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HEREDITARY AND MAXIMAL CROSSED PRODUCT ORDERS

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Abstract. We first deal with classical crossed products $S^f * G$, where G is a finite group acting on a Dedekind domain S and S^G (the G-invariant elements in S) a DVR, admitting a separable residue fields extension. Here $f : G \times G \to S^*$ is a 2-cocycle. We prove that $S^f * G$ is hereditary if and only if $S/\mathrm{Jac}(S)^{\bar{f}} * G$ is semi-simple. In particular, the heredity property may hold even when S/S^G is not tamely ramified (contradicting standard textbook references). For an arbitrary Krull domain S, we use the above to prove that under the same separability assumption, $S^f * G$ is a maximal order if and only if its height one prime ideals are extended from S. We generalize these results by dropping the residual separability assumptions. An application to non-commutative unique factorization rings is also presented.

1. Introduction

Let S be a commutative Krull domain and let G be a finite subgroup of $\mathrm{Aut}(S)$. Then any 2-cocycle $f : G \times G \to S^*$ (where S^* is the group of S-units) gives rise to the crossed product algebra $T := S^f * G = \bigoplus_{g \in G} SU_g$, whose multiplication is defined by

$$sU_g tU_h = sg(t)f(g,h)U_{gh} \quad s,t \in S, \quad g,h \in G. \tag{1.1}$$

Since the action of G on S is faithful, T is a prime ring. Its center is $R := S^G$, the subring of G-invariant elements in S. The crossed product T is an R-order in the central simple algebra $L^f * G$, where L is the field of quotients of S and f is considered as an element in $Z^2(G, L^*)$.

Our main concern here is with the characterization of crossed products T which are hereditary or maximal R-orders. These issues were dealt with by Auslander-Goldman [3], Auslander-Rim [4], Harada [13] and Williamson [24] in the 1960's and many other authors since.

In order to investigate the heredity property of T, we first let S be a Dedekind domain. Let $S_p := (R \setminus p)^{-1}S$, where p is a non-zero prime ideal of R. Then the action of G on S and the 2-cocycle f give rise to an action of G on S_p and to a 2-cocycle with coefficients in S_p^* and so to a crossed product $S_p^f * G$, whose center R_p is a discrete valuation ring (DVR). Moreover, T is hereditary if and only if so is $S_p^f * G$ for every non-zero prime p in R. Thus, we assume here for the sake of simplicity that $R = S^G$ is a DVR.

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It was shown by Rosen [22, Theorem 40.13] (see also [4, P. 578]) that for $f = 1$, the skew group ring $S * G$ is hereditary if and only if S/R is tamely ramified. However, such a result does not hold for arbitrary f, as Example 4.1 shows, even if $S^f * G$ is assumed to be a maximal order.

One of our goals here is to provide the missing characterization in the general case, namely, when S/R is wildly ramified. In doing so, we correct an acute mistake which has crept into standard textbooks [22, Theorem 40.15] and [7, Theorem 28.12] (and possibly into research papers).

Let $\mathrm{Jac}(S)$ be the Jacobson radical of S and let p be the unique maximal ideal of R. Let $\bar{S} := S/\mathrm{Jac}(S)$ and $\bar{R} := R/p$. Denote the projection of f on $\bar{S}$ by $\bar{f}$. Our first result is

Theorem A. *Let S be a Dedekind domain and let G be a finite subgroup of $Aut(S)$ such that R is a DVR. Assume further that the extension $\bar{S}/\bar{R}$ is separable. Then the following are equivalent:*

(1) *$T = S^f * G$ is a hereditary order.*
(2) *$\bar{S}^{\bar{f}} * G$ is a semi-simple algebra, equivalently $Jac(T) = Jac(S)T$.*

Remarks. (i) A precise criterion for semi-simplicity of crossed products over fields can be found in [2, Theorem 2].

(ii) A special case of Theorem A is verified in [13, Theorem 2], assuming in addition that $\bar{R}$ is a perfect field. In this case the semi-simplicity of $\bar{S}^{\bar{f}} * G$ holds if and only if S/R is tamely ramified (see Corollary 2.5).

The separability assumption on $\bar{S}/\bar{R}$ can be omitted in the implication (2)⇒(1) of Theorem A as the proof shows. However, Example 4.2 shows that without this assumption, (1) does not necessarily yield (2). The following condition, which is weaker than (2), is necessary and sufficient for the heredity property to hold in $S^f * G$ when $\bar{S}/\bar{R}$ is an arbitrary extension.

