

## CHEBOTAREV-TYPE THEOREMS IN HOMOLOGY CLASSES

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ABSTRACT. We describe how closed geodesics lying in a prescribed homology class on a negatively curved manifold split when lifted to a finite cover. This generalizes a result of Zelditch in the case of compact hyperbolic surfaces.

### 0. INTRODUCTION

Given a compact manifold of negative curvature, there are geometric analogues of the Chebotarev Theorem in algebraic number theory due to Sunada [13] (cf. also Parry and Pollicott [8] for the generalization to Axiom A flows). More precisely, given a finite Galois cover of the manifold, these theorems describe the proportion of closed geodesics which lift in a prescribed way to the cover.

In this geometric setting, it is also natural to consider infinite covers, and, in particular, the number of closed geodesics lying in a prescribed homology class has been studied by Katsuda and Sunada [4], Phillips and Sarnak [9], Katsuda [3], Lalley [7] and Pollicott [10] (with generalizations to Anosov flows by Katsuda and Sunada [5] and Sharp [12]). In this note we shall combine these points of view, generalizing a result of Zelditch for hyperbolic Riemann surfaces [14].

Let  $M$  be a compact smooth Riemannian manifold with negative curvature. Let  $\widetilde{M}$  be a finite Galois covering of  $M$  with covering group  $G$ . For a closed geodesic  $\gamma$  on  $M$ , let  $l(\gamma)$  denote its length,  $\langle \gamma \rangle$  its Frobenius class in  $G$  and  $[\gamma]$  its homology class in  $H = H_1(M, \mathbb{Z})$ .

We shall examine how the closed geodesics lying in a fixed homology class  $\alpha \in H$ , split when lifted to  $\widetilde{M}$ . More precisely, for a conjugacy class  $C$  in  $G$ , we study the asymptotics of

$$\pi(T, \alpha, C) = \text{Card}\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha, \langle \gamma \rangle = C\}.$$

The problem is complicated by the fact that, in general,  $[\gamma]$  and  $\langle \gamma \rangle$  are not independent quantities. This occurs if the abelian quotient group  $G/[G, G]$  is non-trivial, since this group is also a quotient of  $H$ , the maximal abelian covering group of  $M$ . Let  $\pi_G : G \rightarrow G/[G, G]$  and  $\pi_H : H \rightarrow G/[G, G]$  be the natural projections. In particular, the image  $\pi_G(C)$  of a conjugacy class  $C \subset G$  is a single element in  $G/[G, G]$  and if  $\pi_G(C) \neq \pi_H(\alpha)$ , then  $\pi(T, \alpha, C) = 0$ , for all values of  $T$ .

On the other hand, we have the following result, which extends work of Zelditch for Riemann surfaces [14].

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**Theorem 1.** *If  $\pi_G(C) \neq \pi_H(\alpha)$ , then  $\pi(T, \alpha, C)$  is identically zero. If  $\pi_G(C) = \pi_H(\alpha)$ , then*

$$\frac{\pi(T, \alpha, C)}{\pi(T, \alpha)} \rightarrow \left| \frac{G}{[G, G]} \right| \frac{|C|}{|G|} \text{ as } T \rightarrow +\infty,$$

where  $\pi(T, \alpha) = \text{Card}\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha\}$ .

*Example.* Let  $G$  be a finite nonabelian nilpotent group. Then  $[G, G] \neq G$ . For definiteness, we can let  $G = \{\pm 1, \pm i, \pm j, \pm k\}$  be the quaternion group of eight elements. Then  $[G, G] = \pm 1$ . Let  $\Gamma = \langle a_1, a_2, b_1, b_2 : a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1 \rangle$  be a cocompact Fuchsian group. Define a homomorphism  $\phi : \Gamma \rightarrow G$  by setting  $\phi(a_1) = i$ ,  $\phi(a_2) = j$  and  $\phi(b_1) = \phi(b_2) = 1$  and extending this to  $\Gamma$ . We can then define a normal subgroup by  $\Gamma_0 = \ker(\phi)$ . If we set  $M = \mathbb{H}^2/\Gamma_0$  and  $\widetilde{M} = \mathbb{H}^2/\Gamma$ , then  $\widetilde{M}$  is a finite cover of  $M$  with covering group  $G$ . □

