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A KHINCHIN SEQUENCE

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Abstract. We prove that the geometric and harmonic means of the sequence Z_2 of positive integers proposed by Bailey, Borwein, and Crandall converge to the corresponding Khinchin Constants.

1. Khinchin Sequences

One defines the *Khinchin Constant* K by the following relation:

$$\log(K) = \frac{1}{\log(2)} \sum_{k=1}^{\infty} \log(k) \log\left(\frac{(k+1)^2}{k(k+2)}\right) = \log(2.685452001\cdots).$$

For any sequence $A = (a_j)$,

$$A: \quad a_1, a_2, a_3, \ \ldots\ , a_j, \ \ldots$$

of positive integers, let us refer to A as a *Khinchin Sequence* iff the geometric means of A converge to K:

$$\lim_{n\to\infty} \big(\prod_{j}^{n} a_j\big)^{1/n} = K.$$

That is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_j) = \log(K).$$

For any irrational number x in the interval $(0,1)$, let us refer to x as a *Khinchin Number* iff the continued fraction expansion $A(x) = (a_j(x))$

$$A(x): \quad a_1(x), a_2(x), a_3(x), \ \ldots\ , a_j(x), \ \ldots$$

for x is a Khinchin Sequence. In this paper, our objective is to prove that the particular sequence $C = (c_j)$

$$C: \quad 2, 5, 1, 11, 1, 3, 1, 22, 2, 4, 1, 7, 1, 2, 1, 45, 2, 4, 1, 8, \ \ldots\ , c_j, \ \ldots$$

of positive integers proposed by Bailey, Borwein, and Crandall is a Khinchin Sequence. See reference [1].

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In the paper just cited, the authors denote C by Z_2. They define the sequence C in terms of two auxiliary sequences $U = (u_j)$ and $V = (v_k)$, defined in turn as follows. The first sequence, U, is the *van der Corput Sequence*:

$$U: \quad \frac{1}{2}, \quad \frac{1}{4}, \frac{3}{4}, \quad \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \quad \frac{1}{16}, \frac{9}{16}, \frac{5}{16}, \frac{13}{16}, \frac{3}{16}, \frac{11}{16}, \frac{7}{16}, \frac{15}{16}, \quad \cdots \quad , u_j, \ \ldots .$$

Specifically, for each positive integer j, u_j is the dyadic rational number obtained by reflecting the binary representation of j in the binary point. For example, $u_{12} := 0.0011 = 3/16$ because $12 = 1100.0$. See reference [2]. The second sequence, V, describes a particular partition of the interval $(0, 1]$:

$$V: \quad \cdots < v_k = \frac{1}{\log(2)} \log\left(\frac{k+1}{k}\right) < \ \cdots \ < v_3 < v_2 < v_1 = 1.$$

Now, in terms of U and V, Bailey, Borwein, and Crandall define the sequence C as follows:

$$C: \quad c_j = k \iff v_{k+1} < u_j \leq v_k.$$

For example, $c_{12} = 7$ because $v_8 < u_{12} \leq v_7$.

2. Motivation

To set a context for our study of the sequence C, let us describe a special case of the Ergodic Theorem. Let λ stand for Lebesgue Measure, defined as usual on $\mathbb{R}$. Let X be the set of all irrational numbers in the interval $(0, 1)$. Let μ stand for Gauss Measure, defined on X as follows:

$$\mu(E) := \frac{1}{\log(2)} \int_E \frac{1}{1+x} \lambda(dx)$$

where E is any Borel subset of X. Note that $\mu(E) = 0$ iff $\lambda(E) = 0$. Let F be the mapping carrying X to itself defined as follows:

$$F(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

where x is any number in X. Of course, $\lfloor 1/x \rfloor$ denotes the largest among all integers ℓ for which $\ell \leq 1/x$. Note that F is continuous. One may view the ordered pair (X, F) as a (discrete) dynamical system. For any x in X, one may say that if the system is in state x at time 0, then the system is in state $F(x)$ one unit of time later. By an elementary argument, one can show that μ is invariant under F, in the sense that, for any Borel subset E of X, $\mu(F^{-1}(E)) = \mu(E)$. By a more substantial argument, one can also show that μ is ergodic under F, in the sense that, for any Borel subset E of X, if $F^{-1}(E) = E$, then either $\mu(E) = 0$ or $\mu(E) = 1$. See reference [4]. Let h be the function defined on X as follows:

$$h(x) := \left\lfloor \frac{1}{x} \right\rfloor$$

where x is any number in X. Note that h is continuous and that the values of h are positive integers. One may refer to h as an observable for the given dynamical system.