Theorem B. *Let S be a Dedekind domain and let G be a finite subgroup of $Aut(S)$ such that R is a DVR. Then the following are equivalent:*

(1) *$T = S^f * G$ is a hereditary order.*
(2) *There exists $m \geq 1$ such that $Jac(T)^m = Jac(S)T$.*

Note that if $\bar{S}/\bar{R}$ is separable, then $m = 1$ by Theorem A.

Our next result considers the question of when $S^f * G$ is a maximal R-order for S of possibly higher Krull dimension. The ordinary ideal powers in Theorem B are replaced here by symbolic powers. Let P be a prime two sided ideal of T. Recall [6, P. 125] that the m-th symbolic power of P is $P^{(m)} = P^m_{P\cap R} \cap T$.

Theorem C. *Let S be a commutative Krull domain and let G be a finite subgroup of $Aut(S)$. Then the following are equivalent:*

(1) *$T := S^f * G$ is a maximal order.*
(2) *For every height one prime ideal P in T there exists $m \geq 1$ (which depends on P) such that $P^{(m)} = (P \cap S)T$.*

Furthermore, if in addition $\overline{S_{P\cap R}}/\overline{R_{P\cap R}}$ is separable, then $P = (P \cap S)T$, that is, $m = 1$ in (2).

Remark. When $f = 1$, or more generally, when it is cohomologically trivial, Theorem C is implicitly proven in [18, Theorem 4.6 and Proposition 4.7]. Moreover, it is shown there that the corresponding m in (2) is equal to 1 without the separability assumption. If this assumption is dropped, then for arbitrary f, a maximal crossed product order T may admit a height one prime ideal P which is not extended from S as shown in Examples 4.2 and 4.3. Moreover, Example 4.3 shows that without the separability condition, the symbolic power in Theorem C cannot be replaced by *any* ordinary power (as in Theorem B). It is of interest to analyze why any height one prime of $S * G$ is extended from S independently of the separability condition. Indeed, if the skew group ring $S * G$ is a maximal order, then so is $S_p * G$ for every height one prime p in R. Consequently, $S_p * G \simeq \mathrm{End}_{R_p}(S_p)$, implying by [8, Proposition 1.2, Chapter III] that S_p/R_p is in particular separable. Therefore, by [8, Proposition 1.11, Chapter II] $\bar{S}_p/\bar{R}_p$ is a separable extension. To wit, the separability condition is seen to be a consequence of the maximal order property. This does not necessarily hold for arbitrary f.

The following application is new for arbitrary $f \in Z^2(G, S^*)$; the cohomologically trivial case was noted in [6, Proposition 29].

Corollary D. *Let $T := S^f * G$ be a maximal order, where S is a commutative unique factorization domain (UFD) and G a finite subgroup of $Aut(S)$. Let P be a height one prime ideal in T such that the extension $\overline{S_{P\cap R}}/\overline{R_{P\cap R}}$ is separable. Then P is generated as a one sided ideal by an element of S. In particular, if for every prime ideal $p \subset R$ of height one, the extension $\bar{S}_p/\bar{R}_p$ is separable, then T is a non-commutative unique factorization ring.*

Remark. The separability condition also cannot be dropped in Corollary D. Indeed, Example 4.3 exhibits a maximal crossed product order over a UFD which admits a height one prime ideal that is not principal.

Throughout the paper, we continue to denote by f its restriction to subgroups of G, and by $\bar{f}$ its projection on quotients of S^*. For the definition of Krull domain see [15, P. 82]. Note that any normal Noetherian commutative domain is in particular a Krull domain.

2. Hereditary orders

In this section we prove Theorems A and B. We first deal with the implications (2)⇒(1) in both theorems. The following is a useful well known sufficient condition for the heredity property of $S^f * G$.

Lemma 2.1. *With the above notation, suppose that the Jacobson radical of $S^f * G$ is a projective one sided ideal. Then $S^f * G$ is hereditary.*

Proof. The projectivity assumption on $\mathrm{Jac}(S^f * G)$ implies that the projective dimension of $S^f * G/\mathrm{Jac}(S^f * G)$ over $S^f * G$ is at most 1. In order to prove that $S^f * G$ is hereditary, we need to show that the projective dimension of any $S^f * G$-module does not exceed 1. By [21, Theorem 8], it is enough to show it for simple $S^f * G$-modules. Indeed, let V be an arbitrary simple $S^f * G$-module. Then clearly V is an image of $S^f * G/\mathrm{Jac}(S^f * G)$. Since the algebra $S^f * G/\mathrm{Jac}(S^f * G)$ is semi-simple, it follows that V is a direct summand of $S^f * G/\mathrm{Jac}(S^f * G)$ and thus its projective dimension is at most 1. This shows that the global dimension of $S^f * G$ is lesser or equal to 1, hence $S^f * G$ is hereditary. □