Let us consider a closely related problem. Consider the frame flow  $f_t : \mathcal{FM} \rightarrow \mathcal{FM}$  on the space of orthonormal frames above  $M$ . This is an  $SO(n-1)$ -extension for the geodesic flow. Changing notation slightly, let  $\gamma$  be a periodic orbit of the geodesic flow, to which we associate a holonomy  $\Theta(\gamma) \in SO(n-1)$  which comes from a reference frame being transported around  $\gamma$ . This is defined up to conjugacy. In [8] it was shown that the holonomies were equidistributed on  $SO(n-1)$ . The following shows that the corresponding result holds for geodesics in a fixed homology class. (Recall that a class function is a function that is constant on conjugacy classes.)

**Theorem 2.** *Let  $F : SO(n-1) \rightarrow \mathbb{R}$  be a class function. Then*

$$\frac{1}{\pi(T, \alpha)} \sum_{\substack{l(\gamma) \leq T \\ \gamma = \alpha}} F(\Theta(\gamma)) \rightarrow \int F d\lambda, \text{ as } T \rightarrow +\infty,$$

where  $\lambda$  denotes the Haar measure on  $SO(n-1)$ .

### 1. PRELIMINARIES

Let  $M$  be a compact smooth manifold equipped with a Riemannian metric of negative curvature and let  $X$  denote its universal cover. (In the special case where  $M$  is a surface with constant curvature  $-1$ ,  $X$  is the hyperbolic plane  $\mathbb{H}$ .) Then there is a discrete group of isometries  $\Gamma \cong \pi_1(M)$  of  $X$  such that  $M = X/\Gamma$ . Now let  $\Gamma_0$  be a normal subgroup of  $\Gamma$  with finite index. Then  $\widetilde{M} = X/\Gamma_0$  is a finite (Galois) covering of  $M$ , with covering group  $G = \Gamma/\Gamma_0$  (i.e.,  $G$  acts transitively on the fibres above each point in  $M$ ).

There is a natural dynamical system, the geodesic flow, associated to  $M$ . Let  $SM$  denote the unit-tangent bundle of  $M$  and, for  $(x, v) \in SM$ , let  $\gamma : \mathbb{R} \rightarrow M$  be the unique unit-speed geodesic with  $\gamma(0) = x$  and  $\dot{\gamma}(0) = v$ . Then the geodesic flow  $\phi_t : SM \rightarrow SM$  is defined by  $\phi_t(x, v) = (\gamma(t), \dot{\gamma}(t))$ , and we shall write  $h$  for its topological entropy. There is a one-to-one correspondence between periodic  $\phi$ -orbits and directed closed geodesics on  $M$ . The fact that  $M$  is negatively curved ensures that the geodesic flow is an Anosov flow and that  $h > 0$ . This will enable us to use results proved in the context of Anosov flows in this setting.

We shall make use of  $L$ -functions defined with respect to certain representations of  $\Gamma = \pi_1(M)$ . Let  $\rho : \Gamma \rightarrow U(d)$  be a unitary representation of  $\Gamma$ . We define an

$L$ -function  $L(s, \rho)$  by the product formula

$$(1.1) \quad L(s, \rho) = \prod_{\gamma} \det(I - \rho(\langle\langle\gamma\rangle\rangle))e^{-sl(\gamma)}^{-1},$$

where the product is taken over all (prime) closed geodesics  $\gamma$  on  $M$  and  $\langle\langle\gamma\rangle\rangle$  denotes the homotopy class of  $\gamma$ . Provided  $\text{Re}(s)$  is sufficiently large, this will converge to an analytic function. We shall prove Theorem 1 by studying the behaviour of  $L(s, \rho)$  for a restricted class of representations.

