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PENCILS AND INFINITE DIHEDRAL COVERS OF $\mathbb{P}^2$

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Abstract. In this work we study the connection between the existence of finite dihedral covers of the projective plane ramified along an algebraic curve C, infinite dihedral covers, and pencils of curves containing C.

Introduction

Let us consider a reduced plane curve $C \subset \mathbb{P}^2$. The third author has extensively studied algebraic conditions for the existence of dihedral covers of $\mathbb{P}^2$ ramified along C. In this paper, C will be supposed to have two irreducible components C_1 and C_2 with the purpose of studying the existence of D_{2n}-covers of $\mathbb{P}^2$ branched at $2C_1 + nC_2$, for n odd (see the comments before Theorem 1 for the notation). Such covers are related to epimorphisms $\pi_1(\mathbb{P}^2 \setminus C) \to D_{2n}$ sending a meridian of C_1 (resp. C_2) to a conjugate of σ (resp. τ); see subsection 1.2. Our goal is to derive the existence of $(\mathbb{Z}/2 * \mathbb{Z}/2)$-covers that factorize through such finite dihedral covers. This will be related to the existence of pencils of curves containing C and the existence of infinite dihedral covers of $\mathbb{P}^2$. We will impose some restrictions on the curves C; some of them are necessary conditions for the existence of the above D_{2n}-covers and others will be set for the sake of simplicity.

(i) $\deg C_1$ is even: this is a necessary condition for the existence of the intermediate double cover ramified along C_1; see subsection 1.2.
(ii) C_1 has at most simple singularities: this condition will simplify some proofs.
(iii) $C_2 \cap \mathrm{Sing}(C_1) = \emptyset$.
(iv) For each local branch φ of C_2 at $P \in C_1 \cap C_2$, $(\varphi \cdot C_1)_P$ is even: this is also a necessary condition for the reducibility of the preimage of C_2 by the double cover ramified on C_1; see Proposition 1.5.

Let us introduce the general setting of this work. Let X and Y be normal projective varieties. Let $\pi : X \to Y$ be a finite surjective morphism. Under these conditions, the rational function field $\mathbb{C}(X)$ of X is regarded as a field extension of $\mathbb{C}(Y)$, the function field of Y. We call X a *D_{2n}-cover* of Y if the field extension $\mathbb{C}(X)/\mathbb{C}(Y)$ is Galois and its Galois group is isomorphic to the dihedral group D_{2n} of order $2n$.

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The branch locus of $\pi : X \to Y$, denoted by $\Delta(X/Y)$ or Δ_π, is the subset of Y given by

$$\Delta_\pi := \{y \in Y \mid \pi \text{ is not a local isomorphism at } y\}.$$

It is well known that Δ_π is an algebraic subset of codimension 1 if Y is smooth; see [11]. Suppose that Y is smooth and let $\Delta_\pi = B_1 + \cdots + B_r$ be its irreducible decomposition. We say $\pi : X \to Y$ is branched at $e_1 B_1 + \cdots + e_r B_r$ if the ramification index along B_i is e_i.

Let us state our main results:

Theorem 1. *If D_{2n}-covers of $\mathbb{P}^2$ branched at $2C_1 + nC_2$ exist for enough odd numbers $n \in \mathbb{N}$, then they exist for any $n \in \mathbb{N}$. Moreover, if F_i denote defining equations of C_i, $i = 1, 2$, then there exist homogeneous polynomials G_1 and G_2 such that $F_2 = G_1^2 - G_2^2 F_1$.*

Corollary 2. *Under the hypothesis of Theorem 1, there exists an epimorphism from $\pi_1(\mathbb{P}^2 \setminus (C_1 \cup C_2))$ onto the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2$.*

Remark 3. It is possible to be more precise in the statement of Theorem 1 in terms of the curves C_1, C_2. Consider the standard resolution of the double cover of $\mathbb{P}^2$ ramified along C_1. By Proposition 1.5 the preimage of C_2 under this cover decomposes as $C_2^+ \cup C_2^-$ into two irreducible components. As shown in equation (2), divisibility properties of $C_2^+ - C_2^-$ are required for Theorem 1 to hold. For instance, let ν be the self-intersection of $C_2^+ - C_2^-$ and assume that $\nu \neq 0$; then the existence of a single D_{2n}-cover of $\mathbb{P}^2$ branched at $2C_1 + nC_2$, where n^2 does not divide ν, is enough for Theorem 1 to hold.