Corollary 2.2. *Let S be a Dedekind domain such that R is a DVR. Suppose that $Jac(T)^m = Jac(S)T$ for some m. Then $S^f * G$ is hereditary.*

Proof. Since S is a Dedekind domain, $\mathrm{Jac}(S)$ is invertible in S and hence $\mathrm{Jac}(S)T$ is invertible in T. Since $\mathrm{Jac}(T)^m = \mathrm{Jac}(S)T$, we obtain that $\mathrm{Jac}(T)$ is invertible and in particular projective. By Lemma 2.1, $S^f * G$ is hereditary. □

Corollary 2.2 proves the implications (2)⇒(1) in Theorems A (putting $m = 1$) and B.

We now turn to the direction (1)⇒(2). We first prove this direction in Theorem A. Suppose then that the extension $\bar{S}/\bar{R}$ is separable. The strategy is to prove (1)⇒(2) under the assumption that S is a DVR (Proposition 2.3) and then to drop this assumption (Proposition 2.4). We have

Proposition 2.3. *Let S be a DVR, G a finite subgroup of $Aut(S)$ and $R = S^G$. Assume further that the extension $\bar{S}/\bar{R}$ of the corresponding residue fields is separable. Let $T = S^f * G$ be an hereditary crossed product. Then $T/Jac(S)T = \bar{S}^{\bar{f}} * G$ is a semi-simple (finite dimensional) algebra.*

Proof. We first note that by [2, Theorem B], $\bar{S}^{\bar{f}} * G$ is semi-simple if and only if so is $\bar{S}^{\bar{f}} * G_I$, where G_I is the kernel of the action of G on $\bar{S}$ (the inertia subgroup). Moreover, if $\mathrm{char}(\bar{S}) = p$, then in order to prove that $\bar{S}^{\bar{f}} * G_I$ is semi-simple, it is sufficient to show that the sub-crossed products $\bar{S}^{\bar{f}} * E$ are semi-simple, where E runs over all elementary abelian p-subgroups of G_I [1, Theorem 3].

Let E be any elementary abelian p-subgroup of the inertia group G_I. Note that S^E, the subring of E-invariant elements in S, is still a DVR.

We claim that the sub-crossed product $S^f * E$ is hereditary as well as local. It is hereditary by the monotonicity of the global dimension of sub-crossed products with respect to subgroups. More precisely, since $S^f * G$ is a free $S^f * E$-module then any projective resolution of an $S^f * G$-module V remains projective over $S^f * E$. Thus, the projective dimension of V over $S^f * E$ cannot exceed its projective dimension over $S^f * G$, and the global dimension of $S^f * E$ cannot exceed the global dimension of $S^f * G$. In our case, since the global dimension of $S^f * G$ is 1, then so is the global dimension of $S^f * E$.

As for locality, since $\mathrm{Jac}(S)S^f * E$ is contained in $\mathrm{Jac}(S^f * E)$, it suffices to show that $\bar{S}^{\bar{f}} * E = S^f * E/\mathrm{Jac}(S)(S^f * E)$ is local. Since $E \subset G_I$, it acts trivially on $\bar{S}$. By, e.g., [16, Lemma 2.3.4], the twisted group ring $\bar{S}^{\bar{f}} * E$ and hence $S^f * E$ are local.

The heredity and locality of $S^f * E$, just verified, imply that it is a maximal order [3, Theorem 2.3]. Now, by [22, Theorem 18.7], the Jacobson radical of $S^f * E$ is principally generated. In order to prove that $\bar{S}^{\bar{f}} * E = S^f * E/\mathrm{Jac}(S)(S^f * E)$ is semi-simple, it is clearly enough to show that $\mathrm{Jac}(S^f * E) = \mathrm{Jac}(S)(S^f * E)$. Let $y \in S^f * E$ generate the Jacobson radical. That is,

$$\mathrm{Jac}(S^f * E) = y(S^f * E). \tag{2.1}$$

By [17] (putting $\mathrm{PIdeg}(S^f * E) = |E|$),

$$y^{|E|}(S^f * E) = \mathrm{Norm}(y)(S^f * E) \subset \mathrm{Jac}(S^E)(S^f * E). \tag{2.2}$$

By [22, Theorem 18.3], any ideal of $S^f * E$ is generated by a power of y. Let $n \geq 1$ be such that

$$\mathrm{Jac}(S)(S^f * E) = y^n(S^f * E). \tag{2.3}$$

By (2.2) and (2.3), we obtain that

$$\mathrm{Jac}(S)^{|E|}(S^f * E) = y^{n|E|}(S^f * E) \subset \mathrm{Jac}(S^E)^n(S^f * E). \tag{2.4}$$