For any x in X, one obtains the orbit $O(x) = (x_j)$ of x under F,

$$O(x): \quad x = x_1, x_2, x_3, \ \ldots$$

and one obtains the corresponding (discrete) time sequence $A(x) = (a_j(x))$,

$$A(x): \quad a_1(x), a_2(x), a_3(x), \ \ldots,$$

where

$$x_j := F^{j-1}(x), \qquad a_j(x) := h(x_j).$$

The sequence $A(x)$ is the *Continued Fraction Expansion* for x.

For the assembly X, μ, F, and $\log(h)$, the Ergodic Theorem states that, for almost every x in X,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log(h(F^{j-1}(x))) \ = \ \int_X \log(h(y))\mu(dy).$$

That is, the time average of $\log(h)$ over $O(x)$ equals the space average of $\log(h)$ over X. See reference [5]. Hence,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log(a_j(x)) \ &= \ \int_X \log(h(y))\mu(dy) \\
&= \ \sum_{k=1}^{\infty} \log(k)\mu\left(\frac{1}{k+1}, \frac{1}{k}\right) \\
&= \ \frac{1}{\log(2)} \sum_{k=1}^{\infty} \log(k) \log\left(\frac{(k+1)^2}{k(k+2)}\right) \\
&= \ \log(K).
\end{aligned}$$

Now it is plain that, for almost every irrational number x in the interval $(0,1)$, x is a Khinchin Number. However, no particular examples of such numbers are known. The beguiling cases of $\pi - 3$ and even of $K - 2$ itself have been studied energetically but to no analytic decision as yet. In reference [3], one may find the optimistic opinion that $\pi - 3$ is a Khinchin Number. The graphs displayed in Figures 1 and 2 suggest a more cautious, though still hopeful, opinion on $\pi - 3$ and on $K - 2$ as well. The graphs are list plots of the following data:

$$\frac{1}{n} \sum_{j=1}^{n} \log\left(a_j(x)\right) - \log(K) \qquad (1 \le n \le 4096)$$

where $x = \pi - 3$ and $x = K - 2$.

Failing to identify particular Khinchin Numbers, one naturally turns to the design of particular Khinchin Sequences. One might, for instance, design sequences $A = (a_j)$ such that, for each j, a_j equals 2 or 3 and such that the 2s and 3s occur in A in correct "limiting proportions," specifically, the proportions p and q, where p and q are the positive numbers for which $p + q = 1$ and $\log(K) = p\log(2) + q\log(3)$. However, such a design would be very difficult to implement, since it depends upon the calculation of $\log(K)$ to arbitrary accuracy. In sharp contrast, Bailey, Borwein, and Crandall have proposed a particular candidate for a Khinchin Sequence, namely, the sequence C, which they have defined in constructive and rapidly computable terms. Let us prove formally that C is indeed a Khinchin Sequence.