We shall be interested in how closed geodesics on  $M$  lift to  $\widetilde{M}$ . There are a countable infinity of closed geodesics on  $M$ ; we shall denote a typical one by  $\gamma$  and its length by  $l(\gamma)$ . Each such  $\gamma$  has  $n = |G|$  lifts  $\gamma_1, \dots, \gamma_n$  to  $\widetilde{M}$ . These lifts are not necessarily closed but, for each  $i = 1, \dots, n$ , there is a covering transformation  $g_i \in G$  relating the endpoints of  $\gamma_i$  and, for  $i, j = 1, \dots, n$ ,  $g_i$  and  $g_j$  are conjugate. Hence we may associate to  $\gamma$  a well-defined conjugacy class  $\langle\gamma\rangle \subset G$ , called the Frobenius class of  $\gamma$ . These classes satisfy an analogue to Chebotarev's Theorem in number theory: for a conjugacy class  $C \subset G$ ,

$$(1.2) \quad \lim_{T \rightarrow +\infty} \frac{\#\{\gamma : l(\gamma) \leq T, \langle\gamma\rangle = C\}}{\#\{\gamma : l(\gamma) \leq T\}} = \frac{|C|}{|G|}.$$

The identity (1.2) is proved by considering  $L$ -functions

$$L(s, R_\chi) = \prod_{\gamma} \det(I - R_\chi(\langle\langle\gamma\rangle\rangle))e^{-sl(\gamma)}^{-1},$$

where  $R_\chi$  is an irreducible representation of  $G$  with character  $\chi$ . Since  $R_\chi$  lifts to a representation of  $\Gamma$ ,  $L(s, R_\chi)$  is a special case of the  $L$ -functions defined by (1.1). The geodesic flow on  $SM$  is also covered by the geodesic flow on  $S\widetilde{M}$ , with covering group  $G$ , and hence the analytic properties of  $L(s, R_\chi)$  may be deduced from the results in [8].

- Lemma 1.1.** (i) *Let  $\mathbf{1}$  denote the trivial one-dimensional representation of  $G$ . Then  $L(s, \mathbf{1})$  is analytic and nonzero on a neighbourhood of  $\{s : \text{Re}(s) \geq h\}$ , apart from a simple pole at  $s = h$ .*
- (ii) *If  $R_\chi \neq \mathbf{1}$  is an irreducible representation of  $G$ , then  $L(s, \mathbf{1})$  is analytic and nonzero on a neighbourhood of  $\{s : \text{Re}(s) \geq h\}$ .*

In this paper, we shall refine (1.2) by requiring that  $\gamma$  lies in a prescribed homology class in  $H = H_1(M, \mathbb{Z})$ . More precisely, for  $\alpha \in H$ , we shall write  $\pi(T, \alpha) = \#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha\}$  and  $\pi(T, \alpha, C) = \#\{\gamma : l(\gamma) \leq T, [\gamma] = \alpha, \langle\gamma\rangle = C\}$  and study the ratio

$$(1.3) \quad \frac{\pi(T, \alpha, C)}{\pi(T, \alpha)},$$

where  $[\gamma] \in H_1(M, \mathbb{Z})$  denotes the homology class of  $\gamma$ . Theorem 1 states that either  $\pi(T, \alpha, C)$  is identically zero or (1.3) has a limit as  $T \rightarrow +\infty$ . We shall prove this in the next section; however, to do so, we need to first recall how  $\pi(T, \alpha)$  behaves as  $T \rightarrow +\infty$ .

The asymptotics of  $\pi(T, \alpha)$  are also obtained by considering a family of  $L$ -functions, in this case indexed by the characters of  $H$ . We suppose that  $H$  is

infinite and, for simplicity, we consider  $H$  modulo torsion. Then these characters may be identified with the torus  $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ ,  $d \geq 1$ . For  $\theta \in \mathbb{T}^d$ , we write

$$L(s, \theta) = \prod_{\gamma} \left(1 - e^{-sl(\gamma)+2\pi i\theta \cdot [\gamma]}\right)^{-1}.$$

Since characters of  $H$  lift to  $\Gamma$ , this is again an  $L$ -function of the form defined in (1.1).

We also write

$$\eta_{\alpha}(s) = \int_{\mathbb{T}^d} e^{-2\pi i\theta \cdot \alpha} \frac{d^{\nu+1}}{ds^{\nu+1}}(\log L(s, \theta))d\theta,$$

where  $\nu = [d/2]$ . The following lemma is taken from [5] and [12].

