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BANACH-STONE THEOREM FOR BANACH LATTICE VALUED CONTINUOUS FUNCTIONS

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Abstract. Let X and Y be compact Hausdorff spaces, E be a Banach lattice and F be an AM space with unit. Let $\pi : C(X,E) \to C(Y,F)$ be a Riesz isomorphism such that $0 \notin f(X)$ if and only if $0 \notin \pi(f)(Y)$ for each $f \in C(X,E)$. We prove that X is homeomorphic to Y and E is Riesz isomorphic to F. This generalizes some known results.

Introduction

In this paper we use the standard terminology and notation of the Riesz spaces theory. For unexplained definitions and notation we refer to [1]. Throughout this paper X and Y stand for compact Hausdorff spaces. For a compact Hausdorff space Z and a Banach lattice E, the Banach lattice (under pointwise algebraic operations and order) of continuous functions from Z into E is denoted by $C(Z,E)$. If $E = \mathbb{R}$ we write $C(Z)$ instead of $C(Z,E)$. $\mathbf{1}_Z \in C(Z)$ is defined by $\mathbf{1}_Z(x) = 1$.

One of the versions of the Banach-Stone Theorem states that X and Y are homeomorphic if and only if $C(X)$ and $C(Y)$ are Riesz isomorphic. More precisely, if $\pi : C(X) \longrightarrow C(Y)$ is a Riesz homomorphism, then there exists a homeomorphism $\sigma : Y \longrightarrow X$ such that $\pi(f) = \pi(\mathbf{1}_X)(f\sigma)$. A simple and direct proof of this theorem can be found in [5]. This theorem is generalized in [2] as follows:

Theorem 1. *Let E be a Banach lattice and $\pi : C(X,E) \longrightarrow C(Y)$ be a Riesz isomorphism such that $\pi(f)(y) \neq 0$ for each $y \in Y$ whenever $f(x) \neq 0$ for each $x \in X$. Then X is homeomorphic to Y and E is Riesz isomorphic to $E = \mathbb{R}$.*

A simple and direct proof of the above theorem is given in [3], also providing a positive answer to a conjecture in [2]. By using Theorem 2 of [4] it can also be proved that the above theorem is still true for $d-isomorphism$ π; that is, $x \perp y$ if and only if $\pi(x) \perp \pi(y)$.

The main result

The aim of this paper is to generalize the above theorem as follows:

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Theorem 2. *Let E be a Banach lattice and F be an AM space with unit. Let $\pi : C(X,E) \longrightarrow C(Y,F)$ be a Riesz isomorphism. If*

$$f(x) \neq 0 \quad \text{for each } x \in X \Longleftrightarrow \pi f(y) \neq 0 \quad \text{for each } y \in Y,$$

then X is homeomorphic to Y and E is Riesz isomorphic to F.

To prove this theorem we will need the following lemmas. The proofs of the first two lemmas are clear.

Lemma 3. *Let M and N be compact Hausdorff spaces and $\sigma : X \times M \longrightarrow Y \times N$ be a homeomorphism. If for each $x \in X$ there exists a unique $y \in Y$ such that $\sigma(\{x\} \times M) = \{y\} \times N$, then X and M are homeomorphic to Y and N, respectively.*

Lemma 4. *Let $f : X \longrightarrow Y$ be a continuous open surjective map. If K is a compact subset of Y and O is an open subset of X with $K \subset f(O)$, then there is a closed subset F of X such that $F \subset O$ and $K \subset f(F)$.*

Lemma 5. *Let M and N be compact Hausdorff spaces and $\pi : C(X \times M) \longrightarrow C(Y \times N)$ be a Riesz isomorphism with $\pi(\mathbf{1}_{X\times M}) = \mathbf{1}_{Y\times N}$. If*

$$f(\{x\} \times M) \neq 0 \quad \text{for each } x \in X \Longleftrightarrow \pi f(\{y\} \times N) \neq 0 \text{ for each } y \in Y,$$

then X and M are homeomorphic to Y and N, respectively.

Proof. From the Banach-Stone theorem there exists a homeomorphism $\sigma : Y \times N \longrightarrow X \times M$ such that

$$\pi(f) = f\sigma \quad \text{and} \quad \pi^{-1}(g) = g\sigma^{-1}.$$

Let P_X be the projection of $X \times M$ onto X; that is, $P_X : X \times M \longrightarrow X$ is defined by $P_X(x,m) = x$.

