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UNIQUENESS OF THE KONTSEVICH-VISHIK TRACE

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Dedicated to Boris V. Fedosov on the occasion of his 70th birthday

Abstract. Let M be a closed manifold. We show that the Kontsevich-Vishik trace, which is defined on the set of all classical pseudodifferential operators on M, whose (complex) order is not an integer greater than or equal to $-\dim M$, is the unique functional which (i) is linear on its domain, (ii) has the trace property and (iii) coincides with the L^2-operator trace on trace class operators.

Also the extension to even-even pseudodifferential operators of arbitrary integer order on odd-dimensional manifolds and to even-odd pseudodifferential operators of arbitrary integer order on even-dimensional manifolds is unique.

1. Introduction

We denote by M a compact n-dimensional manifold without boundary. A classical pseudodifferential operator (ψdo) A acting on sections of a vector bundle over M is said to have order $\mu \in \mathbb{C}$ if it belongs to the Hörmander class $S^{\operatorname{Re}\mu}_{1,0}(M)$ and the local symbols $a = a(x,\xi)$ of A have asymptotic expansions

$$a \sim \sum_{j=0}^{\infty} a_{\mu-j}, \tag{1.1}$$

where the $a_{\mu-j}$ are positively homogeneous of degree $\mu - j$ for large ξ. We shall write $\operatorname{ord} A = \mu$ to express that the order of A is μ.

In two remarkable papers, Kontsevich and Vishik in 1994 and 1995 analyzed the properties of determinants of elliptic ψdo's, [7], [8]. One important tool was the construction of a trace-like mapping TR defined on the set of all classical ψdo's whose order is not an element of $\mathbb{Z}_{\geq -n}$, the set of integers greater than or equal to $-n$; see also the Remarks, below.

We shall denote this domain by $\mathcal{D}$. As the sum of two operators of orders μ and μ' in $\mathcal{D}$ is an element of $\mathcal{D}$ only if $\mu - \mu'$ is an integer, $\mathcal{D}$ is not a vector space. Thus it does not make sense to speak about linear functionals on $\mathcal{D}$. The map $\mathrm{TR} : \mathcal{D} \to \mathbb{C}$, however, is as linear as it can be expected to be:

$$\mathrm{TR}(cA + dB) = c\,\mathrm{TR}(A) + d\,\mathrm{TR}(B) \quad \text{for } c, d \in \mathbb{C}, A, B, cA + dB \in \mathcal{D}. \tag{1.2}$$

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Moreover, TR behaves like a trace:

$$\mathrm{TR}(AB) = \mathrm{TR}(BA), \quad \text{whenever } AB, BA \in \mathcal{D}. \tag{1.3}$$

Finally, the Kontsevich-Vishik trace (sometimes also canonical trace) TR coincides with the L^2-operator trace Tr on trace class ψdo's:

$$\mathrm{TR}(A) = \mathrm{Tr}(A) \quad \text{if } \mathrm{Re}\,\mathrm{ord}(A) < -n. \tag{1.4}$$

It is clear that the Kontsevich-Vishik trace cannot be extended to a trace on the algebra of all ψdo's on M: The only trace there (up to multiples) is the Wodzicki residue [13], which is known to vanish on trace class operators. There is also a simple direct way to see this: We know – e.g. from the Atiyah-Singer index theorem – that there exists an elliptic pseudodifferential operator P on M with nonzero index. Using order reducing operators, we may assume the order of P to be zero. Let Q be a parametrix to P modulo smoothing operators. Then

$$\mathrm{Index}\, P = \mathrm{Tr}(1 - PQ) - \mathrm{Tr}(1 - QP).$$

If we could extend TR to a trace on all pseudodifferential operators, the right hand side could be rewritten as the trace of the commutator $[P, Q]$ and therefore would have to be zero – a contradiction.