1. Preliminaries

1.1. Topology of a double cover of $\mathbb{P}^2$.

Let B be a reduced plane curve of even degree d. Assume that singularities of B are all simple. Let $\delta : Z \to \mathbb{P}^2$ be a double cover branched at B and let $\mu : \tilde{Z} \to Z$ be the canonical resolution; see [6].

Lemma 1.1. *$\tilde{Z}$ is simply connected.*

Proof. By using results on the simultaneous resolution ([3, 4]) we know that if $(S, 0) \subset (\mathbb{C}^3, 0)$ is a double simple singularity, then the total space of its resolution is ($\mathcal{C}^\infty$) diffeomorphic to its Milnor fiber; this implies that the surface $\tilde{Z}$ obtained as the minimal resolution of the double covering of $\mathbb{P}^2$ ramified along a curve (of even degree $2m$) having only simple singularities is diffeomorphic to the double covering of $\mathbb{P}^2$ ramified along a smooth curve of degree $2m$. We assume that B is smooth. In this case, $\tilde{Z} = Z$. If B is smooth, $\pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/d\mathbb{Z}$. Hence $\pi_1(Z \setminus \delta^{-1}(B)) \cong \mathbb{Z}/(d/2)\mathbb{Z}$, and it is generated by a meridian around $\delta^{-1}(B)$. In Z, this lasso is homotopic to zero. Hence $\pi_1(Z) = \{1\}$. $\square$

Corollary 1.2. $\operatorname{Pic}(\tilde{Z}) = \operatorname{NS}(\tilde{Z})$*;* $\operatorname{Pic}(\tilde{Z})$ *is a lattice with respect to the intersection pairing.*

1.2. Dihedral covers.

To present D_{2n}, we use the notation

$$D_{2n} = \langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1 \rangle,$$

and fix it throughout this article. Given a D_{2n}-cover $\pi : X \to Y$, we canonically obtain the double cover $D(X/Y)$ of Y by taking the $\mathbb{C}(X)^\tau$-normalization of Y, where $\mathbb{C}(X)^\tau$ is the fixed field of $\langle\tau\rangle$. The variety X is an n-cyclic cover of $D(X/Y)$ by its definition. We denote these cover morphisms by $\beta_1(\pi) : D(X/Y) \to Y$ and $\beta_2(\pi) : X \to D(X/Y)$, respectively. In this context, D_{2n}-covers have been thoroughly considered in [7, 8]. Here we summarize some of the main results that will be needed in this paper. The following is a sufficient condition for the existence of D_{2n}-covers.

Proposition 1.3. *Let Y be a smooth variety and let $n \geq 3$ an integer. Let $\delta : Z \to Y$ be a smooth double cover, and let σ_δ denote the involution. Suppose that there exists a reduced divisor D on Z satisfying the following conditions:*

(1) *$\sigma_\delta^* D$ and D have no common components.*
(2) *There exists a line bundle L on Z such that $D - \sigma_\delta^* D \sim nL$, where $\sim$ means linear equivalence.*

Let us also suppose that either n is odd, or n is even and Y is simply connected. Then there exists a D_{2n}-cover $\pi : X \to Y$ such that $D(X/Y) = Z$ and π is branched at $2\Delta_\delta + n\delta(D)$.