**Lemma 1.2.** *For each  $\alpha \in H$ ,  $\eta_{\alpha}(s)$  is analytic for  $\text{Re}(s) > h$ .*

(i) *If  $d$  is even, then, for some constant  $c_0 > 0$ ,*

$$\lim_{\sigma \rightarrow h} \left( \eta_{\alpha}(\sigma + i\tau) - \frac{(-1)^{\nu+1}c_0}{\sigma + i\tau - h} \right)$$

*exists for almost every  $\tau \in \mathbb{R}$  and is locally integrable. Moreover, there exists a locally integrable function  $f(\tau)$  such that, for  $\sigma > h$ ,*

$$\left| \eta_{\alpha}(\sigma + i\tau) - \frac{(-1)^{\nu+1}c_0}{\sigma + i\tau - h} \right| \leq f(\tau).$$

(ii) *If  $d$  is odd, then, for some constant  $c_0 > 0$ ,*

$$\lim_{\sigma \rightarrow h} \left( \eta_{\alpha}(\sigma + i\tau) - \frac{(-1)^{\nu+1}c_0\sqrt{\pi}}{\sqrt{\sigma + i\tau - h}} \right)$$

*exists for almost every  $\tau \in \mathbb{R}$  and is locally integrable with locally integrable first derivative. Moreover, there exists a locally integrable function  $f(\tau)$  such that, for  $\sigma > h$ ,*

$$\left| \eta_{\alpha}(\sigma + i\tau) - \frac{(-1)^{\nu+1}c_0\sqrt{\pi}}{\sqrt{\sigma + i\tau - h}} \right| \leq f(\tau).$$

The constant  $c_0$  in (i) and (ii) is independent of  $\alpha$ .

When combined with appropriate Tauberian theorems, this lemma is enough to ensure that, for some constant  $c > 0$ , independent of  $\alpha \in H$ ,

$$(1.4) \quad \lim_{T \rightarrow +\infty} T^{1+d/2} e^{-hT} \pi(T, \alpha) = c.$$

(See [5] or [12] for more details.)

## 2. PROOF OF THEOREM 1

It is clear that, in general, (1.3) will depend on the relationship between  $G$  and  $H$  and, particularly,  $C$  and  $\alpha$ . Write  $A = G/[G, G]$ , the abelianization of  $G$ . Clearly,  $A$  is a quotient of  $\pi_1(M)$  and, since  $H$  is the maximal abelian quotient of  $\pi_1(M)$ ,  $A$  is also a quotient of  $H$ . The extreme cases are:

- (a)  $G$  is abelian. Then  $G = A$  and  $G$  itself is a quotient of  $H$ .
- (b)  $G$  is perfect. Then  $G = [G, G]$  and  $A$  is trivial.

We shall write  $\pi_G : G \rightarrow A$  and  $\pi_H : H \rightarrow A$  to denote the respective projections. In particular, it is clear that if  $\pi_G(C) \neq \pi_H(\alpha)$ , then  $\pi(T, \alpha, C) = 0$  for all  $T > 0$ .

The proof of Theorem 1 depends on considering  $L$ -functions defined with respect to unitary representations of  $\Gamma$  of the form  $\theta \otimes R_\chi$ , where  $\theta \in \mathbb{T}^d$  and  $R_\chi$  is an irreducible representation of  $G$  (or, more precisely, the lifts of these quantities to  $\Gamma$ ). However, as we shall describe below, some of these  $\theta \otimes R_\chi$  are trivial. The corresponding  $L$ -functions take the form

$$\begin{aligned} L(s, \theta \otimes R_\chi) &= \prod_{\gamma} \det(I - R_\chi(\langle \gamma \rangle) e^{-snl(\gamma) + 2\pi i n \theta[\gamma]}) \\ &= \exp \sum_{\gamma} \sum_{n=1}^{\infty} \frac{1}{n} \chi(\langle \gamma \rangle^n) e^{-snl(\gamma) + 2\pi i n \theta[\gamma]}, \end{aligned}$$

which converge to analytic functions for  $\text{Re}(s) > h$ . Taking the logarithm and differentiating  $\nu + 1$  times gives

$$\left(\frac{d}{ds}\right)^{\nu+1} (\log L)(s, \theta \otimes R_\chi) = \sum_{\gamma} \sum_{n=1}^{\infty} n^{\nu} (-l(\gamma))^{\nu+1} \chi(\langle \gamma \rangle^n) e^{-snl(\gamma) + 2\pi i n \theta[\gamma]}.$$

