

ONLY ‘FREE’ MEASURES ARE ADMISSABLE ON $F(S)$ WHEN THE INNER PRODUCT SPACE S IS INCOMPLETE

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(Communicated by David Preiss)

ABSTRACT. Using elementary arguments and without having to recall the Gleason Theorem, we prove that the existence of a nonsingular measure on the lattice of orthogonally closed subspaces of an inner product space S is a sufficient (and of course, a necessary) condition for S to be a Hilbert space.

1. NOTIONS AND RESULTS

Incomplete inner product spaces are frequently encountered in mathematics. One can always complete these spaces to form a Hilbert space, but this is sometimes inconvenient since the result will contain elements outside the original space. For example, a maximal orthonormal system for an inner product space S need not be maximal in its completion. There are various properties that characterize completeness of inner product spaces. In [7] the completeness of an inner product space is characterized by its algebraic-topological properties. On the other hand, the Amemiya-Araki Theorem asserts that an inner product space S is complete if, and only if, the lattice of orthogonally closed subspaces of S is orthomodular [1]. In [8] it is shown that for S to be complete, it is sufficient that $F(S)$ admits a σ -additive state. Using the Gleason Theorem [6], it is shown that if $F(S)$ admits a σ -additive state and S is separable, then $F(S)$ is orthomodular and hence, in view of the Amemiya-Araki Theorem, S is complete. In this note we prove that the only measures that can exist on $F(S)$, when S is incomplete, are the ones that vanish identically on the finite-dimensional subspaces of S . Despite the fact of being a proper extension of the Hamhalter-Pták Theorem, this result allows a simpler proof not relying on the Gleason Theorem.

Let us recall some basic notions and facts as we shall use them in the sequel. Let S be an inner product space (real or complex) with inner product $\langle \cdot, \cdot \rangle$. For any subspace $A \subset S$, we let \bar{A} denote its completion and for any vector $x \in S$ we let $[x] = \text{span}\{x\}$. A subspace A of S is called *orthogonally closed* if $A = A^{\perp\perp}$, where $A^\perp = \{x \in S \mid \langle x, y \rangle = 0 \text{ for all } y \in A\}$. Let us denote by $F(S)$ the set of all orthogonally closed subspaces of S . If we order $F(S)$ by the inclusion relation and endow it with the orthocomplementation \perp , then $F(S)$ becomes a complete orthocomplemented lattice (see for example [10]) with the operations \vee and \wedge

Received by the editors May 24, 2006 and, in revised form, October 11, 2006.
2000 *Mathematics Subject Classification*. Primary 46C05, 46C15; Secondary 46L30.

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satisfying the equalities

$$\bigvee_{i \in I} M_i = (\text{span}\{M_i \mid i \in I\})^{\perp\perp} \quad \text{and} \quad \bigwedge_{i \in I} M_i = \bigcap_{i \in I} M_i.$$

If S is complete, then $F(S)$ can be identified with the projection lattice of the algebra of bounded operators on S .

We recall that an orthocomplemented poset $(\mathfrak{L}, \leq, 0, 1, \perp)$ is orthomodular if when $\mathfrak{a}, \mathfrak{b} \in \mathfrak{L}$ such that $\mathfrak{a} \leq \mathfrak{b}$, then $\mathfrak{a} \vee (\mathfrak{b} \wedge \mathfrak{a}^\perp)$ exists in \mathfrak{L} and is equal to \mathfrak{b} . According to the Amemiya-Araki Theorem, an inner product space S is complete if, and only if, $F(S)$ is orthomodular.

In what follows we shall consider measures on $F(S)$. A *charge* on $F(S)$ is an additive real-valued measure, i.e., a map $m : F(S) \rightarrow \mathbb{R}$ such that

$$(1.1) \quad m(A \vee B) = m(A) + m(B)$$

whenever $A, B \in F(S)$ and $A \perp B$. A charge m on $F(S)$ is said to be:

- *completely-additive* if equation (1.1) holds for any collection $\{A_i \mid i \in I\}$ of pairwise orthogonal subspaces;
- *regular*, if for every A in $F(S)$ and every $\epsilon > 0$ there exists a finite-dimensional subspace M , contained in A , such that $|m(A) - m(M)| \leq \epsilon$;
- *free* or *singular* if $m(A)$ is zero for every finite-dimensional subspace A in $F(S)$.

If S is a complete inner product space and $\dim S > 2$, then, by the Generalized Gleason Theorem, every bounded charge on $F(S)$ can be lifted to a bounded linear functional on the algebra of bounded operators on S . (See e.g. [9, Theorem 5.2.4].) Moreover, if the charge is completely-additive, then the corresponding linear functional is normal.