Proof. If n is odd, our statement is a special case of [8]. If n is even, then by [7, Remark 3.1] and a similar argument to the proof of [8, Proposition 1.1], the result follows. □

Corollary 1.4. *Suppose that Y is simply connected. If $\sigma_f^* D \sim D$, then there exists a D_{2n}-cover of Y branched at $2\Delta_f + nf(D)$ for any $n \geq 3$.*

As for a necessary condition for the existence of D_{2n}-covers, we have the following.

Proposition 1.5 ([7, §2])**.** *Let $\pi : X \to Y$ be a D_{2n}-cover such that $D(X/Y)$ is smooth. Then there exist a (possibly empty) effective divisor D_1 and a line bundle L on $D(X/Y)$ satisfying the following conditions:*

(1) *D_1 and $\sigma^* D_1$ have no common components.*
(2) *$D_1 - \sigma^* D_1 \sim nL$.*
(3) *$\Delta(X/D(X/Y)) = \mathrm{Supp}(D_1 + \sigma^* D_1)$.*
(4) *The ramification index along $D_{1,j}$ is $\frac{n}{\gcd(a_j, n)}$, where $D_1 = \sum_j a_j D_{1,j}$ $(a_j > 0)$ is the irreducible decomposition.*

Corollary 1.6. *Let D be an irreducible component of $\beta_1(\pi)(\Delta_{\beta_2(\pi)})$. Then the divisor $\beta_1(\pi)^* D$ is of the form $D' + \sigma^* D'$ for some irreducible divisor on $D(X/Y)$. In other words, $\beta_2(\pi)$ is not branched along any irreducible divisor D with $D = \sigma^* D$.*

2. Certain D_{2n}-covers of algebraic surfaces

Let Σ_o be a smooth projective surface. Let C_1 and C_2 be reduced divisors on Σ_o such that

- C_1 has at most simple singularities;
- C_2 is irreducible;
- $C_2 \cap \mathrm{Sing}(C_1) = \emptyset$;
- there exists a double cover $\delta : Z \to \Sigma_o$ branched at C_1;
- its canonical resolution $\mu : \tilde{Z} \to Z$ is simply connected.

Proposition 2.1. *If there exists a D_{2k}-cover $\pi_k : S_k \to \Sigma_o$ branched at $2C_1 + kC_2$ for finitely many enough odd natural numbers k (see Remark 3), then there exist D_{2n}-covers of Σ_o branched at $2C_1 + nC_2$ for any integer $n \geq 3$.*

Proof. By our assumption, $D(S_k/\Sigma_o) = Z$ and $\beta_1(\pi_k) = \delta$. Let

$$\begin{array}{ccc} Z & \xleftarrow{\mu} & \tilde{Z} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\tilde{\delta}} \\ \Sigma_o & \xleftarrow{q} & \Sigma \end{array}$$

denote the diagram where q is the composition of the minimal sequence of blow-ups such that the pull-back $\tilde{Z}$ is smooth. Let $\tilde{S}_k$ be the $\mathbb{C}(S_k)$-normalization of Σ. The variety $\tilde{S}_k$ is a D_{2k}-cover of Σ and we denote the cover morphism by $\tilde{\pi}_k$. Summing up, we obtain the following commutative diagram:

$$\begin{array}{ccc} S_k & \longleftarrow & \tilde{S}_k \\ \downarrow & & \downarrow \\ Z & \xleftarrow{\mu} & \tilde{Z} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\tilde{\delta}} \\ \Sigma_o & \xleftarrow{q} & \Sigma. \end{array}$$

Note that

$$\Delta_{\tilde{\delta}} = q^{-1}C_1 + \text{ Some irreducible components of the exceptional set of } q,$$

$$\Delta_{\beta_2(\tilde{\pi}_k)} = \tilde{\delta}^{-1}(q^{-1}C_2) + \text{Some irreducible components of the exceptional set of } \mu,$$

where $\bullet^{-1}$ denote proper transforms.

