

ON SOME RANDOM THIN SETS OF INTEGERS

DANIEL LI, HERVÉ QUEFFÉLEC, AND LUIS RODRÍGUEZ-PIAZZA

(Communicated by Michael Lacey)

ABSTRACT. We show how different random thin sets of integers may have different behaviour. First, using a recent deviation inequality of Boucheron, Lugosi and Massart, we give a simpler proof of one of our results in *Some new thin sets of integers in harmonic analysis*, *Journal d'Analyse Mathématique* 86 (2002), 105–138, namely that there exist $\frac{4}{3}$ -Rider sets which are sets of uniform convergence and $\Lambda(q)$ -sets for all $q < \infty$ but which are not Rosenthal sets. In a second part, we show, using an older result of Kashin and Tzafriri, that, for $p > \frac{4}{3}$, the p -Rider sets which we had constructed in that paper are almost surely not of uniform convergence.

1. INTRODUCTION

It is well known that the Fourier series $S_n(f, x) = \sum_{-n}^n \hat{f}(k)e^{ikx}$ of a 2π -periodic continuous function f may be badly behaved: for example, it may diverge on a prescribed set of values of x with measure zero. Similarly, the Fourier series of an integrable function may diverge everywhere. But it is equally well known that, as soon as the spectrum $Sp(f)$ of f (the set of integers k at which the Fourier coefficients of f do not vanish, *i.e.* $\hat{f}(k) \neq 0$) is sufficiently “lacunary”, in the sense of Hadamard *e.g.*, then the Fourier series of f is absolutely convergent if f is continuous and almost everywhere convergent if f is merely integrable (and in this latter case $f \in L^p$ for every $p < \infty$). Those facts have given birth to the theory of thin sets Λ of integers, initiated by Rudin [15]: those sets Λ such that, if $Sp(f) \subseteq \Lambda$ (we shall write $f \in \mathcal{B}_\Lambda$ when f is in some Banach function space \mathcal{B} contained in $L^1(\mathbb{T})$ and $Sp(f) \subseteq \Lambda$), then $S_n(f)$, or f itself, is better behaved than in the general case. Let us for example recall that the set Λ is said to be as follows.

- A p -Sidon set ($1 \leq p < 2$) if $\hat{f} \in l_p$ (and not only $\hat{f} \in l_2$) as soon as f is continuous and $Sp(f) \subseteq \Lambda$. This amounts to an “*a priori* inequality” $\|\hat{f}\|_p \leq C\|f\|_\infty$, for each $f \in \mathcal{C}_\Lambda$. The case $p = 1$ is the celebrated case of Sidon (= 1-Sidon) sets.
- A p -Rider set ($1 \leq p < 2$) if we have an *a priori* inequality $\|\hat{f}\|_p \leq C\|f\|$, for every trigonometric polynomial with spectrum in Λ . Here $\|f\|$ is the so-called Pisier norm of $f = \sum \hat{f}(n)e_n$, where $e_n(x) = e^{inx}$, *i.e.* $\|f\| = \mathbb{E}\|f_\omega\|_\infty$, where $f_\omega = \sum \varepsilon_n(\omega)\hat{f}(n)e_n$, (ε_n) being an *i.i.d.* sequence of

Received by the editors September 19, 2006.

2000 *Mathematics Subject Classification.* Primary 43A46; Secondary 42A55, 42A61.

Key words and phrases. Boucheron-Lugosi-Massart deviation inequality, $\Lambda(q)$ -sets, p -Rider sets, Rosenthal sets, selectors, sets of uniform convergence.

©2007 American Mathematical Society
Reverts to public domain 28 years from publication

centered, ± 1 -valued, random variables defined on some probability space (a Rademacher sequence), and where \mathbb{E} denotes the expectation on that space. This apparently exotic notion (weaker than p -Sidonicity) turned out to be very useful when Rider [12] reformulated a result of Drury (proved in the course of the result that the union of two Sidon sets is a Sidon set) under the form: 1-Rider sets and Sidon sets are the same (in spite of some partial results, it is not yet known whether a p -Rider set is a p -Sidon set: see [5], however, for a partial result).

- A *set of uniform convergence* (in short a UC -set) if the Fourier series of each $f \in \mathcal{C}_\Lambda$ converges uniformly, which amounts to the inequality $\|S_n(f)\|_\infty \leq C\|f\|_\infty$, $\forall f \in \mathcal{C}_\Lambda$. Sidon sets are UC , but the converse is false.
- A $\Lambda(q)$ -set, $1 < q < \infty$, if every $f \in L_\Lambda^1$ is in fact in L^q , which amounts to the inequality $\|f\|_q \leq C_q\|f\|_1$, $\forall f \in L_\Lambda^1$. Sidon sets are $\Lambda(q)$ for every $q < \infty$ (and even $C_q \leq C\sqrt{q}$). The converse is false, except when we require $C_q \leq C\sqrt{q}$ ([11]).
- A *Rosenthal set* if every $f \in L_\Lambda^\infty$ is almost everywhere equal to a continuous function. Sidon sets are Rosenthal, but the converse is false.

