

UNIQUENESS OF E_∞ STRUCTURES FOR CONNECTIVE COVERS

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ABSTRACT. We refine our earlier work on the existence and uniqueness of E_∞ structures on K -theoretic spectra to show that the connective versions of real and complex K -theory as well as the connective Adams summand ℓ at each prime p have unique structures as commutative \mathbb{S} -algebras. For the p -completion ℓ_p we show that the McClure-Staffeldt model for ℓ_p is equivalent as an E_∞ ring spectrum to the connective cover of the periodic Adams summand L_p . We establish a Bousfield equivalence between the connective cover of the Lubin-Tate spectrum E_n and $BP\langle n \rangle$.

INTRODUCTION

The aim of this short note is to establish the uniqueness of E_∞ structures on connective covers of certain periodic commutative \mathbb{S} -algebras E , most prominently for the connective p -complete Adams summand. It is clear that the connective cover of an E_∞ ring spectrum inherits an E_∞ structure; there is even a *functorial* way of assigning a connective cover within the category of E_∞ ring spectra [10, VII.4.3]. But it is not obvious in general that this E_∞ multiplication is unique.

Our main concern is with examples in the vicinity of K -theory; we apply our uniqueness theorem to real and complex K -theory and their localizations and completions and to the Adams summand and its completions.

The existence and uniqueness of E_∞ structures on the periodic spectra KU , KO and L was established in [5] by means of the obstruction theory for E_∞ structures developed by Goerss and Hopkins [9] and Robinson [13]. Note however, that obstruction-theoretic methods would fail in the connective cases. Let e be a commutative ring spectrum. If e satisfies some Künneth and universal coefficient properties [13, proposition 5.4], then the obstruction groups for E_∞ multiplications consist of André-Quillen cohomology groups in the context of differential graded E_∞ algebras applied to the graded commutative e_* -algebra e_*e . Besides problems with non-projectivity of e_*e over e_* , the algebra structures of ku_*ku , ko_*ko and $\ell_*\ell$ are far from being étale, and therefore one would obtain nontrivial obstruction groups. One would then have to identify actual obstruction classes in these obstruction groups in order to establish the uniqueness of the given E_∞ structure,

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but at the moment, this seems to be an intractable problem. Thus an alternative approach is called for.

In Theorem 1.3 we prove that a unique E_∞ structure on E gives rise to a unique structure on the connective cover for spectra E which are obtained from some connective spectrum via a process of Bousfield localization and which satisfy some mild homotopical conditions (see Assumption 1.2). In particular, we identify the E_∞ structure on the p -completed connective Adams summand ℓ_p provided by McClure and Staffeldt in [11] with the one that arises by taking the unique E_∞ structure on the periodic Adams summand $L = E(1)$ developed in [5] and taking its connective cover.

Our theorem applies as well to the connective covers of the Lubin-Tate spectra E_n , and we prove in section 3 that these spectra are Bousfield equivalent to the truncated Brown-Peterson spectra $BP\langle n \rangle$. Unlike other spectra that are Bousfield equivalent to $BP\langle n \rangle$, such as the connective cover of the completed Johnson-Wilson spectrum, $\widehat{E\langle n \rangle}$, the connective cover of E_n is computationally convenient in the sense that its coefficients (see (3.1)) are rather small. So far, only $BP\langle 1 \rangle = \ell$ is known to have an E_∞ structure, and we propose the connective cover of E_n as an E_∞ approximation of $BP\langle n \rangle$.

1. E_∞ STRUCTURES ON CONNECTIVE COVERS

We will give some background on three standard constructions, which we will need later: namely, the connective covers functor, the bar construction that turns an E_∞ -ring spectrum into a commutative \mathbb{S} -algebra and Bousfield localization.

- May *et al.* constructed a connective cover functor in [10, VII.3.2], which we denote by $c(-)$. For every E_∞ ring spectrum E , $c(E)$ is a connective E_∞ ring spectrum E which depends functorially on E and which comes with a morphism of E_∞ ring spectra $c(E) \xrightarrow{\varepsilon} E$ [10, VII.4.3]. This map induces an isomorphism on homotopy groups in nonnegative degrees.
- For any E_∞ ring spectrum E , there is a weakly equivalent commutative \mathbb{S} -algebra $B(E)$, with an equivalence

$$\lambda: B(E) \xrightarrow{\rho} E,$$

in the E_∞ category [8, II.3.6].