Applying the standard orthogonality relations for both for irreducible representations of  $G$  and for  $\mathbb{T}^d$  term-by-term in the above formula we obtain the relation

$$\begin{aligned} (2.1) \quad & \sum_{R_\chi \text{ irred}} \int_{\mathbb{T}^d} e^{-2\pi i \theta \cdot \alpha} \overline{\chi}(C) \left(\frac{d}{ds}\right)^{\nu+1} (\log L)(s, \theta \otimes R_\chi) d\theta \\ &= -\frac{|G|}{|C|} \sum_{\substack{\gamma, n \\ n[\gamma]=\alpha, \langle \gamma^n \rangle=C}} n^{\nu} l(\gamma)^{\nu+1} e^{-snl(\gamma)}, \end{aligned}$$

where the right-hand side vanishes if  $\pi_G(C) \neq \pi_H(\alpha)$ .

The asymptotic behaviour of  $\pi(T, \alpha, C)$  may be deduced from properties of the right-hand side of (2.1); so, to prove Theorem 1, it is enough to study

$$(2.2) \quad \sum_{R_\chi \text{ irred}} \int_{\mathbb{T}^d} e^{-2\pi i \theta \cdot \alpha} \overline{\chi}(C) \left(\frac{d}{ds}\right)^{\nu+1} (\log L)(s, \theta \otimes R_\chi) d\theta$$

and understand its meromorphic extension, via that of  $L(s, \theta \otimes R_\chi)$ , and, in particular, the nature of the singularities on  $\text{Re}(s) = h$ .

First we determine which  $\theta \otimes R_\chi$  are trivial. Let  $m = |G/[G, G]|$  be the cardinality of  $G/[G, G]$  and let  $\mathbf{1} = \chi_0, \dots, \chi_{m-1}$  be the characters of  $G/[G, G]$  (i.e., the 1-dimensional representations of  $G/[G, G]$ ). These lift to  $G$  via  $\pi_G$  but 1-dimensional characters on  $G$  also descend to  $G/[G, G]$ , since any such character annihilates commutators. Thus we may identify 1-dimensional representations of  $G$  with characters of  $G/[G, G]$ . Each  $\chi_i$  also lifts to a character of  $H$ , which we can denote by  $\theta_i$ .

**Lemma 2.1.** *The representation  $\theta \otimes R_\chi$  is trivial precisely when it is of the form  $\theta_i^{-1} \otimes R_{\chi_i}$ ,  $i = 0, \dots, m - 1$ .*

*Proof.* It is clear from their construction that these representations are trivial. On the other hand, if  $\theta \otimes R_\chi$  is trivial, then  $R_\chi$  is one dimensional and hence it corresponds to one of the characters  $\chi_0, \dots, \chi_{m-1}$  of  $G/[G, G]$ . It is easy to see that, for  $i = 0, \dots, m - 1$ ,  $\theta \otimes \chi_i$  is trivial only if  $\theta = \theta_i^{-1}$ .  $\square$

An immediate consequence is the following.

**Lemma 2.2.** *For  $i = 0, \dots, m - 1$ , the function  $\zeta(s) := L(s, \theta_i^{-1} \otimes R_{\chi_i})$  has a simple pole at  $s = h$  and no other poles on  $\text{Re}(s) = h$ .*

Next we can consider  $L(s, \theta \otimes R_\chi)$  for  $\theta \otimes R_\chi$  nontrivial.

**Lemma 2.3.** *If  $L(s, \theta \otimes R_\chi)$  has a pole on  $\text{Re}(s) = h$ , then  $R_\chi$  is 1-dimensional and  $\chi$  is a character of  $G/[G, G]$ , namely, one of the  $\chi_1, \dots, \chi_{m-1}$ .*

*Proof.* This follows from the discussion on page 146 of [8]. □

If  $R_\chi$  is 1-dimensional, then, as above, we can lift  $\theta \otimes R_\chi$  to  $\theta + \theta_i \in \mathbb{T}^{2g}$  and rewrite the  $L$ -function as

$$L(s, \theta \otimes R_\chi) = L(s, \theta + \theta_i) = \prod_{\gamma} (1 - e^{-sl(\gamma) + 2\pi i(\theta + \theta_i) \cdot [\gamma]})^{-1}.$$

However, this is again an  $L$ -function for homology.