This theory has long suffered from a severe lack of examples: those examples were always, more or less, sums of Hadamard sets, and in that case the Banachic properties of the corresponding \mathcal{C}_Λ -spaces were very rigid. The use of random sets (in the sense of the selectors method) of integers has significantly changed the situation (see [8] and our paper [9]). Let us recall more in detail the notation and setting of our previous work [9]. The method of selectors consists in the following: let $(\varepsilon_k)_{k \geq 1}$ be a sequence of independent, $(0, 1)$ -valued random variables, with respective means δ_k , defined on a probability space Ω , and to which we attach the random set of integers $\Lambda = \Lambda(\omega)$, $\omega \in \Omega$, defined by $\Lambda(\omega) = \{k \geq 1; \varepsilon_k(\omega) = 1\}$.

The properties of $\Lambda(\omega)$ of course highly depend on the δ_k 's, and roughly speaking, the smaller the δ_k 's, the better \mathcal{C}_Λ , L_Λ^1 , \dots . In [7], and then, in a much deeper way, in [9], relying on a probabilistic result of J. Bourgain on ergodic means, and on a deterministic result of F. Lust-Piquard ([10]) on those ergodic means, we had randomly built new examples of sets Λ of integers which were both locally thin from the point of view of harmonic analysis (their traces on big segments $[M_n, M_{n+1}]$ of integers were uniformly Sidon sets) and regularly distributed from the point of view of number theory, and therefore globally big from the point of view of Banach space theory, in that the space \mathcal{C}_Λ contained an isomorphic copy of the Banach space c_0 of sequences vanishing at infinity. More precisely, we have constructed subsets $\Lambda \subseteq \mathbb{N}$ which are thin in the following respects: Λ is a UC -set, a p -Rider set for various $p \in [1, 2[$, a $\Lambda(q)$ -set for every $q < \infty$, and large in two respects: the space \mathcal{C}_Λ contains an isomorphic copy of c_0 and, most often, Λ is dense in the integers equipped with the Bohr topology.

Now, taking δ_k bigger and bigger, we had obtained sets Λ which were less and less thin (p -Sidon for every $p > 1$, q -Rider, but s -Rider for no $s < q$, s -Rider for every $s > q$, but not q -Rider), and, in any case $\Lambda(q)$ for every $q < \infty$, and such that \mathcal{C}_Λ contains a subspace isomorphic to c_0 . In particular, in Theorem II.7, page 124, and Theorem II.10, page 130, we take, respectively, $\delta_k \approx \frac{\log k}{k}$ and $\delta_k \approx \frac{(\log k)^\alpha}{k(\log \log k)^{\alpha+1}}$, where $\alpha = \frac{2(p-1)}{2-p}$ is an increasing function of $p \in [1, 2)$ and which becomes ≥ 1 as p becomes $\geq 4/3$. The case $\delta_k = \frac{1}{k}$ would correspond (randomly) to Sidon sets (*i.e.* 1-Sidon sets).

After the proofs of Theorem II.7 and Theorem II.10, we asked two questions:

1) (Page 129) Our construction is very complicated and needs a second random construction of a set E inside the random set Λ . Is it possible to give a simpler proof?

2) (Page 130) In Theorem II.10, can we keep the property for the random set Λ to be a UC -set, with high probability, when $\alpha > 1$ (equivalently when $p > \frac{4}{3}$)?

The goal of this work is to answer affirmatively the first question (relying on a recent deviation inequality of Boucheron, Lugosi and Massart [1]) and negatively the second one (relying on an older result of Kashin and Tzafriri [3]). This work is accordingly divided into three parts. In Section 2, we prove a (one-sided) concentration inequality for norms of Rademacher sums. In Section 3, we apply the concentration inequality to get a substantially simplified proof of Theorem II.7 in [9]. Finally, in Section 4, we give a (stochastically) negative answer to the second question when $p > \frac{4}{3}$: almost surely, Λ will not be a UC -set; here, we use the above mentioned result of Kashin and Tzafriri [3] on the non- UC character of big random subsets of integers.

2. A ONE-SIDED INEQUALITY FOR NORMS OF RADEMACHER SUMS

Let E be a (real or complex) Banach space, let v_1, \dots, v_n be vectors of E , let X_1, \dots, X_n be independent, real-valued, centered, random variables, and let $Z = \left\| \sum_1^n X_j v_j \right\|$.

If $|X_j| \leq 1$ *a.s.*, it is well known (see [6]) that

$$(2.1) \quad \mathbb{P}(|Z - \mathbb{E}(Z)| > t) \leq 2 \exp\left(-\frac{t^2}{8 \sum_1^n \|v_j\|^2}\right), \quad \forall t > 0.$$

But often, the “strong” l_2 -norm of the n -tuple $v = (v_1, \dots, v_n)$, namely $\|v\|_{strong} = (\sum_{j=1}^n \|v_j\|^2)^{1/2}$, is too large for (2.1) to be interesting, and it is advisable to work with the “weak” l_2 -norm of v , defined by

$$(2.2) \quad \sigma = \|v\|_{weak} = \sup_{\varphi \in B_{E^*}} \left(\sum_1^n |\varphi(v_j)|^2 \right)^{1/2} = \sup_{\sum |a_j|^2 \leq 1} \left\| \sum_1^n a_j v_j \right\|,$$

where B_{E^*} denotes the closed unit ball of the dual space E^* .