- For a commutative \mathbb{S} -algebra R and an R -module M , we let $L_M^R(-)$ denote Bousfield localization at M in the category of R -modules and we denote the localization map by $\sigma: E \rightarrow L_M^R(E)$ for any R -module E [8, chapter VIII]. The case that will be most relevant to us is $M = R[X^{-1}]$, where an element $X \in \pi_* R$ is inverted. The case $R = \mathbb{S}$ corresponds to ordinary Bousfield localization.

A ring spectrum for us is a homotopy notion, *i.e.*, it is an object in the homotopy category of spectra with a monoid structure in the homotopy category. We will use the following notion of uniqueness for E_∞ structures.

Definition 1.1. An E_∞ structure on some homotopy commutative and associative ring spectrum E is unique if the following holds. For every E_∞ ring spectrum E' and every map of ring spectra $\varphi: E' \rightarrow E$ that induces an isomorphism on homotopy groups, there is a morphism in the homotopy category of E_∞ ring spectra $\varphi': E' \rightarrow E$ such that φ' is homotopic to φ as maps of ring spectra.

In the appendix to this paper, we relate this notion to the one used in obstruction theory.

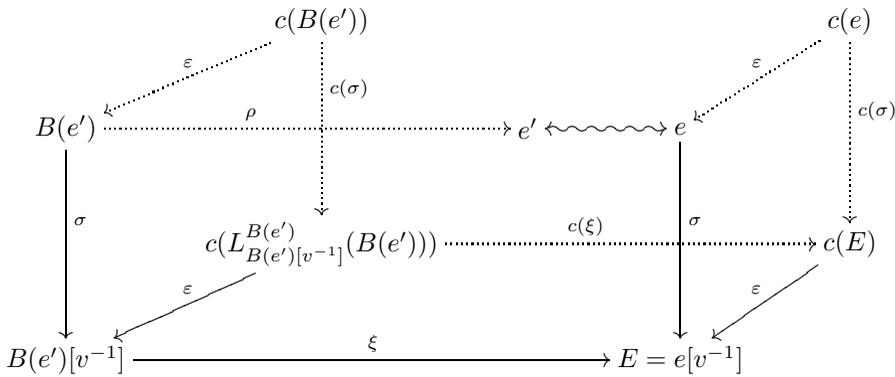
For the rest of the paper we assume the following.

Assumption 1.2. Let E be a periodic commutative \mathbb{S} -algebra with periodicity element $v \in E_*$ of positive degree. We assume that E is obtained from a connective commutative \mathbb{S} -algebra e by Bousfield localization at $e[v^{-1}]$ in the category of e -modules, *i.e.*, $E = L_{e[v^{-1}]}^e(e) = e[v^{-1}]$. Furthermore we assume that the localization map induces an isomorphism between the homotopy groups of e and the homotopy groups of the connective cover $c(E)$ of E and that E satisfies a universal coefficient theorem.

Theorem 1.3. *Assume that we know that the E_∞ structure on E is unique. Then the E_∞ structure on $c(E)$ is unique.*

Proof. Each commutative \mathbb{S} -algebra can be viewed as an E_∞ ring spectrum. Let e' be a model for the connective cover $c(E)$, *i.e.*, e' is an E_∞ ring spectrum with a map of ring spectra φ to $c(E)$, such that $\pi_*(\varphi)$ is an isomorphism. Write $v \in e'_*$ for the isomorphic image of v under the inverse of $\pi_*(\varphi)$. The universal property of $e[v^{-1}]$ [8, V.1.13] asserts that the ring map $\varepsilon \circ \varphi \circ \rho : B(e') \rightarrow E = e[v^{-1}]$ gives rise to a map from $B(e')[v^{-1}]$ to E . Due to the presence of a universal coefficient theorem for E , the homotopy classes of maps from $B(e')[v^{-1}]$ to E are in bijective correspondence to the E_* -module maps from $E_*(B(e')[v^{-1}])$ to E_* . As we started with a ring map φ , we obtain a map of E_* -algebras from $E_*(B(e')[v^{-1}])$ to E_* , and this corresponds to a map of ring spectra from $B(e')[v^{-1}]$ to E .