By Corollary 1.6, $\tilde{\delta}^{-1}(q^{-1}C_2)$ is of the form $C_2^+ + C_2^-$, $\sigma_{\tilde{\delta}}^*(C_2^+) = C_2^-$. Since π_k is branched at $2C_1 + kC_2$, by Proposition 1.5, for all k as in the statement there exists a line bundle L_k such that

$$C_2^+ - C_2^- + R_k - \sigma_{\tilde{\delta}}^* R_k \sim kL_k \tag{1}$$

where $\mathrm{Supp}(R_k \cup \sigma_{\tilde{\delta}}^* R_k)$ is contained in the exceptional set of μ. The subgroup T of $\mathrm{NS}(\tilde{Z})$ generated by the irreducible components of the exceptional divisors of μ is a negative definite sublattice in $\mathrm{NS}(\tilde{Z})$. Let us consider the relation (1) in $\mathrm{NS}(\tilde{Z})/T$. Then we have

$$C_2^+ - C_2^- \equiv kL_k \bmod T. \tag{2}$$

Since $\mathrm{NS}(\tilde{Z})/T$ is a finitely generated Abelian group, the hypothesis implies that $C_2^+ - C_2^-$ is a torsion element of $\mathrm{NS}(\tilde{Z})/T$; we can apply Remark 3 since $C_2^\pm$ is orthogonal to T. Hence there exists a certain $\ell \in \mathbb{N}$ such that $\ell(C_2^+ - C_2^-) \in T$. Put

$$\ell(C_2^+ - C_2^-) = \sum_i c_i \Theta_i,$$

where Θ_i's denote the irreducible components of the exceptional divisor of μ. Since C_2 does not pass through the singularities of C_1 then $\Theta_i \cdot C_2^\pm = 0$ for all i. Hence $\ell(C_2^+ - C_2^-) = 0$, and as T is a free $\mathbb{Z}$-module generated by Θ_i's, then $C_2^+ = C_2^-$ in $\mathrm{NS}(\tilde{Z})$. Since $\tilde{Z}$ is simply connected, $\mathrm{Pic}(\tilde{Z}) = \mathrm{NS}(\tilde{Z})$. This implies $C_2^+ - C_2^- \sim 0$. Hence by Corollary 1.4, our statement follows. $\square$

3. Proof of Theorem 1

Let $\delta : Z \to \mathbb{P}^2$ be a double cover branched at C_1, and let $\mu : \tilde{Z} \to Z$ be its canonical resolution. Since C_1 has at most simple singularities, $\tilde{Z}$ is simply connected by Lemma 1.1. Hence the first half of Theorem 1 follows from Proposition 2.1.

We now go on to the second half. Assume that C_1 and C_2 are given by the equations:

$$C_1 : F_1(U, V, W) = 0,$$
$$C_2 : F_2(U, V, W) = 0.$$

Since $C_2^+ - C_2^- \sim 0$, there exists a rational function $\varphi \in \mathbb{C}(\tilde{Z})(= \mathbb{C}(Z))$ such that

$$(\varphi) = C_2^+ - C_2^-.$$

Put $\theta_n = \sqrt[n]{\varphi}$ $(n \geq 3)$ and consider the $\mathbb{C}(Z)(\theta_n)$-normalization S_n of Z. We denote the induced covering morphism $S_n \to Z$ by g_n.

Lemma 3.1. *S_n is a D_{2n}-cover of $\mathbb{P}^2$ branched at $2C_1 + nC_2$.*

Proof. Since $\varphi \neq 1/\varphi$, $\mathbb{C}(Z) = \mathbb{C}(\mathbb{P}^2)(\varphi)$ and this implies that $\mathbb{C}(S_n) = \mathbb{C}(\mathbb{P}^2)(\theta_n)$ and $[\mathbb{C}(S_n) : \mathbb{C}(\mathbb{P}^2)] = 2n$. One can see that $\mathbb{C}(S_n)/\mathbb{C}(\mathbb{P}^2)$ is a D_{2n}-extension, as a D_{2n}-action over $\mathbb{C}(\mathbb{P}^2)$ is given by $\theta_n^\sigma = 1/\theta_n$ and $\theta_n^\tau = \zeta_n\theta_n$, $\zeta_n = \exp(2\pi\sqrt{-1}/n)$. Hence $\delta \circ g_n : S_n \to \mathbb{P}^2$ is a D_{2n}-cover. As $(\varphi) = C_2^+ - C_2^-$ and $C_2^+ \cup C_2^-$ is contained in the smooth part of Z, the branch locus of g_n is $(C_2^+ + C_2^-)$ and the ramification index along $C_2^\pm$ is n. Since the branch locus of δ is C_1, $\delta \circ g_n$ is branched at $2C_1 + nC_n$. □