Intersecting with S we obtain

$$\mathrm{Jac}(S)^{|E|} \subset \mathrm{Jac}(S^E)^n S. \tag{2.5}$$

We now make use of the separability of $\bar{S}$ over $\overline{S^G}$ and in particular over $\overline{S^E}$. This hypothesis implies that the ramification index of S over S^E is equal to the order of E, since E lies in the inertia subgroup G_I of G. By [23, P. 22, Corollary],

$$\mathrm{Jac}(S)^{|E|} = \mathrm{Jac}(S^E)S. \tag{2.6}$$

Combining equations (2.5) and (2.6), we conclude that $n = 1$ and thus $\mathrm{Jac}(S)$ generates the Jacobson radical of $S^f * E$ proving that $\bar{S}^f * E$ is semi-simple. □

The following proposition completes the proof of Theorem A. It relaxes the DVR assumption on S made in Proposition 2.3.

Proposition 2.4. *Let S be a Dedekind domain and $R = S^G$ a DVR such that $\bar{S} = S/Jac(S)$ is separable over $\bar{R}$. Let $T = S^f * G$ be a hereditary R-order. Then $\bar{S}^{\bar{f}} * G$ is semi-simple.*

Proof. We reduce the hypothesis of the proposition to the conditions of Proposition 2.3, namely where S is itself a DVR. See [24, section 2] for a similar completion argument.

Let p be the unique maximal ideal of R and let $\{q_1, ..., q_k\}$ be the set of all maximal ideals in S. Let $\widehat{S} := \varprojlim_i S/p^iS$. Then $\widehat{S} = S_1 \oplus ... \oplus S_k$, where $S_j = \varprojlim_i S/q_j^i$ is a DVR for each $j = 1, ..., k$. We denote by $e_1, ..., e_k$ the primitive idempotents of $\widehat{S}$, which correspond to the above decomposition, i.e. $\widehat{S}e_j = S_j$. Next, let $\widehat{T} := \varprojlim_i T/p^iT$. Then $\widehat{T} \simeq \widehat{S}^f * G$, where the action and 2-cocycle in the crossed product of G over $\widehat{S}$ are induced from those in T. The heredity property of T implies that $\widehat{T}$ is also hereditary [6, Proposition 31].

For every $j = 1, ..., k$, let $G_j := \{g \in G | g(q_j) = q_j\}$ be the decomposition group corresponding to the maximal ideal q_j. Acting transitively on $\{q_1, ..., q_k\}$, the group G permutes the set $\{e_1, ..., e_k\}$. Further, for each $j = 1, ..., k$, $G_j \subset \mathrm{Aut}(\widehat{S})$ and $\frac{|G|}{|G_j|} = k$. Now, for each $j = 1, ..., k$, $k|G_j| = |G| = \mathrm{rank}_R S = \mathrm{rank}_{\widehat{R}} \widehat{S} = k \cdot \mathrm{rank}_{\widehat{R}} S_j$. Consequently, $|G_j| = \mathrm{rank}_{\widehat{R}} S_j$ and hence $S_j^{G_j} = \widehat{R}, \ \ j = 1, ..., k$. For any $j = 1, ..., k$, let $T_j := S_j^f * G_j \subset \widehat{T}$. Clearly, $T_j = e_j\widehat{T}e_j$ and consequently, by [20, Proposition 5.4.4], T_j is hereditary. Since S_j is a DVR, we can apply Proposition 2.3.

$$\mathrm{Jac}(T_j) = \mathrm{Jac}(S_j)T_j \tag{2.7}$$

and since $\mathrm{Jac}(\widehat{S}) = \bigoplus_{j=1}^k \mathrm{Jac}(S_j)$, we obtain $\mathrm{Jac}(S_j)T_j \subset \mathrm{Jac}(\widehat{S})\widehat{T}$. We claim that $\mathrm{Jac}(\widehat{T}) = \mathrm{Jac}(\widehat{S})\widehat{T}$. Clearly, $\mathrm{Jac}(\widehat{S})\widehat{T} \subset \mathrm{Jac}(\widehat{T})$. We now show that $\mathrm{Jac}(\widehat{T})$ is contained in $\mathrm{Jac}(\widehat{S})\widehat{T}$. Let $x = \sum_{\tau \in G} s_\tau U_\tau \in \mathrm{Jac}(\widehat{T})$. We shall show that for every $\tau \in G$, $s_\tau \in \mathrm{Jac}(\widehat{S})$. If x is non-zero, then we may assume that s_1 is non-zero by multiplying x with an appropriate basis element U_σ. Furthermore, it suffices to prove that $s_1 \in \mathrm{Jac}(\widehat{S})$, again using multiplication of x with such elements. Recall that for any idempotent $e \in \widehat{S}$ we have $\mathrm{Jac}(e\widehat{T}e) = e\mathrm{Jac}(\widehat{T})e = e\widehat{T}e \cap \mathrm{Jac}(\widehat{T})$. Hence,