As the E_∞ structure on E is unique by assumption, this ring map can be replaced by an equivalent equivalence, ξ , of E_∞ ring spectra. We consider the following diagram whose dotted lines provide a zigzag of E_∞ equivalences and hence a map in the homotopy category of E_∞ ring spectra.



□

Real and complex K -theory, ko and ku , have E_∞ structures obtained using algebraic K -theory models [10, VIII, §2]. The connective Adams summand ℓ has an E_∞ structure because it is the connective cover of the Johnson-Wilson spectrum $E(1)$ with $E(1)_* = \mathbb{Z}_{(p)}[v_1^\pm]$, $|v_1| = 2p - 2$. In the following we will refer to these models as the standard ones. The E_∞ structures on KO , KU and $E(1)$ are unique by [5, theorems 7.2, 6.2]. In all of these cases, the periodic versions are obtained by Bousfield localization [8, VIII.4.3].

Corollary 1.4. *The E_∞ structures on ko , ku and ℓ are unique.*

2. THE p -COMPLETE CONNECTIVE ADAMS SUMMAND

In [11], McClure and Staffeldt construct a model for the p -completed connective Adams summand using the algebraic K -theory of fields. Let $\tilde{\ell} = K(\mathbf{k}')$, the algebraic K -theory spectrum of $\mathbf{k}' = \bigcup_i \mathbb{F}_{q^{p^i}}$, where q is a prime which generates the p -adic units \mathbb{Z}_p^\times . Then the p -completion of $\tilde{\ell}$ is additively equivalent to the p -completed connective Adams summand ℓ_p [11, proposition 9.2]. For further details, see also [2, §2]. Note that the p -completion ℓ_p inherits an E_∞ structure from ℓ because p -completion is Bousfield localization with respect to the mod p Moore spectrum [7, proposition 2.5] and therefore preserves commutative \mathbb{S} -algebra structures [8, VIII.2.2].

An *a priori* different model for the p -completion of the connective Adams summand can be obtained by taking the connective cover of the p -complete periodic version $L = E(1)$. For the following we denote the composition $B \circ c$ by \bar{c} .

Note that p -completion and Bousfield localization of ℓ in the category of ℓ -modules with respect to L are compatible in the following sense. Consider $\ell = \bar{c}(L)$ and its p -completion

$$\lambda_\ell: \ell \longrightarrow \ell_p = (\bar{c}(L))_p.$$

The p -completion map λ is functorial in the spectrum; therefore the following diagram of solid arrows commutes.

$$\begin{array}{ccccc} \ell = \bar{c}(L) & \xrightarrow{\lambda_\ell} & \ell_p = \bar{c}(L)_p & \cdots \cdots \cdots & \bar{c}(L_p) \\ & \searrow & \swarrow & & \swarrow \\ & & L & \xrightarrow{\lambda_L} & L_p \end{array}$$

The universal property of the connective cover functor ensures that there is a map in the homotopy category of commutative \mathbb{S} -algebras from ℓ_p to $\bar{c}(L_p)$ which is a weak equivalence. In the following we will no longer distinguish ℓ_p from $\bar{c}(L_p)$ and will denote this model simply by ℓ_p .

Proposition 2.1. *The McClure-Staffeldt model $\tilde{\ell}_p$ of the p -complete connective Adams summand is equivalent as an E_∞ ring spectrum to ℓ_p .*

Remark 2.2. If E is a commutative \mathbb{S} -algebra with naive G -action for some group G , then neither the connective cover functor $\bar{c}(-)$ nor Bousfield localization of E has to commute with taking homotopy fixed points. As an example, consider connective complex K -theory ku with the conjugation action by C_2 . The homotopy fixed points ku^{hC_2} are not equivalent to ko , but on the periodic versions we obtain $KU^{hC_2} \simeq KO$.