A *state* s on $F(S)$ is a normalized (i.e., $s(S) = 1$) positive charge. The set of all states on $F(S)$, denoted by $\mathcal{S}(F(S))$, is a convex subset of the cube $[0, 1]^{F(S)}$. When endowed with the product topology, $[0, 1]^{F(S)}$ is a compact space, and since $\mathcal{S}(F(S))$ is closed in $[0, 1]^{F(S)}$, it follows that the state space of $F(S)$ is a compact, convex topological space.

The relation between the completeness of S and $\mathcal{S}(F(S))$ was first investigated in [8]: S is complete if, and only if, $\mathcal{S}(F(S))$ contains a completely-additive state. Then, in [5], it was shown that it is sufficient that $F(S)$ admits a regular state to imply completeness of S . (See also [4].) Recently, in [2], it was shown that for incomplete inner product spaces, $F(S)$ can admit only free states. All the proofs of these results make use of the Gleason Theorem and, moreover, the proof of the last result holds only for positive measures.

Using very simple arguments and without having to recall the Gleason Theorem, we show that every charge on $F(S)$ is free when S is incomplete. In view of the results in [3], it can be seen that this result is in a sense the best possible.

Lemma 1.1. *An inner product space S is complete if, and only if, for any $M \in F(S)$ and any maximal orthonormal system $\{x_i \mid i \in I\}$ of M we have $M = \text{span}\{x_i \mid i \in I\}^{\perp\perp}$.*

Proof. We only need to show sufficiency. Let $A \subseteq B$ be elements of $F(S)$ and suppose that $\{a_i \mid i \in I\}$ and $\{c_j \mid j \in J\}$ are maximal orthonormal systems in A

and $A^\perp \cap B$, respectively. Then

$$A = \text{span}\{a_i \mid i \in I\}^{\perp\perp} \quad \text{and} \quad A^\perp \cap B = \text{span}\{c_j \mid j \in J\}^{\perp\perp}.$$

One can easily verify that $\{a_i, c_j \mid i \in I, j \in J\}$ is a maximal orthonormal system in B and therefore,

$$\begin{aligned} B &= \text{span}\{a_i, c_j \mid i \in I, j \in J\}^{\perp\perp} \\ &= \text{span}\{a_i \mid i \in I\}^{\perp\perp} \vee \text{span}\{c_j \mid j \in J\}^{\perp\perp} \\ &= A \vee (A^\perp \cap B). \end{aligned}$$

This implies that $F(S)$ is orthomodular, and therefore S is complete by the Amemiya-Araki Theorem. □

Lemma 1.2. *Let m be a charge on $F(S)$. For any $A, B \in F(S)$, $A \subseteq B$, we have $m(B) = m(A) + m(A^\perp \cap B)$.*

Proof. This follows directly from the following equalities:

$$\begin{aligned} m(A) + m(A^\perp \cap B) &= m(A) + m((A \vee B^\perp)^\perp) = m(A) + m(S) - m(A \vee B^\perp) \\ &= m(A) + m(S) - m(A) - m(B^\perp) = m(S) - m(B^\perp) = m(B). \end{aligned} \quad \square$$

As a consequence of the previous proposition observe that whenever A and B are elements of $F(S)$ such that $A \subseteq B$ and $A^\perp \cap B = \{0\}$, then $m(A) = m(B)$ for every charge m on $F(S)$. We can now prove the statement announced in the title.

Theorem 1.3. *Let S be an incomplete inner product space. Then, every charge on $F(S)$ is free.*

Proof. By Lemma 1.1, if S is incomplete, there exists a subspace M in $F(S)$ and a maximal orthonormal system $\{x_i \mid i \in I\}$ in M such that $\text{span}\{x_i \mid i \in I\}^{\perp\perp} \subsetneq M$. Let $X = \text{span}\{x_i \mid i \in I\}^{\perp\perp}$. Then $M^\perp \subsetneq X^\perp$. Let u be a unit vector in $X^\perp \setminus M^\perp$. Observe that $u \notin M \oplus M^\perp$ and that

$$X^\perp \cap M = (X \oplus [u])^\perp \cap (M + [u]) = M^\perp \cap (M + [u]) = \{0\}.$$

Suppose that m is a charge on $F(S)$ such that $m([v]) \neq 0$ for some unit vector v of S . Let U be a unitary operator on S such that $U(u) = v$ and let \tilde{m} be the charge on $F(S)$ defined by $\tilde{m}(M) = m(UM)$. Then $\tilde{m}([u]) = m([v]) \neq 0$. By Lemma 1.1 we obtain

$$\tilde{m}(M) + \tilde{m}([u]) = \tilde{m}(X) + \tilde{m}([u]) = \tilde{m}(X \oplus [u]) = \tilde{m}(M + [u]) = \tilde{m}(M),$$

which is a contradiction. □

Corollary 1.4. *Let S be an inner product space. The following statements are equivalent.*

- (1) S is complete.
- (2) $F(S)$ admits a regular nonzero charge.
- (3) $F(S)$ admits a completely-additive nonzero charge.

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