$$T_j \cap \mathrm{Jac}(\widehat{T}) \subset \mathrm{Jac}(T_j). \tag{2.8}$$

By (2.8) $e_j x e_j = e_j \sum_{\tau \in G} s_\tau U_\tau e_j \in \mathrm{Jac}(T_j)$, and by (2.7), we have that $e_j s_1 \in \mathrm{Jac}(S_j)$ for each $j = 1, ..., k$. Consequently, $s_1 = (\sum_{j=1}^k e_j)s_1 \in \mathrm{Jac}(\widehat{S})$ and hence for every $\tau \in G$, $s_\tau \in \mathrm{Jac}(\widehat{S})$. Thus, $x = \sum_{\tau \in G} s_\tau U_\tau \in \mathrm{Jac}(\widehat{S})\widehat{T}$. We obtain

$$\mathrm{Jac}(\widehat{T}) = \mathrm{Jac}(\widehat{S})\widehat{T}. \tag{2.9}$$

Finally, since the left hand side of (2.9) equals $\widehat{\mathrm{Jac}(T)}$ whereas the right hand side of (2.9) equals $\widehat{\mathrm{Jac}(S)}T$, we have $\widehat{\mathrm{Jac}(T)} = \widehat{\mathrm{Jac}(S)}T$. Since the completion here is faithfully flat [19, Theorem 8.14], we get $\mathrm{Jac}(T) = \mathrm{Jac}(S)T$ as needed. $\square$

Corollary 2.5 ([13, Theorem 2]). *With the notation of Theorem A. Suppose that $\bar{R}$ is a perfect field. Then $S^f * G$ is hereditary if and only if S/R is tamely ramified.*

Proof. By [2, Theorem 2], the additional condition on the field $\bar{R}$ implies that $\bar{S}^{\bar{f}} * G$ is semi-simple if and only if the kernel of the action of G on $\bar{S}$ is a p'-group. This condition is equivalent to tameness of the extension S/S^G (see e.g. [23, P.22, Corollary]). $\square$

We now prove the implication (1)⇒(2) in Theorem B. Recall that there is no restriction here on the extension $\bar{S}/\bar{R}$. Throughout the proof we make use of the fact that left invertibility is the same as right invertibility [5, Theorem 1.7]. Denote the maximal two sided ideals of T by $M_1, ..., M_n$. By [9, Theorem 4.13], $J := \mathrm{Jac}(T) = \bigcap_{i=1}^n M_i$ is invertible. For every $j \geq 0$, let $x_j := J^{-j}(\mathrm{Jac}(S)T)$. The proof will be completed if we show that there exists m such that $x_m = T$. If this is not true, then we first claim by induction on $j \geq 0$ that $x_j \subset J$. This is true for $j = 0$, since $\mathrm{Jac}(S)T \subset J$. Suppose that $x_{j-1} \subset J$. Consequently, $x_j = J^{-1}x_{j-1} \subset J^{-1}J = T$. Now, since $\mathrm{Jac}(S)$ is invertible in S, then $\mathrm{Jac}(S)T$ is invertible in T. It follows that x_j is invertible (with inverse $(\mathrm{Jac}(S)T)^{-1}J^j$). By the assumption, $x_j \neq T$. Then by [9, Proposition 2.4], x_j is contained in a sub-cycle $C_j \subset \{M_1, ..., M_n\}$. Let $y_j := \bigcap\{M_i \in C_j\}$. By [9, Proposition 2.5], y_j is invertible and hence localizable. Now, $R = S^G$, the center of T, is local by the hypothesis. By a result of Müller [12, Theorem 11.20], all the maximal two sided ideals of T consist of a single clique. Consequently, any intersection of a proper subset of $\{M_1, ..., M_n\}$ cannot be localizable and hence $y_j = \bigcap_{i=1}^n M_i = J$. This implies that $x_j \subset J$. Next, since $J^{-1} \supset T$, then $\{x_j\}$ is an ascending sequence of ideals in T. By the Noetherian property of T, $x_j = x_{j+1} = J^{-1}x_j$ for some j, contradicting Nakayama's lemma. Therefore, $x_m = T$ for some m. Consequently, $J^m = \mathrm{Jac}(S)T$ as required.

3. Maximal orders with an arbitrary Krull dimension

In this section we prove Theorem C and Corollary D. We first deal with the implication (2)⇒(1) in Theorem C.