If $(X_j)_j$ is a standard Gaussian sequence ($\mathbb{E} X_j = 0, \mathbb{E} X_j^2 = 1$), this is what Maurey and Pisier succeeded in doing, using either the Itô formula or the rotational invariance of the X_j 's. They proved the following (see [8], Chapitre 8, Théorème I.4):

$$(2.3) \quad \mathbb{P}(|Z - \mathbb{E} Z| > t) \leq 2 \exp\left(-\frac{t^2}{C\sigma^2}\right), \quad \forall t > 0,$$

where σ is as in (2.2) and C is a numerical constant, *e.g.* $C = \pi^2/2$.

To the best of our knowledge, no inequality as simple and direct as (2.3) is available for non-Gaussian (*e.g.* Rademacher) variables, although several more complicated deviation inequalities are known: see *e.g.* [2], [6].

For the applications to harmonic analysis which we have in view, where we use the so-called “selectors method”, we precisely need an analogue of (2.3), in the non-Gaussian, uniformly bounded (and centered) case; we shall prove that at least a one-sided version of (2.3) holds in this case, by showing the following result, which is interesting in itself.

Theorem 2.1. *With the previous notation, assume that $|X_j| \leq 1$ a.s. Then, we have the one-sided estimate*

$$(2.4) \quad \mathbb{P}(Z - \mathbb{E}Z > t) \leq \exp\left(-\frac{t^2}{C\sigma^2}\right), \quad \forall t > 0,$$

where $C > 0$ is a numerical constant ($C = 32$, for example).

The proof of (2.4) will make use of a recent deviation inequality due to Boucheron, Lugosi and Massart [1]. Before stating this inequality, we need some notation.

Let X_1, \dots, X_n be independent, real-valued random variables (here, we temporarily forget the assumptions of the previous theorem), and let (X'_1, \dots, X'_n) be an independent copy of (X_1, \dots, X_n) .

If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a given measurable function, we set $Z = f(X_1, \dots, X_n)$ and $Z'_i = f(X_1, \dots, X_{i-1}, X'_i, X_{i+1}, \dots, X_n)$, $1 \leq i \leq n$. With this notation, the Boucheron-Lugosi-Massart Theorem goes as follows:

Theorem 2.2. *Assume that there are some constants $a, b \geq 0$, not both zero, such that*

$$(2.5) \quad \sum_{i=1}^n (Z - Z'_i)^2 \mathbb{1}_{(Z > Z'_i)} \leq aZ + b \quad \text{a.s.}$$

Then, we have the following one-sided deviation inequality:

$$(2.6) \quad \mathbb{P}(Z > \mathbb{E}Z + t) \leq \exp\left(-\frac{t^2}{4a\mathbb{E}Z + 4b + 2at}\right), \quad \forall t > 0.$$

Proof of Theorem 2.1. We shall in fact use a very special case of Theorem 2.2, the case when $a = 0$; but, as the three before-named authors remarked, this special case is already very useful and far from trivial to prove! To prove (2.4), we are going to check that, for $f(X_1, \dots, X_n) = \|\sum_1^n X_j v_j\| = Z$, the assumption (2.5) holds for $a = 0$ and $b = 4\sigma^2$. In fact, fix $\omega \in \Omega$ and denote by $I = I_\omega$ the set of indices i such that $Z(\omega) > Z'_i(\omega)$. For simplicity of notation, we assume that the Banach space E is real. Let $\varphi = \varphi_\omega \in E^*$ such that $\|\varphi\| = 1$ and $Z = \varphi(\sum_{j=1}^n X_j v_j) = \sum_{j=1}^n X_j \varphi(v_j)$.

For $i \in I$, we have $Z'_i(\omega) = Z'_i \geq \varphi(\sum_{j \neq i} X_j v_j + X'_i v_i)$, so that $0 \leq Z - Z'_i \leq \sum_{j=1}^n X_j \varphi(v_j) - \sum_{j \neq i} X_j \varphi(v_j) - X'_i \varphi(v_i) = (X_i - X'_i) \varphi(v_i)$, implying $(Z - Z'_i)^2 \leq 4|\varphi(v_i)|^2$. By summing those inequalities, we get

$$\begin{aligned} \sum_{i=1}^n (Z - Z'_i)^2 \mathbb{1}_{(Z > Z'_i)} &= \sum_{i \in I} (Z - Z'_i)^2 \leq 4 \sum_{i \in I} |\varphi(v_i)|^2 \leq 4 \sum_{i=1}^n |\varphi(v_i)|^2 \leq 4\sigma^2 \\ &= 0 \cdot Z + 4\sigma^2. \end{aligned}$$

Let us observe the crucial role of the “conditioning” $Z > Z'_i$ when we want to check that (2.5) holds. Now, (2.4) is an immediate consequence of (2.6). \square

