{x}]$ such that $\deg\overline{P} = \deg\widetilde{P}$ and $\text{length}_{R_0}(H^a_{R_+}(M)_n/\mathfrak{q}_0 H^a_{R_+}(M)_n) = \overline{P}(n)$ for all $n \ll 0$.*

Proof. The proof of Artinianess of $H^a_{R_+}(M)/\mathfrak{q}_0 H^a_{R_+}(M)$ is similar to [BFT, Corollary 2.6] and $\deg\overline{P} = \deg\widetilde{P}$ by [BRS, Proposition 2.6]. □

2.8. **Definition.** Let M be a finitely generated graded R-module and let $\mathfrak{a}$ be a graded ideal of R. We say that M is $\mathfrak{a}$-cofinite if $\mathrm{Supp}(M) \subset V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated graded for all $i \geq 0$. We also introduce the non-negative integer s as follows:

$$s = c_{\mathfrak{a}}(M) = \inf\{i | H^i_{\mathfrak{a}}(M) \text{ is not } \mathfrak{a}\text{-cofinite}\}.$$

Moreover, if there in no such integer, then we define $c_{\mathfrak{a}}(M) = -\infty$.

2.9. **Theorem.** *Let $s = c_{R_+}(M)$ and $i \in \mathbb{N}_0$ with $i \leq s$. Then $\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M))$ is Artinian.*

Proof. The strategy of the proof for all $i \leq s$ is the same, and hence we consider the case $i = s$. As M is a finitely generated graded R-module, using [DY, Theorem 2.1], $\mathrm{Hom}_R(R/R_+, H^s_{R_+}(M))$ is a finitely generated graded R-module. One can easily show that $\Gamma_{\mathfrak{m}_0}(\mathrm{Hom}_R(R/R_+, H^s_{R_+}(M))) \cong \Gamma_{\mathfrak{m}_0}((0 :_{H^s_{R_+}(M)} R_+)) = (0 :_{\Gamma_{\mathfrak{m}_0}(H^s_{R_+}(M))} R_+)$. As $\mathrm{Hom}_R(R/R_+, H^s_{R_+}(M))$ is $(0 :_{\Gamma_{\mathfrak{m}_0}(H^s_{R_+}(M))} R_+)$ it is Artinian, and now since $\Gamma_{\mathfrak{m}_0}(H^s_{R_+}(M))$ is R_+-torsion, the result follows by the Melkersson Lemma. ☐

2.10. **Corollary.** *Let $s < \infty$. Then we have the following conditions:*

(a) *There exists a polynomial $\widetilde{P} \in \mathbb{Q}[\mathbf{x}]$ such that*

$$\mathrm{length}_{R_0}(\Gamma_{\mathfrak{m}_0}(H^s_{R_+}(M)_n)) = \widetilde{P}(n) \quad \text{for all } n \ll 0.$$

(b) *If $\mathfrak{q}_0$ is an $\mathfrak{m}_0$-primary ideal of R_0, then there is a polynomial $\overline{P} \in \mathbb{Q}[\mathbf{x}]$ such that $\deg(\overline{P}) = \deg(\widetilde{P}) = \dim_{\widehat{R}}({}^*\widehat{D}(0 :_{\Gamma_{\mathfrak{m}_0}(H^s_{R_+}(M))} \mathfrak{m}_0))$ and*

$$\mathrm{length}_{R_0}((0 :_{H^s_{R_+}(M)_n} \mathfrak{q}_0)) = \overline{P}(n) \quad \text{for all } n \ll 0.$$

Proof. (a) As $\Gamma_{\mathfrak{m}_0}(H^s_{R_+}(M)$ is Artinian, the result is obtained by [K].

(b) Apply [BRS, Corollary 2.5] with $A = \Gamma_{\mathfrak{m}_0}(H^s_{R_+}(M))$. ☐

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