Proof of Proposition 2.1. Consider the algebraic K -theory model for connective complex K -theory, $ku = K(\mathbf{k})$, with $\mathbf{k} = \bigcup_i \mathbb{F}_{q^{p^i(p-1)}}$. The canonical inclusions $\mathbb{F}_{q^{p^i}} \hookrightarrow \mathbb{F}_{q^{p^i(p-1)}}$ assemble into a map $j: \mathbf{k}' \longrightarrow \mathbf{k}$. The Galois group C_{p-1} of \mathbf{k} over \mathbf{k}' acts on \mathbf{k} and induces an action on algebraic K -theory. As \mathbf{k}' is fixed under the

action of C_{p-1} there is a factorization of $K(j)_p$ as

$$\begin{array}{ccc}
 K(\mathbf{k}')_p & \xrightarrow{K(j)_p} & K(\mathbf{k})_p \\
 & \searrow i & \nearrow \\
 & K(\mathbf{k})_p^{hC_{p-1}} &
 \end{array}$$

and i yields a weak equivalence of commutative \mathbb{S} -algebras, where $K(\mathbf{k})_p^{hC_{p-1}}$ is a model for the connective p -complete Adams summand which is weakly equivalent to $\tilde{\ell}_p$ (see [2, §2]).

Consider the composition of the following chain of maps between commutative \mathbb{S} -algebras:

$$K(\mathbf{k}')_p \xrightarrow{i} (K(\mathbf{k})_p)^{hC_{p-1}} \longrightarrow K(\mathbf{k})_p \longrightarrow KU_p.$$

The target KU_p is as well the target of the map $\bar{c}(KU_p) \longrightarrow KU_p$. Note that the universal property of $\bar{c}(-)$ yields a zigzag $\zeta: K(\mathbf{k})_p \rightleftarrows \bar{c}(KU_p)$ of equivalences between $K(\mathbf{k})_p$ and $\bar{c}(KU_p)$ in the category of commutative \mathbb{S} -algebras.

As KU_p is the Bousfield localization of $K(\mathbf{k})_p$ in the category of $K(\mathbf{k})_p$ -modules with respect to the Bott element,

$$KU_p = L_{K(\mathbf{k})_p^{[\beta-1]}}^{K(\mathbf{k})_p} K(\mathbf{k})_p,$$

it inherits the C_{p-1} -action on $K(\mathbf{k})_p$. The functoriality of the connective cover lifts this action to an action on $\bar{c}(KU_p)$.

The connective cover functor is in fact a functor in the category of commutative \mathbb{S} -algebras with multiplicative naive G -action for any group G . To see this we have to show that the map $\bar{c}(A) \longrightarrow A$ is G -equivariant if A is a commutative \mathbb{S} -algebra with an underlying naive G -spectrum. The functor $B(-)$ does not cause any problems. Proving the claim for the functor c involves chasing the definition given in [10, VII, §3].

The prespectrum underlying $c(A)$ applied to an inner product space V is defined as $T(A_0)(V)$, where A_0 is the zeroth space of the spectrum A and T is a certain bar construction involving suspensions and a monad consisting of the product of a fixed E_∞ operad with the partial operad of little convex bodies \mathcal{K} . For a fixed V the suspension Σ^V and the operadic term \mathcal{K}_V are used. As the G -action is compatible with the E_∞ and the additive structure of A , the evaluation map $T(A_0)(V) \longrightarrow A(V)$ is G -equivariant. For varying V , these maps constitute a map of prespectra and its adjoint on the level of spectra is $c(A) \longrightarrow A$. As the spectrification functor preserves G -equivariance, the claim follows. Therefore the resulting zigzag $\zeta: K(\mathbf{k})_p \rightleftarrows \bar{c}(KU_p)$ is C_{p-1} -equivariant, and we obtain an induced zigzag on homotopy fixed points,

$$\zeta^{hC_{p-1}}: (K(\mathbf{k})_p)^{hC_{p-1}} \rightleftarrows (\bar{c}(KU_p))^{hC_{p-1}}.$$

As ζ is an isomorphism in the homotopy category and is C_{p-1} -equivariant, $\zeta^{hC_{p-1}}$ yields an isomorphism as well. □

3. CONNECTIVE LUBIN-TATE SPECTRA

Goerss and Hopkins proved in [9] that the Lubin-Tate spectra E_n with

$$(E_n)_* = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][[u^{\pm 1}]] \quad \text{with } |u_i| = 0 \text{ and } |u| = -2$$

possess unique E_∞ structures for all primes p and all $n \geq 1$. The connective cover $c(E_n)$ has coefficients

$$(3.1) \quad (c(E_n))_* = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] [u^{-1}] \quad \text{with } |u_i| = 0 \text{ and } |u| = -2.$$

Of course $\bar{c}(E_n)[(u^{-1})^{-1}] \sim E_n$.