3. CONSTRUCTION OF 4/3-RIDER SETS

We first recall some notation of [9]. Again, Ψ_2 denotes the Orlicz function $\Psi_2(x) = e^{x^2} - 1$, and $\|\cdot\|_{\Psi_2}$ is the corresponding Luxemburg norm. If A is a finite subset of the integers, Ψ_A denotes the quantity $\|\sum_{n \in A} e_n\|_{\Psi_2}$, where $e_n(t) = e^{int}$, $t \in \mathbb{R}/2\pi\mathbb{Z} = \mathbb{T}$, and \mathbb{T} is equipped with its Haar measure m . Also, Λ will always be a subset of the positive integers \mathbb{N} . Recall that Λ is *uniformly distributed* if

the ergodic means $A_N(t) = \frac{1}{|\Lambda_N|} \sum_{n \in \Lambda_N} e_n(t)$ tend to zero as $N \rightarrow \infty$, for each $t \in \mathbb{T}$, $t \neq 0$. Here, $\Lambda_N = \Lambda \cap [1, N]$. If Λ is uniformly distributed, \mathcal{C}_Λ contains c_0 , and if \mathcal{C}_Λ contains c_0 , Λ cannot be a Rosenthal set (see [9]). According to results of J. Bourgain (see [9]) and F. Lust-Piquard ([10]), respectively, a random set Λ corresponding to selectors of mean δ_k with $k\delta_k \rightarrow \infty$ is almost surely uniformly distributed and if a subset E of a uniformly distributed set Λ has positive upper density in Λ , *i.e.* if $\limsup_N \frac{|E \cap [1, N]|}{|\Lambda \cap [1, N]|} > 0$, then \mathcal{C}_E contains c_0 and E is non-Rosenthal.

In [9], we gave a fairly complicated proof of the following theorem (labelled as Theorem II.7):

Theorem 3.1. *There exists a subset Λ of the integers which is uniformly distributed and contains a subset E of positive integers with the following properties:*

- 1) E is a $\frac{4}{3}$ -Rider set, but it is not q -Rider for $q < 4/3$, a UC-set, and a $\Lambda(q)$ -set for all $q < \infty$;
- 2) E is of positive upper density inside Λ ; in particular, \mathcal{C}_E contains c_0 and E is not a Rosenthal set.

We shall show here that the use of Theorem 2.1 allows a substantially simplified proof, which avoids a double random selection. We first need the following simple lemma.

Lemma 3.2. *Let A be a finite subset of the integers, of cardinality $n \geq 2$, let $v = (e_j)_{j \in A}$, considered as an n -tuple of elements of the Banach space $E = L^{\Psi_2} = L^{\Psi_2}(\mathbb{T}, m)$, and let σ be its weak l_2 -norm. Then*

$$(3.1) \quad \sigma \leq C_0 \sqrt{\frac{n}{\log n}},$$

where C_0 is a numerical constant.

Proof. Let $a = (a_j)_{j \in A}$ be such that $\sum_{j \in A} |a_j|^2 = 1$. Let $f = f_a = \sum_{j \in A} a_j e_j$ and $M = \|f\|_\infty$. By Hölder's inequality, we have $\frac{\|f\|_p}{\sqrt{p}} \leq \frac{M}{\sqrt{p} M^{2/p}}$ for $2 < p < \infty$. Since $M \leq \sqrt{n}$, we get $\frac{\|f\|_p}{\sqrt{p}} \leq \frac{\sqrt{n}}{\sqrt{p} n^{1/p}} \leq C \sqrt{\frac{n}{\log n}}$. By Stirling's formula, $\|f\|_{\Psi_2} \approx \sup_{p>2} \frac{\|f\|_p}{\sqrt{p}}$, so the lemma is proved, since $\sigma = \sup_a \|f_a\|_{\Psi_2}$. \square

We now turn to the shortened proof of Theorem 3.1.

Let $I_n = [2^n, 2^{n+1}[$, $n \geq 2$ and let $\delta_k = c \frac{n}{2^n}$ if $k \in I_n$ ($c > 0$).

Let $(\varepsilon_k)_k$ be a sequence of "selectors", *i.e.* independent, $(0, 1)$ -valued, random variables of expectation $\mathbb{E} \varepsilon_k = \delta_k$, and let $\Lambda = \Lambda(\omega)$ be the random set of positive integers defined by $\Lambda = \{k \geq 1; \varepsilon_k = 1\}$. We also set $\Lambda_n = \Lambda \cap I_n$ and $\sigma_n = \mathbb{E} |\Lambda_n| = \sum_{k \in I_n} \delta_k = cn$.

We shall now need the following lemma (the notation Ψ_A is defined at the beginning of the section).