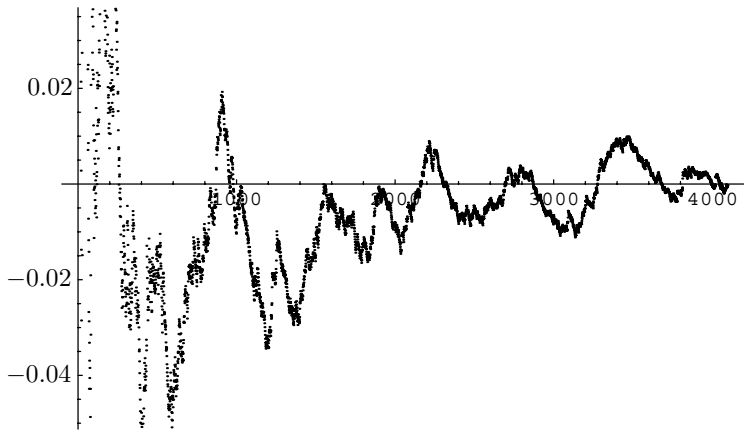

FIGURE 1. $x = \pi - 3$

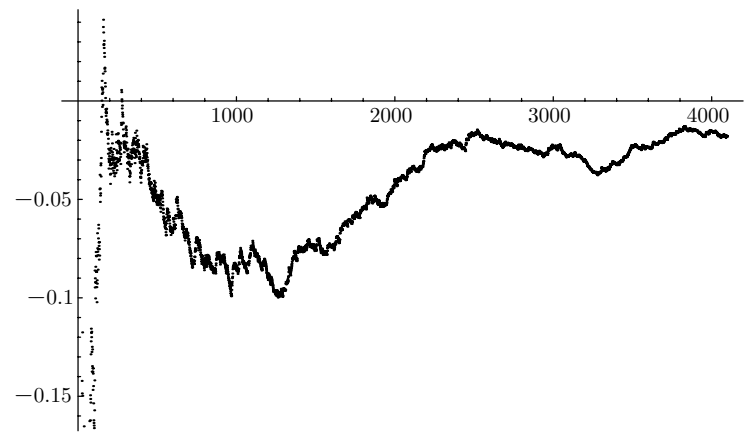

FIGURE 2. $x = K - 2$

3. THE FUNCTION ϕ

Let ϕ be the function defined on the interval $J = (0, 1]$ as follows. For each x in J and for any positive integer k,

$$\phi(x) = \log(k) \Longleftrightarrow v_{k+1} < x \leq v_k.$$

In particular, for each positive integer j, $\phi(u_j) = \log(c_j)$. See Figure 3. Clearly,

$$\begin{aligned}\int_J \phi(x)\lambda(dx) &= \frac{1}{\log(2)} \sum_{k=1}^{\infty} \log(k) \left(\log\left(\frac{k+1}{k}\right) - \log\left(\frac{k+2}{k+1}\right)\right) \\ &= \frac{1}{\log(2)} \sum_{k=1}^{\infty} \log(k) \log\left(\frac{(k+1)^2}{k(k+2)}\right) \\ &= \log(K).\end{aligned}$$

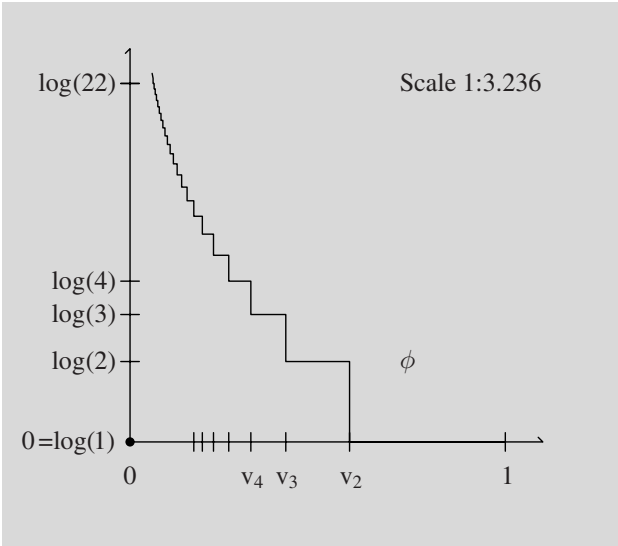

FIGURE 3. The function ϕ

Now it is plain that C is a Khinchin Sequence iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \phi(u_j) = \int_J \phi(x)\lambda(dx). \tag{1}$$

We proceed to prove relation (1).