Proposition 3.1. *Let $T = S^f * G$, where S is a commutative Krull domain, G a finite subgroup of $Aut(S)$ and $f \in Z^2(G, S^*)$. Suppose that for every height one prime ideal P in T there exists $m \geq 1$ such that $P^{(m)} = (P \bigcap S)T$. Then T is a maximal order.*

Proof. It is well known [10, Proposition 1.2] that $R := S^G$ is a Krull domain and hence $R = \bigcap_{p\in\mathcal{I}} R_p$, where $\mathcal{I} = \{p \in \mathrm{Spec}(R) | \ \ \mathrm{height}(p) = 1\}$. Moreover, R_p is a DVR for every $p \in \mathcal{I}$. Let q be a height one prime ideal in S. Then by the "Going-Down" property between S and R, we get that height$(q \cap R) = 1$. Consequently, $S = \bigcap_{p\in\mathcal{I}} S_p$. Now, since T is a free S-module generated by $\{U_\sigma\}_{\sigma\in G}$, it follows that $\bigcap_{p\in\mathcal{I}} T_p = \bigcap_{p\in\mathcal{I}} S_p^f * G = \bigcap_{p\in\mathcal{I}}(\bigoplus_{\sigma\in G} S_pU_\sigma) = \bigoplus_{\sigma\in G}(\bigcap_{p\in\mathcal{I}} S_p)U_\sigma = \bigoplus_{\sigma\in G} SU_\sigma = T$. Therefore, in order to show that T is a maximal R-order, it suffices to show that T_p is a maximal R_p-order for every $p \in \mathcal{I}$ [11, Proposition 1.3]. Indeed, the hypothesis implies that for every height one prime P of T there is an m such that $P_p^m = (P \cap S)_pT_p$. Since $(P \cap S)_p$ is invertible, then so is $(P \cap S)_pT_p$ and hence so is P_p. This readily shows that every two sided ideal in T_p is invertible. Consequently, by [20, Proposition 5.2.6], we obtain that T_p is a maximal order for every $p \in \mathcal{I}$. □

We proceed with the direction (1)⇒(2) of Theorem C.

Let $q := P \cap S$. Since q is G-prime, then $q = \bigcap_{\sigma\in G} \sigma(q')$ for some height one prime $q' \lhd S$. Let $p := P \cap R = q' \cap R$. Consider the localization $T_p := S_p^f * G$. Since T is a maximal order, then T_p is a hereditary R_p-order. Note that R_p is a DVR, $\mathrm{Jac}(S_p) = q_p$ and $\mathrm{Jac}(T_p) = P_p$. We can apply Theorem B to deduce that there exists $m \geq 1$ such that $P_p^m = q_pT_p$. Furthermore, if in addition the extension $\bar{S}_p/\bar{R}_p$ is separable, then by Theorem A, $m = 1$. Now, $q_pT_p = \sum_{\sigma\in G} q_pU_\sigma$. Hence, $q_pT_p \cap T = (\sum_{\sigma\in G} q_pU_\sigma) \cap (\sum_{\sigma\in G} SU_\sigma) = \sum_{\sigma\in G}(q_p \cap S)U_\sigma = \sum_{\sigma\in G} qU_\sigma = qT$, where $q_p \cap S = q$ follows from the semi-primeness of q. Thus, $P^{(m)} = P_p^m \cap T = q_pT_p \cap T = qT$ as required.

Proof of Corollary D. Let P be a height one prime ideal in T and let $q := P \cap S$. We show that q is a reflexive ideal in S. Indeed, q is G-stable as an ideal in S. Furthermore, it is a G-prime ideal. Hence, $q = \bigcap_{\sigma\in G} \sigma(q')$, for some prime ideal $q' \subset S$. Now, since by "Going Down", $P \cap R$ is a height one prime in R, the same holds for $\sigma(q')$ for every $\sigma \in G$. This shows that q is a reflexive ideal. Therefore, by the factorial property of S, $q = aS$ for some a in S. Hence, by Theorem C, $P = qT = aT$, as needed. □

4. Examples

4.1. **Example.** The following example shows that a crossed product $S^f * G$ may be hereditary even when S/R is not tamely ramified (compare with [22, Theorem 40.15], and [7, Theorem 28.12]).