Lemma 3.3. *Almost surely, for n large enough,*

$$(3.2) \quad \frac{c}{2} n \leq |\Lambda_n| \leq 2cn,$$

$$(3.3) \quad \Psi_{\Lambda_n} \leq C'' |\Lambda_n|^{1/2}.$$

Proof. Inequality (3.2) is the easier part of Lemma II.9 in [9]. To prove (3.3), we recall an inequality due to G. Pisier [11]: if (X_k) is a sequence of independent,

centered and square-integrable, random variables of respective variances $V(X_k)$, we have

$$(3.4) \quad \mathbb{E} \left\| \sum_k X_k e_k \right\|_{\Psi_2} \leq C_1 \left(\sum_k V(X_k) \right)^{1/2}.$$

Applying (3.4) to the centered variables $X_k = \varepsilon_k - \delta_k$, we get, assuming $c \leq 1$,

$$\mathbb{E} \left\| \sum_{k \in I_n} (\varepsilon_k - \delta_k) e_k \right\|_{\Psi_2} \leq C_1 \left(\sum_{k \in I_n} \delta_k (1 - \delta_k) \right)^{1/2} \leq C_1 \left(\sum_{k \in I_n} \delta_k \right)^{1/2} \leq C_1 \sqrt{n}.$$

Now, set $Z_n = \left\| \sum_{k \in I_n} (\varepsilon_k - \delta_k) e_k \right\|_{\Psi_2}$. Let λ be a fixed real number > 1 , and let C_0 be as in Lemma 3.2. Applying Theorem 2.1 with $C = 32$ and setting $t_n = \lambda \sqrt{32 C_0^2 n}$, we get, using Lemma 3.2,

$$\mathbb{P}(Z_n - \mathbb{E} Z_n > t_n) \leq \exp\left(-\frac{t_n^2}{32\sigma^2}\right) \leq \exp\left(-\frac{32\lambda^2 C_0^2 n \log n}{32 C_0^2 n}\right) = n^{-\lambda^2}.$$

By the Borel-Cantelli Lemma, we have almost surely, for n large enough,

$$Z_n \leq \mathbb{E} Z_n + t_n \leq (C_1 + 4C_0\lambda)\sqrt{n} = C_2\sqrt{n}.$$

For such ω 's and n 's, it follows that

$$\begin{aligned} \Psi_{\Lambda_n} &= \left\| \sum_{k \in I_n} \varepsilon_k e_k \right\|_{\Psi_2} \leq Z_n + \left\| \sum_{k \in I_n} \delta_k e_k \right\|_{\Psi_2} \leq Z_n + \frac{n}{2^n} \left\| \sum_{k \in I_n} e_k \right\|_{\Psi_2} \\ &\leq C_2\sqrt{n} + \frac{n}{2^n} C_0 \frac{2^n}{\sqrt{\log 2^n}} =: C_3\sqrt{n}, \end{aligned}$$

because, with the notation of Lemma 3.2, we have

$$\left\| \sum_{k \in I_n} e_k \right\|_{\Psi_2} \leq \sqrt{|I_n|} \sigma \leq 2^{n/2} C_0 \frac{2^{n/2}}{\sqrt{\log 2^n}}.$$

This ends the proof of Lemma 3.3, because we know that $n \leq \frac{2}{c} |\Lambda_n|$ for large n , almost surely, and therefore $\Psi_{\Lambda_n} \leq C_3 \sqrt{\frac{2}{c} |\Lambda_n|}^{1/2} =: c'' |\Lambda_n|^{1/2}$, *a.s.* \square

We now prove Theorem 3.1 as follows: let us fix a point $\omega \in \Omega$ in such a way that $\Lambda = \Lambda(\omega)$ is uniformly distributed and that Λ_n verifies (3.2) and (3.3) for $n \geq n_0$; this is possible from [9] and from Lemma 3.3. We then use a result of the third author ([13]), asserting that there is a numerical constant $\delta > 0$ such that each finite subset A of \mathbb{Z}^* contains a quasi-independent subset B such that $|B| \geq \delta \left(\frac{|A|}{\Psi_A}\right)^2$ (recall that a subset Q of \mathbb{Z} is said to be quasi-independent if, whenever $n_1, \dots, n_k \in Q$, the equality $\sum_{j=1}^k \theta_j n_j = 0$ with $\theta_j = 0, -1, +1$ holds only when $\theta_j = 0$ for all j). This allows us to select inside each Λ_n a quasi-independent subset E_n such that

$$(3.5) \quad |E_n| \geq \delta \left(\frac{|\Lambda_n|}{\Psi_{\Lambda_n}}\right)^2 \geq \frac{\delta}{c''^2} |\Lambda_n| =: \delta' |\Lambda_n|.$$

A combinatorial argument (see [9], pp. 128–129) shows that, if $E = \bigcup_{n > n_0} E_n$, then each finite $A \subset E$ contains a quasi-independent subset $B \subseteq A$ such that $|B| \geq \delta |A|^{1/2}$. By [13], E is a $\frac{4}{3}$ -Rider set. The set E has all the required properties. Indeed, it follows from (3.2) in Lemma 3.3 that $|E \cap [1, N]| \geq \delta (\log N)^2$. If now E is p -Rider, we must have $|E \cap [1, N]| \leq C (\log N)^{\frac{p}{2-p}}$; therefore $2 \leq \frac{p}{2-p}$,

so $p \geq 4/3$. The fact that E is both UC and $\Lambda(q)$ is due to the local character of these notions and to the fact that the sets $E \cap [2^n, 2^{n+1}[= E_n$ are by construction quasi-independent (as detailed in [9]). On the other hand, since each E_n is approximately proportional to Λ_n , E is of positive upper density in Λ . Now Λ is uniformly distributed (by Bourgain’s criterion; see [9], p. 115). Therefore, by the result of F. Lust-Piquard ([10] and see Theorem I.9, p. 114 in [9]), \mathcal{C}_E contains c_0 , which prevents E from being a Rosenthal set.