4. Integrating sequences

Let ψ be a real-valued Borel function defined on J and integrable with respect to λ. Let us say that the sequence U *integrates* ψ iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \psi(u_j) = \int_J \psi(x)\lambda(dx).$$

To prove relation (1), we must prove that U integrates ϕ. Obviously, the functions integrated by U comprise a linear space. By an elementary argument, one can show that, for each subinterval I of J, U integrates the characteristic function χ_I of I. One summarizes this property of U by saying that U is *uniformly distributed* in J. We will prove this property in an appendix to this paper. Presuming the property, let us prove that U integrates ϕ. To that end, we require only that:

(1) ϕ is nonnegative and decreasing;
(2) for each positive integer p, U integrates the function

$$\phi_p := \chi_{[1/2^p,1]}\phi.$$

Let n be any positive integer. Let α_n be the average of the values of ϕ at the first n terms of U:

$$\alpha_n := \frac{1}{n} \sum_{j=1}^{n} \phi(u_j).$$

Let

$$\beta := \int_J \phi(x)\lambda(dx).$$

We must prove that

$$\lim_{n\to\infty} \alpha_n = \beta.$$

Let p be any positive integer. Let ϕ_p be the function defined on J by truncation of ϕ, as follows:

$$\phi_p := \chi_{[1/2^p,1]}\phi.$$

That is,

$$\phi_p(x) := \begin{cases} 0 & \text{if} \quad 0 < x < 1/2^p, \\ \phi(x) & \text{if} \quad 1/2^p \le x \le 1. \end{cases}$$

Obviously, for each x in J,

$$\phi_1(x) \le \phi_2(x) \le \ \cdots \ \le \phi_p(x) \le \ \cdots \ \uparrow \ \phi(x).$$

Let $\alpha_{n,p}$ be the corresponding average of the values of ϕ_p at the first n terms of U:

$$\alpha_{n,p} := \frac{1}{n}\sum_{j=1}^{n} \phi_p(u_j).$$

Let

$$\beta_p := \int_J \phi_p(x)\lambda(dx).$$

Clearly, ϕ_p is a linear combination of characteristic functions of subintervals of J. By our initial remarks, it is plain that U integrates ϕ_p:

$$\lim_{n\to\infty} \alpha_{p,n} = \beta_p. \tag{2}$$

Now let ϵ be any positive real number. By the Monotone Convergence Theorem, we may introduce a positive integer p such that

$$\beta - \epsilon < \beta_p \le \beta, \tag{3}$$

from which it follows that

$$\int_{(0,1/2^p)} \phi(x)\lambda(dx) < \epsilon. \tag{4}$$

By relation (2), we may introduce a positive integer m such that, for every positive integer n, if $m \le n$, then

$$\beta_p - \epsilon < \alpha_{n,p} < \beta_p + \epsilon. \tag{5}$$

We may as well arrange that $2^p \le m$. Let n be any positive integer for which $m \le n$. Let q be the positive integer for which $2^{q-1} - 1 < n \le 2^q - 1$. Clearly, $p < q$. One may say that the first n terms of U have run through the first $q-1$ "cycles" of U and have at least begun (perhaps even finished) the q-th cycle. The smallest term of the q-th cycle is $1/2^q$. Hence, for each positive integer j, if $1 \le j \le n$, then $1/2^q \le u_j$. Consequently,