It has been observed, however, by Kontsevich-Vishik and Grubb [4] that TR extends to a slightly larger domain. Recall that the symbol a of an integer order operator A is said to be even-even if the homogeneous components satisfy

$$a_{\mu-j}(x, -\xi) = (-1)^{\mu-j} a_{\mu-j}(x, \xi). \tag{1.5}$$

It is called even-odd, if

$$a_{\mu-j}(x, -\xi) = (-1)^{\mu-j+1} a_{\mu-j}(x, \xi). \tag{1.6}$$

The Kontsevich-Vishik trace $\mathrm{TR}(A)$ for a ψdo A of order μ then can also be defined if $\mu \in \mathbb{Z}_{\geq -n}$, provided that

(EE) n is odd, and the symbol of A is even-even, or
(EO) n is even, and the symbol of A is even-odd.

For the sake of brevity we shall denote this larger domain (depending on n) by $\mathcal{D}^+$.

In both cases, the component a_{-n} in the asymptotic expansion of the symbol of A is odd in ξ for large $|\xi|$, say for $|\xi| \geq 1$:

$$a_{-n}(x, -\xi) = -a_{-n}(x, \xi).$$

Hence the density for the Wodzicki residue of the operator A vanishes pointwise, i.e.

$$\mathrm{res}_x(A) = \int_{S_x^* M} \mathrm{tr}\, a_{-n}(x, \xi)\, d\sigma(\xi) = 0 \quad \text{for each } x \in M.$$

Here, $d\sigma$ is the surface measure on the unit sphere $S_x^* M$ over x in the cotangent bundle and tr is the fiber trace. The Wodzicki residue of A is given by integration of $\mathrm{res}_x A$ over M and therefore also vanishes.

The trace property (1.3) extends to the case where A and B have integer order and AB and BA belong to $\mathcal{D}^+$.

The Kontsevich-Vishik trace has received considerable attention and found interesting applications; see e.g. [5, 9, 10, 11, 12]. Moreover, it has been extended to boundary value problems in Boutet de Monvel's calculus [3].

It seems, however, that it has never been noticed that the above properties make the Kontsevich-Vishik trace unique. This is what we show in this short note. The proof, which will be given in the next section, relies on ideas in [2].

Theorem. (a) *Let* $\tau : \mathcal{D} \to \mathbb{C}$ *be a map with properties* (1.2), (1.3), *and* (1.4). *Then* $\tau = \mathrm{TR}$.

(b) *Also the extension of* τ *to* $\mathcal{D}^+$ *is unique. In fact,* τ *is already unique on the space of all integer order* ψ*do's which satisfy* (EE) *or* (EO) *when* $\mu \geq -n$.

2. Proof

In order to establish (a), choose a ψdo A of order $\mu \in \mathbb{C} \setminus \mathbb{Z}_{\geq -n}$ on M.

We find a covering of M by open neighborhoods and a finite subordinate partition of unity $\{\varphi_j\}$ such that for every pair (j,k), both φ_j and φ_k have support in one coordinate neighborhood. We write

$$A = \sum_{j,k} \varphi_j A \varphi_k.$$

Each operator $\varphi_j A \varphi_k$ may be considered a ψdo on $\mathbb{R}^n$. As the map τ has the linearity property (1.2), we may confine ourselves to the case where $A = \operatorname{op} a$ with a symbol a on $\mathbb{R}^n$ having an expansion (1.1). Moreover, we can assume that $A = \varphi A \psi$ whenever $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$ are equal to one on a sufficiently large set.

To simplify further, we write

$$A = \operatorname{op} a_\mu + \operatorname{op} a_{\mu-1} + \ldots + \operatorname{op} a_{\mu-K} + \operatorname{op} r, \tag{2.7}$$

where $a_{\mu-j}$ is a symbol on $\mathbb{R}^n$, homogeneous in ξ of degree $\mu - j$ for $|\xi| \geq 1$, and K is so large that $r \in S_{1,0}^{-n-1}$. For $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$ as above we then have

$$\tau(\operatorname{op} a) = \tau(\varphi \operatorname{op}(a)\psi) = \sum_{j=0}^{K} \tau(\varphi \operatorname{op}(a_{\mu-j})\psi) + \tau(\varphi \operatorname{op}(r)\psi).$$

Since $\tau(\varphi \operatorname{op}(r)\psi) = \operatorname{tr}(\varphi \operatorname{op}(r)\psi)$ by (1.4), we will know $\tau(\operatorname{op} a)$ as soon as we know $\tau(\varphi \operatorname{op}(a_{\mu-j})\psi)$ for $j = 0, \ldots, K$.