4. p -RIDER SETS, WITH $p > 4/3$, WHICH ARE NOT UC -SETS

Let $p \in]\frac{4}{3}, 2[$, so that $\alpha = \frac{2(p-1)}{2-p} > 1$. As we mentioned in the Introduction, the random set $\Lambda = \Lambda(\omega)$ of integers in Theorem II.10 of [9] corresponds to selectors ε_k with mean $\delta_k = c \frac{(\log k)^\alpha}{k(\log \log k)^{\alpha+1}}$. We shall prove the following:

Theorem 4.1. *The random set Λ corresponding to selectors of mean*

$$\delta_k = c \frac{(\log k)^\alpha}{k(\log \log k)^{\alpha+1}}$$

has almost surely the following properties:

- a) Λ is p -Rider, but it is q -Rider for no $q < p$.
- b) Λ is $\Lambda(q)$ for all $q < \infty$.
- c) Λ is uniformly distributed; in particular, it is dense in the Bohr group and \mathcal{C}_Λ contains c_0 .
- d) Λ is **not** a UC -set.

Remark. This supports the conjecture that p -Rider sets with $p > 4/3$ are not of the same nature as p -Rider sets for $p < 4/3$ (see also [4], Theorem 3.1. and [5]).

The novelty here is *d*), which answers in the negative a question of [9] and we shall mainly concentrate on it, although we shall add some details for *a*), *b*), *c*), since the proof of Theorem II.10 in [9] is too sketchy and contains two small misprints (namely (*) and (**), p. 130).

Recall that the UC -constant $U(E)$ of a set E of positive integers is the smallest constant M such that $\|S_N f\|_\infty \leq M \|f\|_\infty$ for every $f \in \mathcal{C}_E$ and every non-negative integer N , where $S_N f = \sum_{-N}^N \hat{f}(k) e_k$. We shall use the following result of Kashin and Tzafriri [3]:

Theorem 4.2. *Let $N \geq 1$ be an integer and let $\varepsilon'_1, \dots, \varepsilon'_N$ be selectors of equal mean δ . Set $\sigma(\omega) = \{k \leq N ; \varepsilon'_k(\omega) = 1\}$. Then*

$$(4.1) \quad \mathbb{P} \left(U(\sigma(\omega)) \leq \gamma \log \left(2 + \frac{\delta N}{\log N} \right) \right) \leq \frac{5}{N^3},$$

where γ is a positive numerical constant.

We now turn to the proof of Theorem 4.1. As in [9], we set, for a fixed $\beta > \alpha$,

$$(4.2) \quad M_n = n^{\beta n}, \quad \Lambda_n = \Lambda \cap [1, n], \quad \Lambda_n^* = \Lambda \cap [M_n, M_{n+1}[.$$

We need the following technical lemma, whose proof is postponed (and is needed only for Theorem 4.1 *a*), *b*), *c*)).

Lemma 4.3. *We have almost surely for large n*

$$(4.3) \quad |\Lambda_{M_n}| \approx n^{\alpha+1}, \quad |\Lambda_n^*| \approx n^\alpha.$$

Observe that, for $k \in \Lambda_n^*$, one has

$$\delta_k = c \frac{(\log k)^\alpha}{k(\log \log k)^{\alpha+1}} \gg \frac{(n \log n)^\alpha}{M_{n+1}(\log n)^{\alpha+1}} = \frac{n^\alpha}{M_{n+1} \log n} =: \frac{q_n}{N_n},$$

where $N_n = M_{n+1} - M_n$ is the number of elements of the support of Λ_n^* (note that $N_n \sim M_{n+1}$) and where q_n is such that

$$(4.4) \quad q_n \approx \frac{n^\alpha}{\log n}.$$

We can adjust the constants so as to have $\delta_k \geq q_n/N_n$ for $k \in \Lambda_n^*$. Now, we introduce selectors (ε_k'') independent of the ε_j 's, of respective means $\delta_k'' = q_n/(N_n \delta_k)$. Then the selectors $\varepsilon_k' = \varepsilon_k \varepsilon_k''$ have means $\delta_k' = q_n/N_n$ for $k \in \Lambda_n^*$, and we have $\delta_k \geq \delta_k'$ for each $k \geq 1$.