Put $u := \varphi + 1/\varphi$. As u is σ-invariant, there exists a rational function $\psi \in \mathbb{C}(\mathbb{P}^2)$ such that $\delta^*\psi = u$.

Lemma 3.2. *The polar divisor of ψ is C_2.*

Proof. Let C_∞ be the polar divisor of ψ. Since the polar divisor of $\varphi + 1/\varphi$ is $C_2^+ + C_2^-$, we have $\delta^* C_\infty = C_2^+ + C_2^-$. □

Let $\varpi_n : \mathbb{P}^1 \to \mathbb{P}^1$ be a D_{2n}-cover given by

$$t \mapsto \frac{1}{2}\left(t^n + \frac{1}{t^n}\right) =: s,$$

where t, s are inhomogeneous coordinates. Let $\Phi_n : S_n \dashrightarrow \mathbb{P}^1$ and $\overline{\Phi}_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be rational maps given by θ_n and ψ, respectively. The rational map Φ_n is D_{2n}-equivariant, and we have the following commutative diagram:

$$\begin{array}{ccccc} & & \Phi_n & & \\ & S_n & \dashrightarrow & \mathbb{P}^1 & \\ \delta \circ g_n & \downarrow & & \downarrow & \varpi_n \\ & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 & \\ & & \overline{\Phi}_n & & \end{array}$$

From this diagram, we can infer that S_n is obtained as a *rational* pullback by $\overline{\Phi}_n$; note that any D_{2n}-cover is obtained as a rational pullback as above if n is odd; see [9, Theorem 1]. Since ϖ_n is branched at $2[1 : \pm 1] + n[0 : 1]$, $[a : b] \equiv [1 : s]$ being a homogenous coordinate of $\mathbb{P}^1$, we may assume that the images of C_1 and C_2 are $[1 : 1]$ and $[0 : 1]$, respectively.

Following Lemma 3.2, we can write $\psi := F_0/F_2$, where F_0 is a homogeneous polynomial, $\deg F_0 = \deg F_2$. Then the images of the curves given by $F_0 - F_2 = 0$ and $F_0 + F_2 = 0$ under $\overline{\Phi}_n$ are $[1:1]$ and $[1:-1]$. This implies that the divisors given by $F_0 - F_2 = 0$ and $F_0 + F_2 = 0$ are of the form $C_1 + 2D_1$ and $2D_2$. Hence there exist homogeneous polynomials G_1 and G_2 such that $F_0 + F_2 = G_1^2$ and $F_0 - F_2 = G_2^2 F_1$, and we deduce

$$F_2 = \frac{G_1^2 - G_2^2 F_1}{2}.$$

The second half of Theorem 1 follows.

4. Pencils and fundamental groups

Let C be a complex projective plane curve. In this section we intend to exhibit the connection between the existence of pencils of curves related to C and the fundamental group of its complement $X_C := \mathbb{P}^2 \setminus C$ from a topological point of view. We will apply it to curves satisfying the statement of Theorem 1.

Definition 4.1. Let D be a compact algebraic curve, let $p_1, \ldots, p_r, q_1, \ldots, q_s \in D$ be distinct points and let $n_1, \ldots, n_r \in \mathbb{Z}_{\geq 2}$. An *orbifold* $D^{q_1,\ldots,q_s}_{(p_1,n_1),\ldots,(p_r,n_r)}$ is a punctured curve $D \setminus \{q_1, \ldots, q_s\}$ where the points p_i are weighted with the integers n_i, $i = 1, \ldots, r$. For the sake of simplicity sometimes it will be denoted by $D^{q_1,\ldots,q_s}_{n_1,\ldots,n_r}$.