We may assume that μ is not an integer, since the operator trace determines τ on all operators of order $\mu < -n$. Now we let, similar to the proof of [2, Lemma 1.3(i)],

$$b_{\mu-j}(x,\xi) = \frac{1}{n+\mu-j} \sum_{k=1}^{n} \partial_{\xi_k}(\xi_k a_{\mu-j}(x,\xi)). \tag{2.8}$$

Euler's relation for homogenous functions implies that, for $|\xi| \geq 1$,

$$b_{\mu-j} = \frac{1}{n+\mu-j}\,(n a_{\mu-j} + (\mu-j)a_{\mu-j}) = a_{\mu-j}.$$

Hence we can write

$$\tau(\varphi \operatorname{op}(a_{\mu-j})\psi) = \tau(\varphi \operatorname{op}(a_{\mu-j} - b_{\mu-j})\psi) + \tau(\varphi \operatorname{op}(b_{\mu-j})\psi). \tag{2.9}$$

Since $a_{\mu-j} - b_{\mu-j}$ is regularizing, the first term on the right hand side is determined by property (1.4). Now we additionally choose $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi\varphi = \varphi$ and

$\chi\psi = \psi$. The fact that $\operatorname{op}(\partial_{\xi_k} p) = -i\,[x_k, \operatorname{op} p]$ for an arbitrary symbol p implies that

$$\varphi \operatorname{op}(b_{\mu-j})\psi = -i \sum_{k=1}^{n} [\chi x_k, \varphi \operatorname{op}(\xi_k a_{\mu-j})\psi].$$

Assuming that τ has property (1.3), it vanishes on the last term in (2.9) which is a sum of commutators. Hence the proof of (a) is complete.

Next let us show (b). With the same considerations as before we may assume that $A = \operatorname{op} a$ is a pseudodifferential operator on $\mathbb{R}^n$ with a representation as in (2.7), where now μ is an integer $\geq -n$ and the $a_{\mu-j}$ have property (EE) or (EO). We only have to show that $\tau(\varphi \operatorname{op}(a_{\mu-j})\psi)$ is uniquely determined, $j = 0, \dots, \mu+n$. For $\mu - j \neq -n$ the argument is as before, using the symbols in (2.8) and noting that $a_{\mu-j}(x,\xi)\xi_k$ is even-even or even-odd whenever this is the case for $a_{\mu-j}$.

So let us consider a_{-n}. Now we apply the technique used in the proof of [2, Lemma 1.3(ii)]. The assumption that n is odd and a_{-n} even-even or n is even and a_{-n} even-odd implies that a_{-n} is odd in ξ:

$$a_{-n}(x, -\xi) = -a_{-n}(x, \xi) \quad \text{for } |\xi| \geq 1.$$

Hence, for each fixed x, the integral over the unit sphere $S = \{|\xi| = 1\}$ vanishes:

$$\int_S a_{-n}(x,\xi)\, d\sigma(\xi) = 0. \tag{2.10}$$

The Laplace operator $\Delta = \sum_{k=1}^n \partial^2/\partial\xi_k^2$ in polar coordinates takes the form

$$\Delta = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_S,$$

where $r = |\xi|$ is the radial variable and Δ_S is the Laplace-Beltrami operator on S.