It is well known that $\mathbb{Z}_{(2)}[\sqrt{2}]$ is the integral closure of $\mathbb{Z}_{(2)}$ in $\mathbb{Q}(\sqrt{2})$ and is endowed with the $G := \langle 1, \sigma\rangle$ action: $\sigma(\sqrt{2}) = -\sqrt{2}$. Consider the corresponding polynomial rings $\mathbb{Z}_{(2)}[x] \subset \mathbb{Z}_{(2)}[\sqrt{2}][x]$, where σ acts trivially on x, and let $p_0 := 2\mathbb{Z}_{(2)}[x]$ and $q_0 := \sqrt{2}\mathbb{Z}_{(2)}[\sqrt{2}][x]$. Then

$$\mathbb{Z}_{(2)}[x]/p_0 \simeq \mathbb{F}_2[x], \quad \mathbb{Z}_{(2)}[\sqrt{2}][x]/q_0 \simeq \mathbb{F}_2[x]. \tag{4.1}$$

Furthermore, $p_0\mathbb{Z}_{(2)}[\sqrt{2}][x] = q_0^2$. Denote the localization $S := \mathbb{Z}_{(2)}[\sqrt{2}][x]_{p_0}$. Clearly, $S = \mathbb{Z}_{(2)}[\sqrt{2}][x]_{q_0}$. Then S is a DVR with a maximal ideal $q := \sqrt{2}S$.

Let $R := S^G = \mathbb{Z}_{(2)}[x]_{p_0}$. Then R is also a DVR with maximal ideal $p := 2R$. Note that $S/q = R/p = \mathbb{F}_2(x)$, in particular the extension of the residue fields is separable.

Consider the crossed product $T := S^f * G = SU_1 \oplus SU_\sigma$, where f is given via $f(\sigma,\sigma) = x, f(1,\sigma) = f(\sigma,1) = f(1,1) = 1$. So $U_\sigma^2 = x$.

Claim 4.1. (1) The extension S/R is not tamely ramified.
(2) T is a hereditary R-order, moreover, it is a maximal R-order.

Proof. (1) Since $pS = q^2$, the ramification index of S over R is 2. The index is not invertible in the residue field $R/p \simeq \mathbb{F}_2(x)$ and hence the extension is not tamely ramified.

(2) Observe that $q := (q_0)_{p_0}$. Then $T/qT = (S/q)^{\bar{f}} * G \simeq \mathbb{F}_2(x)^{\bar{f}} * G$. Since G acts trivially on $\mathbb{F}_2(x)$ (the extension is totally ramified), then $\mathbb{F}_2(x)^{\bar{f}} * G$ is the field $\mathbb{F}_2(\sqrt{x})$ (by identifying U_σ with $\sqrt{x}$) and in particular semisimple. By the implication (2)⇒(1) of Theorem A, T is hereditary. Moreover, since $\mathbb{F}_2(x)^{\bar{f}} * G$ is actually simple, then qT is the unique maximal ideal in T and is principally generated by $\sqrt{2}$. We therefore obtain that T is a maximal order. □

4.2. **Example.** This example shows the significance of the separability assumption in the implication (1)⇒(2) of Theorem A. Moreover, it shows that without separability assumption in Theorem C, a height one prime of a maximal order T is not necessarily extended from S.

Let $S := \mathbb{Z}_{(2)}[x]_{q_0}$, where $q_0 := 2\mathbb{Z}_{(2)}[x]$. Let $G := \langle 1, \sigma\rangle$ act on S by $\sigma(x) = -x$. Then $R = S^G = \mathbb{Z}_{(2)}[x^2]_{p_0}$, where $p_0 := 2\mathbb{Z}_{(2)}[x^2]$. Clearly, S and R are DVR's with unique maximal ideals $q := (q_0)_{q_0} = (q_0)_{p_0}$ and $p := (p_0)_{p_0}$ respectively. However, $\bar{S} = S/q = \mathbb{F}_2(x)$ is a non-separable extension of the field $\bar{R} = R/p = \mathbb{F}_2(x^2)$.

Consider the crossed product $T = S^f * G = SU_1 \oplus SU_\sigma$, where f is given via $f(\sigma,\sigma) = -1, f(1,\sigma) = f(\sigma,1) = f(1,1) = 1$. So $U_\sigma^2 = -1$.

Claim 4.2. (1) T is a local maximal R-order (and in particular hereditary).
(2) $\bar{S}^{\bar{f}}G$ is not semi-simple.
(3) $M \neq (M \cap S)T$, where M is the unique maximal ideal of T.