We may think that the charts around the points p_i are obtained as the quotient of disks in $\mathbb{C}$ by the action of the n_i-roots of unity. This justifies the following definition.

Definition 4.2. The orbifold-fundamental group $\pi_1^{\text{orb}}(D^{q_1,\ldots,q_s}_{(p_1,n_1),\ldots,(p_r,n_r)}; *)$, $* \in \check{D} := D \setminus \{p_1, \ldots, p_r, q_1, \ldots, q_s\}$ is defined as the quotient of $\pi_1(\check{D}; *)$ by the normal subgroup generated by $\mu_i^{n_i}$, $i = 1, \ldots, r$, where μ_i is a meridian of p_i.

Examples 4.3. We will fix $D = \mathbb{P}^1$.

(1) $\pi_1^{\text{orb}}((\mathbb{P}^1)_{2,2,n}; *)$ is the dihedral group D_{2n}.
(2) $\mathbb{T}_{p,q,r} := \pi_1^{\text{orb}}((\mathbb{P}^1)_{p,q,r}; *)$ is the corresponding triangle group.
(3) $\mathbb{F}_{n_1,\ldots,n_r} := \pi_1^{\text{orb}}((\mathbb{P}^1)^{\infty}_{n_1,\ldots,n_r}; *)$ is the free product $\mathbb{Z}/n_1 * \cdots * \mathbb{Z}/n_r$.

Let us now fix a connected smooth projective surface X, a connected smooth projective curve Γ and a non-constant rational map $\widetilde{\rho} : X \dashrightarrow \Gamma$. Let $C \subset X$ be a compact curve such that $\widetilde{\rho}$ is well defined on $X \setminus C$ and let $A := \Gamma \setminus \widetilde{\rho}(X \setminus C)$, which is a finite set of points. We denote by $\rho : X \setminus C \to \Gamma \setminus A$ the restriction of $\widetilde{\rho}$, which is assumed to have connected fibers.

Let $p \in \Gamma \setminus A$; we consider the divisor $\rho^*(p)$, which is the restriction of $\widetilde{\rho}^*(p)$ to $X \setminus C$. For each p we denote n_p as the gcd of the multiplicities of $\rho^*(p)$. We consider the orbifold $\Gamma_\rho := \Gamma^A_{\{(p,n_p) \mid n_p > 1\}}$. Fix $q \in \Gamma \setminus A$ such that $n_q = 1$ and $* \in \rho^{-1}(q)$.

Proposition 4.4. *The mapping ρ induces a natural epimorphism*

$$\rho_* : \pi_1(X \setminus C; *) \twoheadrightarrow \pi_1^{\text{orb}}(\Gamma_\rho; q).$$

Proof. Let us denote $\widetilde{C} := C \cup \bigcup_{n_p > 1} \widetilde{\rho}^*(p)$ and $\Gamma_1 := \Gamma \setminus (A \cup \{(p, n_p) \mid n_p > 1\})$. The rational map $\widetilde{\rho}$ induces a well-defined surjective morphism $\rho_1 : X \setminus \widetilde{C} \to \Gamma_1$. It is a standard fact that ρ induces an epimorphism

$$\pi_1(X \setminus \widetilde{C}; *) \twoheadrightarrow \pi_1(\Gamma_1; q).$$

Recall that $\pi_1(X \setminus C; *)$ is the quotient of $\pi_1(X \setminus \widetilde{C}; *)$ by the subgroup generated by the components of $\widetilde{C}$ not in C. The condition on the gcd of multiplicities guarantees the following commutative diagram which gives the result:

$$\begin{array}{ccc} \pi_1(X \setminus \widetilde{C}; *) & \longrightarrow & \pi_1(\Gamma_1; q) \\ \big\downarrow & & \big\downarrow \\ \pi_1(X \setminus C; *) & \longrightarrow & \pi_1^{\text{orb}}(\Gamma_\rho; q). \end{array}$$