Claim a: For each $y_0 \in Y$, $P_X\sigma((Y - \{y_0\}) \times N) \neq X$. Indeed, suppose it does not hold. Note that $P_X\sigma : Y \times N \longrightarrow X$ is a surjective open map and $(Y - \{y_0\}) \times N$ is open in $Y \times N$. Since $X \subset P_X\sigma((Y - \{y_0\}) \times N)$ there exists a closed set F in $Y \times N$ such that

$$F \subset Y - \{y_0\} \times N \quad \text{and} \quad X \subset P_X\sigma(F).$$

Let $A = \sigma(F)$ and $B = \sigma(\{y_0\} \times N)$. Clearly A and B are closed in $Y \times N$ and $A \cap B = \emptyset$. Then there exists $f \in C(Y \times N)$ such that

$$f(A) = \{1\} \quad \text{and} \quad f(B) = \{0\}.$$

Let $x \in X$ be given. Since $X = P_X\sigma(F)$, there exists $m \in M$, $(y,n) \in F$ such that

$$x = P_X(x,m) = P_X\sigma(y,n).$$

Then $\sigma(y,n) \in A$ and so $1 = f(\sigma(y,n)) = f(x,m)$. From the hypothesis, $\pi f(\{y\} \times N) \neq \{0\}$ for each $y \in Y$, in particular,

$$\{0\} = f(B) = \pi f(\{y_0\} \times N) \neq \{0\}.$$

This is a contradiction.

Claim b: For each $n \in N$, $P_X\sigma(Y \times \{n\}) = X$. To see this, suppose that $x_0 \notin P_X\sigma(Y \times \{n\})$ for some $x_0 \in X$ and $n \in N$. Since $\{x_0\} \times M$ and $\sigma(Y \times \{n\})$ are nonempty disjoint closed sets, there exists $f \in C(X \times M)$ such that

$$f\sigma(Y \times \{n\}) = \{1\} \quad \text{and} \quad f(\{x_0\} \times N) = \{0\}.$$

That implies that $\pi f(Y \times \{n_0\}) = \{1\}$, so $f(\{x_0\} \times N) \neq \{0\}$. This contradiction shows that Claim b is also true.

Claim c: For each $y \in Y$ there exists a unique $x \in X$ such that $\sigma^{-1}(\{y\} \times N) = \{x\} \times M$. Let $y \in Y$ be given. Since for each $n \in N$,

$$P_X\sigma((Y - \{y\}) \times N) \neq P_X\sigma(Y \times \{n\}) = X,$$

there exists $x \in X$ such that $(x, m) \notin \sigma(Y - \{y\} \times N)$ for each $m \in M$, so $\{x\} \times M \subset \sigma(\{y\} \times N)$. Hence $\sigma^{-1}(\{x\} \times M) \subset \{y\} \times N$. Similarly there exists $y_1 \in Y$ such that $\sigma(\{y_1\} \times N) \subset \{x\} \times M$. From

$$\{y_1\} \times N \subset \sigma^{-1}(\{x\} \times M) \subset \{y\} \times N$$

we have $y_1 = y$. Now, it is clear that $\sigma^{-1}(\{x\} \times M) = \{y\} \times N$ and y must be unique. Now Lemma 3 is applied to complete the proof. □

Now we are ready to give the proof of the theorem.

Proof of Theorem 2. From the Kakutani Representation Theorem there exists a compact Hausdorf space N such that F is isometrically Riesz isomorphic to $C(N)$. Since for each $x \in X$, $\pi^{-1}(\mathbf{1}_Y \otimes e)(x)$ is a strong order unit for E ($\mathbf{1}_Y \otimes e$ is defined by $\mathbf{1}_Y \otimes e(y) = e$), E is (norm) isomorphic and Riesz isomorphic to $C(M)$ for some compact Hausdorff space. Since for compact Hausdorff spaces Y and Z, the Banach lattices $C(Y, C(Z))$ and $C(Y \times Z)$ are Riesz isomorphic spaces, we have that $C(X \times M)$ and $C(Y \times N)$ are Riesz isomorphic spaces. Moreover, it is clear that from the hypotheses of the theorem there exists a Riesz isomorphism $\pi_1 : C(X \times M) \longrightarrow C(Y \times N)$ such that $\pi_1(\mathbf{1}_{X\times M}) = \mathbf{1}_{Y\times N}$ and

$$f(\{x\} \times M) \neq 0 \quad \text{for each } x \in X \Longleftrightarrow \pi_1 f(\{y\} \times N) \neq 0 \quad \text{for each } y \in Y.$$

Hence from Lemma 5, X and M are homeomorphic to Y and N respectively, so E is (norm) isomorphic and Riesz isomorphic to F. □

Let X and Y be compact Haudorff spaces, and E and F be Banach lattices. Let $\pi : C(X, E) \to C(Y, F)$ be a Riesz isomorphism such that $0 \notin f(X)$ if and only if $0 \notin \pi(f)(Y)$. We conjecture that X and Y are homeomorphic and E and F are Riesz isomorphic.

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