Proof. We show that the Jacobson radical of T is $M := (U_\sigma - 1)T$. Since $x(U_\sigma - 1) = (U_\sigma - 1)U_\sigma x$, then $U_\sigma - 1$ is a normal element. Consequently, M is a two sided ideal. Now, note that $2 = (U_\sigma - 1)^2 U_\sigma$ implies that $M^2 = 2T$. Therefore, $T/M \simeq \bar{S}$, so M is a maximal ideal. Note that $2T$ is properly contained in the Jacobson radical of T and hence $\bar{S}^{\bar{f}}G = T/2T$ is not semi-simple. Moreover, we obtain that $\mathrm{Jac}(T) = M = (U_\sigma - 1)T \supsetneq (M \cap S)T = 2T$. In particular, the Jacobson radical is projective from both sides. By Lemma 2.1, T is hereditary. □

4.3. **Example.** The following example shows that Theorem C is not necessarily true if the symbolic powers are replaced by ordinary powers (as in Theorem B). Furthermore, it also shows the need of the separability assumption in Corollary D by introducing a maximal crossed product order over a UFD admitting a height one prime ideal which is not principal.

Let $G = \langle 1, \sigma\rangle$ act on $\mathbb{Z}_{(2)}[x]$ by $\sigma(x) = -x$. Then $(\mathbb{Z}_{(2)}[x])^G = \mathbb{Z}_{(2)}[x^2]$. Let $w := 2\mathbb{Z}_{(2)}[x^2] + (x^2 - 1)\mathbb{Z}_{(2)}[x^2]$ be a maximal ideal in $\mathbb{Z}_{(2)}[x^2]$. Then G continues to act on $S := (\mathbb{Z}_{(2)}[x])_w$ with $R := S^G = (\mathbb{Z}_{(2)}[x^2])_w$. Consider the crossed product $T = S^f * G = SU_1 \oplus SU_\sigma$, where f is given via $f(\sigma,\sigma) = 2x^2 - 1, f(1,\sigma) = f(\sigma,1) = f(1,1) = 1$. Let $P := 2T + (U_\sigma - 1)T$. Clearly $p := P \cap R = 2R$ is a height one prime ideal in R. Then S is local with a maximal ideal $2S + (x-1)S$. Since S is also regular, it follows from the Auslander-Buchsbaum Theorem that it is a UFD. We shall verify the following properties.

Claim 4.3. (1) $T = S^f * G$ is a maximal order.
(2) P is a two sided height one prime ideal in T.
(3) $P^{(2)} = 2T = (P \cap S)T \neq P^2$.
(4) P is not projective as a T-module and hence not principal.

Proof. The fact that $x(U_\sigma - 1) = (U_\sigma - 1)(-x) - 2x \in P$ shows that P is a two sided ideal in T. Since $T/P \simeq (\mathbb{F}_2[x])_w$ is a domain, then P is a height one prime, proving (2). Recall that T satisfies the intersection property, that is $T = \bigcap T_v$, where v runs over all height one primes of R. So, (1) will hold once we show that T_v is a maximal order for any such v. We begin with the case $v = p$. Clearly $x^2 - 1 \notin p = 2R$. Hence, $x^2 - 1$ is invertible in R_p. Consequently, $U_\sigma^2 - 1 = 2(x^2 - 1)$ shows that $2 \in (U_\sigma - 1)T_p$ and therefore $P_p = (U_\sigma - 1)T_p$ is a principal, hence localizable, maximal ideal in T_p. By [12, Theorem 11.20] and [3, Theorem 2.3], T_p is a maximal order. Next, let $v \neq p$. Then clearly $2 \notin v$. Now, since $x^2 - 1 \in w$, then x^2 is invertible in R and hence in R_v. Consequently, $[x, U_\sigma]^2 = -4x^2U_\sigma^2 \notin v$, yielding that v does not contain all the squares of commutators in T. Thus by the Artin-Procesi Theorem [20, Theorem 13.7.14], T_v is an Azumaya algebra and in particular a maximal order.

Now, $P^2 = 4T + 2(U_\sigma - 1)T + (U_\sigma - 1)^2T = 4T + 2(U_\sigma - 1)T + 2(x^2 - 1)T$. In particular $P^2 \subset 2T$. If $2 \in P^2$, then $1 \in P'$ where $P' := 2T + (U_\sigma - 1)T + (x^2 - 1)T$. This cannot happen since P' is a proper ideal in T as observed from the isomorphism $T/P' \simeq \mathbb{F}_2[x]/\langle x^2 - 1\rangle$. Hence $2 \notin P^2$ proving the inequality in (3). Now, $P^{(2)} = P_p^2 \cap T \subset 2T_p \cap T = 2T$, where the last equality holds since $2S_p \cap S = 2S$ (as in the proof of Theorem C (1)⇒(2)). We apply Theorem C to the maximal order T and obtain that $2T$ either equals P or $P^{(2)}$. Now, by comparison of coefficients we clearly have $1 - U_\sigma \notin 2T$, and hence $P \neq 2T$. This confirms the equality in (3). Finally, (4) holds since by [6, Lemma 13], symbolic powers and ordinary powers of projective prime ideals are equal. □

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