Let us note that a meridian of a component of $\widetilde{C}$ not in C is sent by ρ to the power of a meridian μ_i; the power is a multiple of n_i. □

We say that a pencil $\mathcal{P} := \{F_p\}_{p \in \mathbb{P}^1}$ *contains* C if each irreducible component of C is contained in a member of $\mathcal{P}$. Let $A \subset \mathbb{P}^1$ be the subset of $p \in \mathbb{P}^1$ such that $F_p^{\text{red}} \subset C$. Let n_p denote the gcd of the multiplicities of the components in F_p not contained in C. We define the set $B = \{p \in \mathbb{P}^1 \setminus A \mid n_p > 1\} \subset \mathbb{P}^1$. Let us assume that $\#A = n$ and $B := \{p_1, \ldots, p_r\}$, $n_i := n_{p_i}$.

Corollary 4.5. *There is a surjection from* $\pi_1(X_C)$ *onto*

$$\mathbb{F}_{n;(n_1,\ldots,n_r)} := \langle x_1, \ldots, x_n, y_1, \ldots, y_r : \prod_{j=1}^{n} x_j \cdot \prod_{i=1}^{r} y_i = y_1^{n_1} = \ldots = y_r^{n_r} = 1 \rangle.$$

Remark 4.6. If $n_i' | n_i$, then $\mathbb{F}_{n;(n_1,\ldots,n_r)}$ surjects onto $\mathbb{F}_{n;(n_1',\ldots,n_r')}$. Any n_i' equal 1 will be dropped. By doing so, we only add some ambiguity about the surjection, but this is not relevant for our purposes.

Example 4.7. Let C_6 be a Zariski sextic, that is, of equation $D_2^3 + D_3^2 = 0$, where D_i is a homogeneous polynomial in $\mathbb{C}[x, y, z]$ of degree i. The pencil generated by D_2^3 and D_3^2 has at least these two as special fibers. According to the notation of Corollary 4.5 we have that $\pi_1(X_{C_6})$ surjects, onto a group $\mathbb{F}_{1;(2,3,n_3,\ldots,n_r)}$ and therefore (Remark 4.6) onto $\mathbb{F}_{1;(2,3)} = \mathbb{Z}_2 * \mathbb{Z}_3$. Zariski proved in [10] that this is an isomorphism for generic choices.

Proof of Corollary 2. By Theorem 1 the pencil generated by G_1^2 and $G_2^2 F_1$ contains F_2, therefore, using Corollary 4.5, there exists a surjection from $\pi_1(\mathbb{P}^2 \setminus (C_1 \cup C_2))$ onto $\mathbb{F}_{1;(2,2,n_3,\ldots,n_r)}$, and hence, by Remark 4.6, there exists a surjection onto $\mathbb{F}_{1;(2,2)} = \mathbb{Z}_2 * \mathbb{Z}_2$. □

5. Examples

Example 5.1. Let us suppose that there exists a pencil with three fibers $2A_1 + B_1$, $2A_2 + B_2$, $nA_3 + B_3$. Then the fundamental group of the complement of $B_1 \cup B_2 \cup B_3$ surjects onto $\mathbb{T}_{2,2,n} = D_{2n}$. The simplest example is the tricuspidal quartic. Zariski proved in [10] that it lives in a pencil as in Example 4.7 if we add the double of the bitangent line. Then we have a surjection onto D_6 which is not an isomorphism since it is also proved in [10] that its fundamental group has order 12.