$$\alpha_{n,q} = \alpha_n. \tag{6}$$

Now let

$$t_1, t_2, \ \ldots \ , t_\ell \qquad (\ell = 2^{q-p} - 1)$$

be the terms among

$$u_1, u_2, \ \ldots \ , u_r \qquad (r = 2^q - 1)$$

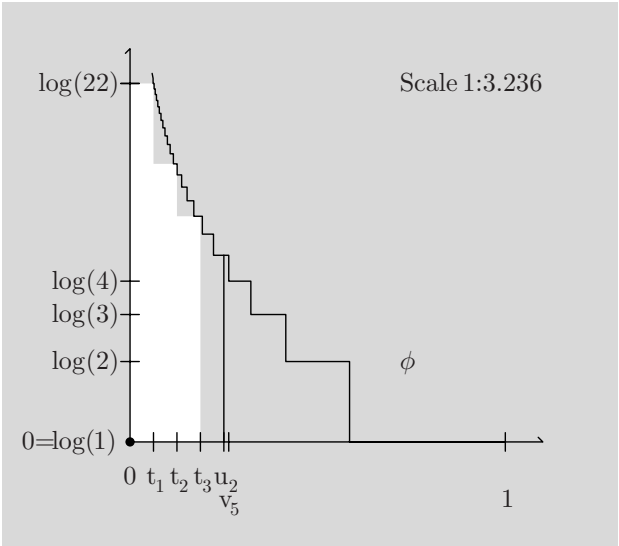

FIGURE 4. Comparison of areas

that are less than $1/2^p$. Figure 4, $p = 2$, $q = 4$, and $\ell = 3$. Since ϕ is nonnegative and decreasing, we find that

$$\begin{aligned}
\alpha_{n,q} - \alpha_{n,p} &\le \frac{1}{n}\sum_{j=1}^{\ell}\phi(t_j) \\
&= \frac{2^q}{n}\frac{1}{2^q}\sum_{j=1}^{\ell}\phi(t_j) \\
&\le 4\int_{(0,1/2^p)}\phi(x)\lambda(dx) && \text{(since } 2\,2^{q-1} < 2(n+1)\text{)} \\
&< 4\epsilon && \text{(by relation (4)).}
\end{aligned}$$

Hence, by relations (3) and (5) and by the foregoing computation,

$$\beta - 2\epsilon < \beta_p - \epsilon < \alpha_{n,p} \le \alpha_{n,q} < \alpha_{n,p} + 4\epsilon < \beta_p + 5\epsilon \le \beta + 5\epsilon$$

so that, by relation (6), $\beta - 2\epsilon < \alpha_n < \beta + 5\epsilon$. Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\phi(u_j) = \int_J \phi(x)\lambda(dv). \tag{1}$$

5. QUESTIONS

The number x in $(0,1)$ of which C is the continued fraction expansion is approximately equal to 0.46107049595671951935. Of course, it is a Khinchin Number. Can one identify x in "familiar" terms?

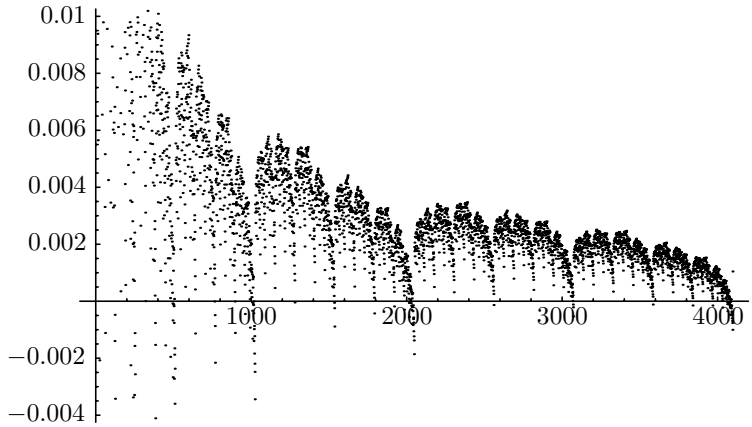

FIGURE 5. $N = 4096$

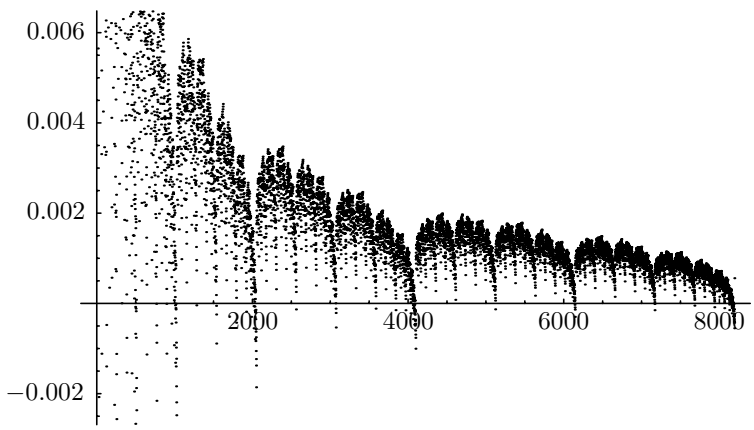

FIGURE 6. $N = 8192$

In Figures 5 and 6, we display list plots of the following data:

$$\frac{1}{n}\sum_{j=1}^{n}\log(c_j) - \log(K) \qquad (1 \le n \le N),$$

where $N = 4096$ and $N = 8192$. Can one explain, in formally precise terms, the apparent self-similarity of the data?