The spectra $BP\langle n \rangle$ can be built from the Brown-Peterson spectrum BP by killing all generators of the form v_m with $m > n$ in $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$. Using for instance Angeltveit’s result [1, theorem 4.2], one can prove that the $BP\langle n \rangle$ are A_∞ spectra and from [4] it is known that this \mathbb{S} -algebra structure can be improved to an MU -algebra structure. On the other hand, Strickland showed in [14] that $BP\langle n \rangle$ with $n \geq 2$ is not a homotopy commutative MU -ring spectrum for $p = 2$. We offer $c(E_n)$ as a replacement for the p -completion $BP\langle n \rangle_p$ of $BP\langle n \rangle$.

We also need to recall that in the category of MU -modules, $E(n)$ is the Bousfield localization of $BP\langle n \rangle$ with respect to $BP\langle n \rangle[v_n^{-1}]$; hence by [8] it inherits the structure of an MU -algebra and the natural map $BP\langle n \rangle \rightarrow E(n)$ is a morphism of MU -algebras. Furthermore, the Bousfield localization of $E(n)$ with respect to the MU -algebra $K(n)$ is the I_n -adic completion $\widehat{E(n)}$, which was shown to be a commutative \mathbb{S} -algebra in [5], and the natural map $\widehat{E(n)} \rightarrow E_n$ is a morphism of commutative \mathbb{S} -algebras; see for example [6, example 2.2.9]. Thus there is a morphism of ring spectra $BP\langle n \rangle \rightarrow E_n$ that lifts to a map $BP\langle n \rangle \rightarrow c(E_n)$ in the homotopy category.

Proposition 3.1. *The spectra $BP\langle n \rangle$ and $BP\langle n \rangle_p$ are Bousfield equivalent to $c(E_n)$.*

Proof. On coefficients, we obtain a ring homomorphism $(BP\langle n \rangle_p)_* \rightarrow (c(E_n))_*$, which is given by

$$v_k \mapsto \begin{cases} u^{1-p^k} u_k & \text{for } 1 \leq k \leq n-1, \\ u^{1-p^n} & \text{for } k = n, \end{cases}$$

extending the natural inclusion of the p -adic integers $\mathbb{Z}_p = W(\mathbb{F}_p)$ into $W(\mathbb{F}_{p^n})$. This homomorphism is induced by a map of ring spectra.

Recall from [3] that $E(n)$ and $\widehat{E(n)}$ are Bousfield equivalent as \mathbb{S} -modules, and it follows that $\widehat{E_n}$ is Bousfield equivalent to these since it is a finite wedge of suspensions of $\widehat{E(n)}$.

If X is a p -local spectrum with torsion free homotopy groups, then its p -completion X_p is Bousfield equivalent to X , i.e., $\langle X_p \rangle = \langle X \rangle$. This follows using the cofibre triangles (in which $M(p)$ is the mod p Moore spectrum and the circled arrow indicates a map of degree one)

$$\begin{array}{ccc} X & \xrightarrow{p} & X \\ & \searrow \circlearrowleft & \swarrow \\ & X \wedge M(p) & \end{array} \qquad \begin{array}{ccc} X_p & \xrightarrow{p} & X_p \\ & \searrow \circlearrowleft & \swarrow \\ & X \wedge M(p) & \end{array}$$

together with the fact that the rationalization $p^{-1}X$ is a retract of $p^{-1}(X_p)$. In particular, we have $\langle BP\langle n \rangle_p \rangle = \langle BP\langle n \rangle \rangle$ and $\langle E(n)_p \rangle = \langle E(n) \rangle$.

From [12, theorem 2.1], the Bousfield class of $BP\langle n \rangle$ is

$$\langle BP\langle n \rangle \rangle = \langle E(n) \rangle \vee \langle H\mathbb{F}_p \rangle.$$