**Lemma 2.4.** *If  $\theta \otimes R_\chi$  is nontrivial, then  $L(s, \theta \otimes R_\chi)$  is analytic on  $\text{Re}(s) = h$ .*

*Proof.* By the above, we only need to consider the case when  $L(s, \theta \otimes R_\chi) = L(s, \theta + \theta_i)$ . However, if  $\theta \otimes R_\chi$  is nontrivial, then  $\theta \neq -\theta_i$ , so the lemma follows from standard results in [3], [5], [10], [12]. □

To proceed, we return to the expression (2.1). We can rewrite this as

$$\frac{|C|}{|G|} \sum_{i=0}^{m-1} \bar{\chi}_i(C) \int_{\mathbb{T}^{2g}} e^{-2\pi i \theta \cdot \alpha} \left( \frac{d}{ds} \right)^{g+1} (\log L)(s, \theta \otimes R_{\chi_i}) d\theta + \phi(s),$$

where  $\phi(s)$  is a function analytic in a neighbourhood of  $\text{Re}(s) = h$  and, from (2.2),  $L(s, \theta \otimes R_{\chi_i}) = L(s, \theta + \theta_i)$ . We also note that, since we are assuming that  $\pi_G(C) = \pi_H(\alpha)$ , we have that

$$\bar{\chi}_i(C) e^{-2\pi i(-\theta_i) \cdot \alpha} = 1, \text{ for } i = 0, \dots, m - 1.$$

Hence one sees that the function in (2.1) satisfies an analogue of Lemma 1.2 in which that  $c_0$  is replaced by  $c_0 m |G|/|C|$ .

From this one deduces, as in [5] or [12], that

$$\lim_{T \rightarrow +\infty} T^{1+d/2} e^{-hT} \pi(T, \alpha, C) = cm \frac{|C|}{|G|},$$

with  $c$  as in (1.4).

Finally, recalling that  $m = |G/[G, G]|$ , this is enough to prove Theorem 1.

*Remarks.* (i) If  $M$  is either a surface or has curvature that is 1/4-pinned, then, working along the lines of [1], [6], [11], one can get an  $O(T^{-1})$  error term, as Zelditch obtained for a hyperbolic surface. It is also possible to prove analogous results where a fixed homology class is replaced by one which changes linearly in  $T$  (cf. [2], [7]).

(ii) There is a natural extension of Theorem 1 to Anosov flows which are homologically full in the sense of [12], i.e., ones for which every homology class is represented by a periodic orbit. □

## 3. PROOF OF THEOREM 2

We can easily adapt the proof of Theorem 1 to prove Theorem 2. Since we are replacing a finite group  $G$  by a compact group  $SO(n-1)$  we need to consider a countable family of representations  $R_\chi$ , rather than a finite family. However, by approximation it suffices to consider each representation separately. As in the proof in the last section, one can consider representations  $\theta \otimes R_\chi$ . However, a significant advantage here is that the groups  $H$  and  $SO(n-1)$  can be treated independently.

In the case of the trivial representation, we have that  $F = \chi = \mathbf{1}$  and we see that

$$\left(\frac{d}{ds}\right)^{\nu+1} (\log L)(s, \theta \otimes \mathbf{1})$$

has a singularity of the form

$$\text{Const.} \times \frac{1}{(s - s(\theta))}.$$

The analysis reduces to that in [5], [12], from which we get an asymptotic formula

$$\lim_{T \rightarrow +\infty} \frac{1}{\pi(T, \alpha)} \sum_{\substack{l(\gamma) \leq T \\ \gamma = \alpha}} F(\Theta(\gamma)) = 1.$$

However, in the case of nontrivial representations we have that the  $L$ -function  $L(s, \theta \otimes \mathbf{1})$  is always analytic on  $\text{Re}(s) = h$ , from which one sees that

$$\sum_{\substack{l(\gamma) \leq T \\ \gamma = \alpha}} F(\Theta(\gamma)) = o(\pi(T, \alpha)).$$

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