Equation (2.10) implies that, for each x, the function $a_{-n}(x,\cdot)|_S$ is orthogonal to the constants which form the kernel of the symmetric operator Δ_S. Hence there is a unique function $q(x,\cdot) \in C^\infty(S)$, orthogonal to the constants, such that $\Delta_S q(x,\cdot) = a_{-n}(x,\cdot)|_S$. As Δ_S commutes with the antipodal map $\eta \mapsto -\eta$, we have $\Delta_S(q(x,-\cdot)) = a_{-n}(x,-\cdot)|_S = -a_{-n}(x,\cdot)|_S$. Hence $q(x,\cdot) + q(x,-\cdot)$ belongs to the kernel of Δ_S, and thus is constant. On the other hand, both $q(x,\cdot)$ and $q(x,-\cdot)$ are orthogonal to the constants. Therefore $q(x,\cdot)+q(x,-\cdot)$ is zero, i.e., $q(x,\cdot)$ is an odd function on S.

Now we choose a smooth function ω on $\mathbb{R}$ which vanishes for small r and is equal to 1 for $r \geq 1/2$. We let

$$b_{-n} = \omega(r) r^{2-n} q = \omega(|\xi|)\, |\xi|^{2-n}\, q(x, \xi/|\xi|).$$

This is a smooth function on $\mathbb{R}^n$ which is homogeneous of degree $2-n$ in ξ for $|\xi| \geq 1$. As $a_{-n}(x,\xi)$ vanishes for x outside a compact set, so does $b_{-n}(x,\xi)$. In particular, b_{-n} is an element of $S^{2-n}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover, we have for $|\xi| \geq 1$

$$\Delta b_{-n} = \Delta(r^{2-n} q(x,\cdot)) = r^{-n} a_{-n}(x,\cdot)|_S = a_{-n}.$$

We write $a_{-n} = (a_{-n} - \Delta b_{-n}) + \Delta b_{-n}$. The symbol $a_{-n} - \Delta b_{-n}$ is regularizing and thus $\tau(\varphi \operatorname{op}(a_{-n} - \Delta b_{-n})\psi)$ is determined by (1.4). The operator associated with $\operatorname{op}(\varphi(\Delta b_{-n})\psi)$ on the other hand is a sum of commutators:

$$\varphi \operatorname{op}(\Delta b_{-n})\psi = -i \sum_{k=1}^{n} [\chi x_k, \varphi \operatorname{op}(\partial_{\xi_k} b_{-n})\psi], \tag{2.11}$$

where χ is chosen as in the proof of (a). Hence τ vanishes on $\varphi\,\mathrm{op}(\Delta b_{-n})\psi$. This concludes the argument.

Remarks. (a) One way of defining TR is as follows [7], [4]: Choose an invertible positive ψdo P of order $m > 0$ with scalar principal symbol and define the complex powers P^z, $z \in \mathbb{C}$. The generalized 'zeta function' $\zeta(A, P, z) = \mathrm{Tr}(AP^{-z})$ is holomorphic on $\{\mathrm{Re}\, z > (n+\mu)/m\}$, $\mu = \mathrm{ord}\, A$, and extends meromorphically to $\mathbb{C}$ with at most simple poles in the points $z_j = (n+\mu-j)/m$. If $\mu \notin \mathbb{Z}_{\geq -n}$ or if A satisfies (EE) or (EO), then there is no pole in $z = 0$, and $\mathrm{TR}(A) := \zeta(A; P, 0)$ is independent of P.

One can also define TR for $\mu \notin \mathbb{Z}_{\leq -n}$ by regularizing the integral $\int k(x,x)\,dx$ over the local distributional kernel of A, thus generalizing Lidskij's formula for trace class operators. In this spirit (and a more general framework) Connes and Moscovici prove another uniqueness result, [1, Lemma I.5]: TR is the unique holomorphic extension of the classical Lidskij formula through holomorphic families of ψdo's of noninteger order.

(b) For noninteger μ, the uniqueness of the Kontsevich-Vishik trace can also be derived from a result by Lesch, [9, Proposition 4.7], which implies that a ψdo A of order $\mu \notin \mathbb{Z}$ can be written in the form $A = \sum [P_j, Q_j] + R$ with finitely many ψdo's P_j, Q_j, and R of orders 1, μ, and $-\infty$, respectively. His proof relies on a construction by Guillemin [6, Theorem 6.2] which makes the argument less elementary than the one given here.

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