There is a cofibre triangle

$$\begin{array}{ccc} \Sigma^2 c(E_n) & \xrightarrow{u^{-1}} & c(E_n) \\ & \swarrow \circ \searrow & \swarrow \searrow \\ & HW(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] & \end{array}$$

in which $HW(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]]$ is the Eilenberg-Mac Lane spectrum. More generally we can construct a family of Eilenberg-Mac Lane spectra with $W(\mathbb{F}_{p^n})[[u_1, \dots, u_k]]$ as coefficients for $k = 0, \dots, n - 1$ which are related by cofibre triangles

$$\begin{array}{ccc} HW(\mathbb{F}_{p^n})[[u_1, \dots, u_k]] & \xrightarrow{u_k} & HW(\mathbb{F}_{p^n})[[u_1, \dots, u_k]] \\ & \swarrow \circ \searrow & \swarrow \searrow \\ & HW(\mathbb{F}_{p^n})[[u_1, \dots, u_{k-1}]] & \end{array}$$

such that for $k = 0$ we obtain $HW(\mathbb{F}_{p^n})$. With the help of these cofibre sequences we can identify $\langle HW(\mathbb{F}_{p^n})[[u_1, \dots, u_k]] \rangle$ with $\langle HW(\mathbb{F}_{p^n})[[u_1, \dots, u_{k-1}]] \rangle \vee \langle HW(\mathbb{F}_{p^n})[[u_1, \dots, u_k]][[u_k^{-1}]] \rangle$.

In general, if R is a commutative ring, then the ring of finite tailed Laurent series $R((x))$ is faithfully flat over R , and therefore we have

$$\langle HR((x)) \rangle = \langle HR \rangle.$$

Using this auxiliary fact we inductively get that

$$\langle HW(\mathbb{F}_{p^n})[[u_1, \dots, u_k]] \rangle = \langle HW(\mathbb{F}_{p^n})[[u_1, \dots, u_{k-1}]] \rangle.$$

This reduces the Bousfield class of $c(E_n)$ to $\langle E_n \rangle \vee \langle HW(\mathbb{F}_{p^n}) \rangle$. As $W(\mathbb{F}_{p^n})$ is a finitely generated free \mathbb{Z}_p -module and as $\langle H\mathbb{Z}_p \rangle = \langle H\mathbb{Q} \rangle \vee \langle H\mathbb{F}_p \rangle$, this leads to

$$\begin{aligned} \langle c(E_n) \rangle &= \langle E(n) \vee H\mathbb{Q} \vee H\mathbb{F}_p \rangle \\ &= \langle E(n) \vee H\mathbb{F}_p \rangle = \langle BP\langle n \rangle \rangle. \end{aligned} \quad \square$$

4. APPENDIX: UNIQUENESS AND E_∞ -MAPPING SPACES

If E and F are spectra whose E_∞ structure was provided by the obstruction theory of Goerss and Hopkins [9], then we can compare our uniqueness notion with theirs. Note that examples of such E_∞ ring spectra include ring spectra such as E_n [9, 7.6], KO , KU , L and $\widehat{E}(n)$ [5]. In such cases the Hurewicz map

$$(4.1) \quad \pi_0 \text{Hom}_{E_\infty}(E, F) \xrightarrow{h} \text{Hom}_{F_*\text{-alg}}(F_*E, F_*)$$

from the connected components of the derived space of E_∞ -maps between E and F and the F_* -algebra homomorphisms from F_*E to F_* is a bijection [9, corollary 4.4, theorem 4.5]. Assume that we have a mere ring map φ as above between E and F . This gives rise to a map of F_* -algebras from F_*E to F_* by composing $F_*(\varphi)$ with the multiplication μ in F_*F . In the presence of a universal coefficient theorem we have $\text{Hom}_{F_*\text{-hom}}(F_*E, F_*) = [E, F]$; therefore the element $\mu \circ F_*(\varphi)$ gives rise to a homotopy class of maps of ring spectra $\tilde{\varphi}$ from E to F . We can assume that we have a functorial cofibrant replacement $Q(-)$; hence we obtain a ring map $Q(\tilde{\varphi})$ from

$Q(E)$ to $Q(F)$. Via the isomorphism of (4.1) this gives a map Φ , of E_∞ spectra from $Q(E)$ to $Q(F)$; therefore we obtain a zigzag

$$\begin{array}{ccc} Q(E) & \xrightarrow{\Phi} & Q(F) \\ \downarrow \sim & & \downarrow \sim \\ E & \xrightarrow{\varphi} & F \end{array}$$

of weak equivalences of E_∞ spectra from E to F . Thus in such cases our definition agrees with the uniqueness notion that is natural in the Goerss–Hopkins setting.

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