6. Harmonic means

Let r be any real number for which $r < 1$ and $r \neq 0$. With reference to Section 4, let us define the function ϕ_r on J as follows. For each x in J and for any positive integer k,

$$\phi_r(x) = k^r \iff v_{k+1} < x \le v_k.$$

In particular, for each positive integer j, $\phi_r(u_j) = c_j^r$. If $r < 0$, then $1-\phi_r$ is similar to ϕ, in the sense that it meets the conditions (1) and (2) stated in Section 4. If

$0 < r < 1$, then ϕ_r itself is similar to ϕ. In either case, U integrates ϕ_r. Hence,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} c_j^r &= \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \phi_r(u_j) \\ &= \int_J \phi_r(x)\lambda(dx) \\ &= \frac{1}{\log(2)}\sum_{k=1}^{\infty} k^r \log\left(\frac{(k+1)^2}{k(k+2)}\right).\end{aligned}$$

One defines the *Khinchin Constant* K_r by the following relation:

$$K_r^r = \frac{1}{\log(2)}\sum_{k=1}^{\infty} k^r \log\left(\frac{(k+1)^2}{k(k+2)}\right).$$

We infer that the r-th harmonic means of C converge to K_r:

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{j=1}^{n} c_j^r\right)^{1/r} = K_r.$$

7. Appendix

The van der Corput Sequence U falls into cycles:

$$U: \quad \frac{1}{2}, \quad \frac{1}{4}, \frac{3}{4}, \quad \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \quad \frac{1}{16}, \frac{9}{16}, \frac{5}{16}, \frac{13}{16}, \frac{3}{16}, \frac{11}{16}, \frac{7}{16}, \frac{15}{16}, \quad \ldots, u_j, \ldots$$

For each positive integer p, the first term of the p-th cycle is $1/2^p$ and the length of the p-th cycle is 2^{p-1}. The sum of the lengths of the first p cycles is $2^p - 1$. Moreover,

$$u_{2^p+j} = \frac{1}{2^{p+1}} + u_j \qquad (0 < j < 2^p).$$

By these observations, it is plain that, for any dyadic interval I of the form

$$I = [j/2^p, (j+1)/2^p) \qquad (p \in \mathbb{Z}^+,\ 0 < j < 2^p),$$

the sequence U visits I precisely once in the course of its first p cycles. Thereafter, it visits I periodically with period 2^p. Hence, for any positive integer n, if $2^p \le n$, then

$$\frac{m}{n} \le \frac{1}{n}\sum_{j=1}^{n} \chi_I(u_j) \le \frac{m+1}{n},$$

where m is the positive integer for which

$$m2^p - 1 < n \le (m+1)2^p - 1.$$

Consequently,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\chi_I(u_j) = \frac{1}{2^p} = \lambda(I),$$

which is to say that U integrates χ_I.

In turn, for any subinterval I of the interval $(0,1)$ and for any positive real number ϵ, we may introduce finite disjoint unions I' and I'' of dyadic intervals of the foregoing form such that $I' \subseteq I \subseteq I''$ and $\lambda(I''\backslash I') < \epsilon$. Hence,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_I(u_j) &\leq \lim_{j\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_{I''}(u_j) \\
&= \lambda(I'') \\
&< \lambda(I') + \epsilon \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_{I'}(u_j) + \epsilon \\
&\leq \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_I(u_j) + \epsilon.
\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \chi_I(u_j) = \lambda(I),$$

which is to say that U integrates χ_I.

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