Let $\Lambda' = \{k; \varepsilon_k' = 1\}$ and $\Lambda_n' = \Lambda' \cap [M_n, M_{n+1}[$. It follows from (4.1) and the fact that $U(E+a) = U(E)$ for any set E of positive integers and any non-negative integer a that

$$\mathbb{P} \left(U(\Lambda_n') \leq \gamma \log \left(2 + \frac{q_n}{\log N_n} \right) \right) \leq 5N_n^{-3}.$$

By the Borel-Cantelli Lemma, we have almost surely $U(\Lambda_n') > \gamma \log \left(2 + \frac{q_n}{\log N_n} \right)$ for n large enough. But we see from (4.3) and (4.2) that:

$$\frac{q_n}{\log N_n} \approx \frac{n^\alpha}{(\log n)(n \log n)} = \frac{n^{\alpha-1}}{(\log n)^2},$$

and this tends to infinity since $\alpha > 1$. This shows that Λ' is almost surely non- UC . Also, due to the construction of the ε_k' 's, we have $\Lambda \supseteq \Lambda'$ almost surely. This of course implies that Λ is not a UC -set either (almost surely), ending the proof of d) in Theorem 4.1. \square

We now indicate a proof of the lemma. Almost surely, $|\Lambda_{M_n}|$ behaves for large n as

$$\begin{aligned} \mathbb{E} (|\Lambda_{M_n}|) &= \sum_1^{M_n} \frac{(\log k)^\alpha}{k(\log \log k)^{\alpha+1}} \approx \int_{e^2}^{M_n} \frac{(\log t)^\alpha}{t(\log \log t)^{\alpha+1}} dt \\ &= \int_2^{\log M_n} \frac{x^\alpha dx}{(\log x)^{\alpha+1}} \approx \frac{1}{(\log n)^{\alpha+1}} \int_2^{\log M_n} x^\alpha dx \approx \frac{(\log M_n)^{\alpha+1}}{(\log n)^{\alpha+1}} \approx n^{\alpha+1}. \end{aligned}$$

Similarly, $|\Lambda_n^*|$ behaves almost surely as

$$\begin{aligned} \int_{M_n}^{M_{n+1}} \frac{(\log t)^\alpha}{t(\log \log t)^{\alpha+1}} dt &= \int_{\log M_n}^{\log M_{n+1}} \frac{x^\alpha}{(\log x)^{\alpha+1}} dx \approx \frac{1}{(\log n)^{\alpha+1}} \int_{\log M_n}^{\log M_{n+1}} x^\alpha dx \\ &\approx \frac{1}{(\log n)^{\alpha+1}} (\log M_{n+1} - \log M_n) (\log M_n)^\alpha \\ &\approx \frac{1}{(\log n)^{\alpha+1}} \log n (n \log n)^\alpha \approx n^\alpha. \end{aligned}$$

To finish the proof, we shall use a lemma of [9] (recall that a *relation of length n* in $A \subseteq \mathbb{Z}^*$ is a $(-1, 0, +1)$ -valued sequence $(\theta_k)_{k \in A}$ such that $\sum_{k \in A} \theta_k k = 0$ and $\sum_{k \in A} |\theta_k| = n$):

Lemma 4.4. *Let $n \geq 2$ and M be integers. Set*

$$\Omega_n(M) = \{\omega \mid \Lambda(\omega) \cap [M, \infty[\text{ contains at least a relation of length } n\}.$$

Then

$$\mathbb{P} [\Omega_n(M)] \leq \frac{C^n}{n^n} \sum_{j>M} \delta_j^2 \sigma_j^{n-2},$$

where $\sigma_j = \delta_1 + \dots + \delta_j$ and C is a numerical constant.

In our case, with $M = M_n$, this lemma gives

$$\begin{aligned} \mathbb{P}[\Omega_n(M)] &\ll \frac{C^n}{n^n} \sum_{j>M} \frac{(\log j)^{2\alpha}}{j^2 (\log \log j)^{2\alpha+2}} \left[\frac{(\log j)^{\alpha+1}}{(\log \log j)^{\alpha+1}} \right]^{n-2} \\ &\ll \frac{C^n}{n^n} \int_M^\infty \frac{(\log t)^{(\alpha+1)n+2\alpha}}{(\log \log t)^{(\alpha+1)n+2\alpha+2} t^2} dt \end{aligned}$$

and an integration by parts (see [9], pp. 117–118) now gives

$$\begin{aligned} \mathbb{P}[\Omega_n(M)] &\ll \frac{C^n}{n^n} \frac{1}{M} \frac{(\log M)^{(\alpha+1)n+2\alpha}}{(\log \log M)^{(\alpha+1)n+2\alpha+2}} \\ &\ll \frac{C^n}{n^n} \frac{1}{n^{\beta n}} \frac{(n \log n)^{(\alpha+1)n+2\alpha}}{(\log n)^{(\alpha+1)n+2\alpha+2}} \ll \frac{n^{2\alpha} C^n}{n^{(\beta-\alpha)n} (\log n)^2}; \end{aligned}$$

then the assumption $\beta > \alpha$ (which reveals its importance here!) shows that $\sum_n \mathbb{P}[\Omega_n(M_n)] < \infty$, so that, almost surely $\Lambda(\omega) \cap [M_n, \infty[$ contains no relation of length n , for $n \geq n_0$. Having this property at our disposal, we prove (exactly as in [9], pp. 119–120) that Λ is p -Rider. It is not q -Rider for $q < p$, because then $|\Lambda_{M_n}| \ll (\log M_n)^{\frac{q}{2-q}} \ll (n \log n)^{\frac{q}{2-q}}$, whereas (4.3) of Lemma 4.3 shows that $|\Lambda_{M_n}| \gg n^{\alpha+1}$, with $\alpha + 1 = \frac{p}{2-p} > \frac{q}{2-q}$. This proves *a*). Conditions *b*) and *c*) are clearly explained in [9]. \square