Example 5.2. Let C be a smooth conic and L_1, L_2, L_3 tangent lines at three different points P_1, P_2 and P_3 of C. The pencil $\mathcal{P}$ generated by C and $L_1 + L_2$ contains as a special fiber $2L$, where L is the line passing through P_1 and P_2. Let f_n be the cover map $f_n : \mathbb{P}^2 \to \mathbb{P}^2$, $f_n(L_1, L_2, L_3) := [L_1^n : L_2^n : L_3^n]$. The pull-back $f^*\mathcal{P}$ of the pencil $\mathcal{P}$ is generated by $f_n^* C$ and $f_n^* L_1 + f_n^* L_2 = n(L_1 + L_2)$ and contains the curve $2f_n^* L$. A description of the curve $f_n^* C$ and a presentation of its fundamental group $\pi_1(\mathbb{P}^2 \setminus f_n^* C)$ can be found in [5]. By Corollary 4.5, $\pi_1(\mathbb{P}^2 \setminus f_n^* C)$ has a surjection onto $\mathbb{F}_{1;(2,n)} = \mathbb{Z}_2 * \mathbb{Z}_n$.

Example 5.3. In [1] we have studied curves having two irreducible components: a quartic C_1 having two singular points of types $\mathbb{A}_3$ and $\mathbb{A}_1$ and a smooth conic C_2 such that its intersection with C_1 produces a singular point of type $\mathbb{A}_{15}$. Let us drop the $\mathbb{A}_1$ point. Then it is easily seen that the moduli space of such curves has three connected components. Let us describe two of them:

- The tangent line T at $\mathbb{A}_{15}$ passes through $\mathbb{A}_3$. In this case there is a pencil of quartics containing C_1 and $4T$ such that another element of the pencil is $C_2 + 2L$, where L is the tangent line at $\mathbb{A}_3$. By Corollary 4.5, $\pi_1(\mathbb{P}^2 \setminus (C_1 \cup C_2))$ has a surjection onto $\mathbb{F}_{1;(2,4)} = \mathbb{Z}_2 * \mathbb{Z}_4$.
- There exists a smooth conic Q having four infinitely near points in common with $\mathbb{A}_{15}$ and tangent at $\mathbb{A}_3$. In this case there is again a pencil of quartics containing C_1, $2Q$ and C_2+2L. Therefore, $\pi_1(\mathbb{P}^2 \setminus (C_1 \cup C_2))$ has a surjection onto $\mathbb{F}_{1;(2,2)} = \mathbb{Z}_2 * \mathbb{Z}_2$.

Example 5.4. Let us consider the family of curves of type I described in [2]. When D is rational, they satisfy the conditions of Theorem 1 and therefore there is a surjection $\pi_1(\mathbb{P}^2 \setminus (D \cup L_1 \cup L_2)) \twoheadrightarrow \mathbb{Z}_2 * \mathbb{Z}_2$ (Corollary 2). Note that there is yet another pencil that produces a surjection $\pi_1(\mathbb{P}^2 \setminus (D \cup L_1 \cup L_2)) \twoheadrightarrow \mathbb{Z}_2 * \mathbb{Z}_2$. Consider the most general case, that is:

(1) C a rational arrangement of degree $2k + 1$ with an ordinary multiple point P of multiplicity $2k - 1$ and at least $2k - 1$ nodes $Q_1, ..., Q_{2k-1}$.
(2) L_i a line tangent to C at a point P_i, $i = 1, 2$.
(3) D_i a curve of degree k with an ordinary multiple point at P of multiplicity $k - 1$, passing through $P_i, Q_1, ..., Q_{2k-1}$.

The pencil generated by $L_1 + 2D_2$ and $L_2 + 2D_1$ contains C. Using a third line L_3 and the cover $f_n : \mathbb{P}^2 \to \mathbb{P}^2$ described above, one obtains curves $f_n^* C$ whose fundamental group $\pi_1(\mathbb{P}^2 \setminus f_n^* C)$ surjects onto $\mathbb{Z}_2 * \mathbb{Z}_2$ (for n even) and such that $\pi_1(\mathbb{P}^2 \setminus (f_n^* C \cup f_n^* D_1 \cup f_n^* D_2))$ surjects onto $\mathbb{Z}_n * \mathbb{Z}_n$.

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