REFERENCES

- [1] S. BOUCHERON, G. LUGOSI, P. MASSART, *Concentration inequalities using the entropy method*, Annals of Probability 31 (3) (2003), 1583–1614. MR1989444 (2004i:60023)
- [2] G.B. JOHNSON, G. SCHECHTMAN, *Remarks on Talagrand's deviation inequality for Rademacher functions*, Texas Functional Analysis Seminar 1988-1989. The University of Texas (1989). Lecture Notes in Math., 1470, Springer, Berlin, 1991. MR1126739 (92m:60017)
- [3] B.S. KASHIN, L.A. TZAFRIRI, *Random sets of uniform convergence*, Mat. Zametki (Math. Notes) 54 (1) (1993), 17–33 [677–687]; *Letter to the Editor*, Math. Notes 58 (4) (1995), 1124–1125. MR1244478 (95c:42009); MR1378341
- [4] P. LEFÈVRE, D. LI, H. QUEFFÉLEC, L. RODRÍGUEZ-PIAZZA, *Lacunary sets and function spaces with finite cotype*, J. Funct. Anal. 188 (2002), 272–291. MR1878638 (2002j:42033)
- [5] P. LEFÈVRE, L. RODRÍGUEZ-PIAZZA, *p -Rider sets are q -Sidon sets*, Proc. Amer. Math. Soc. 131 (2002), 1829–1838. MR1955271 (2004c:43007)
- [6] M. LEDOUX, M. TALAGRAND, *Probability in Banach spaces*, Ergeb. der Math. 23, Springer (1991). MR1102015 (93c:60001)
- [7] D. LI, *A remark about $\Lambda(p)$ -sets and Rosenthal sets*, Proc. Amer. Math. Soc. 126 (1998), 3329–3333. MR1459133 (99a:43005)
- [8] D. LI, H. QUEFFÉLEC, *Introduction à l'Étude des espaces de Banach*, Analyse et probabilités. Cours Spécialisés 12, Société Mathématique de France (2004). MR2124356 (2006c:46012)
- [9] D. LI, H. QUEFFÉLEC, L. RODRÍGUEZ-PIAZZA, *Some new thin sets of integers in Harmonic Analysis*, Journal d'Analyse Mathématique 86 (2002), 105–138. MR1894479 (2003e:43007)
- [10] F. LUST-PIQUARD, *Bohr local properties of $C_\Lambda(\mathbb{T})$* , Colloq. Math. 58 (1989), 29–38. MR1028158 (91c:43009)
- [11] G. PISIER, *Sur l'espace de Banach des séries de Fourier aléatoires presque sûrement continues*, Séminaire sur la géométrie des espaces de Banach 1977-1978, Ecole Polytechnique Paris (1978), exposés 17–18. MR520216 (80j:60054)
- [12] D. RIDER, *Randomly continuous functions and Sidon sets*, Duke Math. J. 42 (1975), 759–764. MR0387965 (52:8802)
- [13] L. RODRÍGUEZ-PIAZZA, *Caractérisation des ensembles p -Sidon $p.s.$* , C.R.A.S. Paris Sér. I Math. 305 (1987), 237–240. MR907951 (88h:43005)

- [14] H.P. ROSENTHAL, *On Trigonometric Series Associated with Weak* closed Subspaces of Continuous Functions*, J. Math. and Mech. 17 (1967), 485–490. MR0216229 (35:7064)
- [15] W. RUDIN, *Trigonometric series with gaps*, J. Math. and Mech. 9 (1960), 203–228. MR0116177 (22:6972)

UNIVERSITÉ D'ARTOIS, LABORATOIRE DE MATHÉMATIQUES DE LENS EA 2462–FR 2956, FACULTÉ DES SCIENCES JEAN PERRIN, 23, RUE J. SOUVRAZ SP 18, F-62307 LENS CEDEX, FRANCE
E-mail address: `daniel.li@euler.univ-artois.fr`

LABORATOIRE PAUL PAINLEVÉ UMR CNRS 8524, U.F.R. DE MATHÉMATIQUES PURES ET APPLIQUÉES, BÂT. M2, UNIVERSITÉ DES SCIENCES ET TECHNOLOGIES DE LILLE 1, F-59665 VILLENEUVE D'ASCQ CEDEX, FRANCE
E-mail address: `Herve.Queffelec@math.univ-lille1.fr`

UNIVERSIDAD DE SEVILLA, FACULTAD DE MATEMÁTICAS, DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, APARTADO DE CORREOS 1160, 41080 SEVILLA, SPAIN
E-mail address: `piazza@us.es`