THE DUAL FOLIATION IN OPEN MANIFOLDS
WITH NONNEGATIVE SECTIONAL CURVATURE

LUIS GUIJARRO AND GERARD WALSCHAP

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ABSTRACT. We study several properties of the Sharafutdinov dual foliation in
open manifolds with nonnegative sectional curvature.

1. INTRODUCTION

A foliation in a Riemannian manifold M is called *metric* when its leaves are
locally equidistant. A generic Riemannian structure on M does not admit such
foliations, except in codimension one (and even then, usually only locally), so it
should come as no surprise that the existence of a metric foliation has strong reper-
cussions for the geometry of M.

One of the nicest examples consists of the collection of fibers of a Riemannian
submersion $\pi : M \to N$, for in this case, the leaves are closed, globally equidis-
tant, and the foliation is in fact a fibration, provided M is complete. This occurs
in particular for every open manifold M of nonnegative sectional curvature: the
metric projection $\pi : M \to S$ of the ambient space onto a soul S which assigns to
$p \in M$ that point $\pi(p) \in S$ in the soul that is closest to p is a C^∞ Riemannian
submersion (as shown by Perelman for the C^1 case in [5], one of the authors for the
C^2 case in [2], and by Wilking in [9] for the full regularity). π is also referred to as
the Sharafutdinov map, as Perelman proved that it coincides with the retraction
constructed in [7].

In [9], Wilking associates to a given metric foliation $\mathcal{F}$ another (possibly singu-
lar) so-called *dual* foliation $\mathcal{F}^\#$, which is also metric whenever M has nonnegative
sectional curvature and the leaves are closed. It is therefore natural to look at the
foliation dual to the metric projection of an open manifold with nonnegative curva-
ture onto a soul. The main purpose of this note is to study some of its properties.

We will use the standard terminology for metric foliations throughout: $\mathcal{F}$ induces
an orthogonal splitting $TM = \mathcal{V} \oplus \mathcal{H}$ of the tangent bundle of M, with $\mathcal{V}$ (the *vertical*
distribution) denoting the distribution tangent to the leaves, and $\mathcal{H}$ (the *horizontal*
distribution) its orthogonal complement. We write $e = e^v + e^h$ for the corresponding

decomposition of a vector $e \in TM$ into its vertical and horizontal components. The *O'Neill tensor* of $\mathcal{F}$ is the skew-symmetric tensor field $A : \mathcal{H} \times \mathcal{H} \to \mathcal{V}$ given by $A_x y = (1/2)[X,Y]^v$, with X, Y denoting local horizontal fields extending x, y.

2. THE SHARAFUTDINOV DUAL FOLIATION

Let M be an open (i.e., complete, noncompact) manifold with nonnegative sectional curvature. Recall that Cheeger and Gromoll [1] showed the existence of a compact, totally convex, totally geodesic submanifold S, known as a soul of M, whose normal bundle $\nu(S)$ is diffeomorphic to M. As mentioned above, there is a smooth Riemannian submersion $\pi : M \to S$ that is a left inverse to the normal exponential map $\exp : \nu(S) \to M$, i.e, $\pi \circ \exp(u) = \pi_S(u)$, where $\pi_S : \nu(S) \to S$ is the vector bundle projection. Furthermore, recall that if $\bar{\gamma} : I \to S$ is a geodesic, and U is a parallel section of $\nu(S)$ along $\bar{\gamma}$, then the map

$$I \times \mathbb{R} \to M,$$

$$(t,s) \mapsto R(t,s) := \exp(sU(t))$$

is a flat totally geodesic immersion of $I \times \mathbb{R}$ into M. Finally, each curve $R_s : I \to M$, where $R_s(t) := R(t,s)$, is a horizontal geodesic for the submersion π. The reader is invited to consult [3] for further details. We should also point out that since the fiber $\nu_{\bar{p}}(S)$ of π_S over $\bar{p}$ is a vector space, we often identify tangent vectors to it at different points with elements of $\nu_{\bar{p}}(S)$, hoping that this will not create any confusion.

2.1. **Description of the dual leaf.**

Recall from [9] that in general, given a metric foliation on a Riemannian manifold M, the dual leaf through a point p consists of all points that can be joined to p by a piecewise smooth horizontal curve. Since any curve can be approximated by piecewise smooth geodesics, it suffices to consider the latter. By the previous paragraph, exponentiating parallel vector fields along curves in the soul yields horizontal curves. On the other hand, uniqueness of horizontal lifts implies that all horizontal curves are obtained in this way. As a result, if $p = \exp(u)$, $u \in \nu(S)$, then the dual leaf $\mathcal{F}^{\#}(p)$ agrees with the image by exp of the holonomy bundle $H(u)$ through u, which is the subset of the normal bundle of S consisting of all parallel translates of u along curves in S; i.e., $\mathcal{F}^{\#}(p) = \exp(H(u))$. Of course, as was already noted by Wilking, these leaves stay at a constant distance to the soul.

Our main tool to explore this leaf is the Ambrose-Singer theorem, which in terms of vector bundles can be stated as follows: Consider a Euclidean vector bundle $E \to S$ over S together with a Riemannian connection, and a point $\bar{p}$ in S. Parallel translation along piecewise-smooth loops at $\bar{p}$ in S induces a group of orthogonal transformations of the fiber over $\bar{p}$, called the *holonomy group* at $\bar{p}$. Its Lie algebra $\mathfrak{g}(\bar{p})$ is called the *holonomy algebra* of the connection at $\bar{p}$. The theorem says that if R denotes the curvature tensor of the connection, then the holonomy algebra at $\bar{p}$ is the union over all points $\bar{q} \in S$ of the set of all skew-adjoint endomorphisms $P_{\bar{\gamma}} R(\bar{x}, \bar{y})$, $\bar{x}$, $\bar{y} \in T_{\bar{q}} S$, where $\bar{\gamma}$ is a C^{∞} curve in S joining $\bar{q}$ to $\bar{p}$, and $P_{\bar{\gamma}}$ denotes parallel translation along $\bar{\gamma}$. But the holonomy algebra through $\bar{p}$ is the Lie algebra of the holonomy group at $\bar{p}$, and in our situation, given $u \in \nu_{\bar{p}}$, the intersection of $H(u)$ with $\nu_{\bar{p}}(S)$ is precisely the orbit of u under the action of this group. Hence the vertical part of the tangent space to $H(u)$ at u is obtained as $\mathfrak{g}(\bar{p}) \cdot u$, which by the Ambrose-Singer theorem is just the set of all possible vectors of the form

$P_{\bar{\gamma}}R(\bar{x}, \bar{y})u$. Finally, since the derivative of the normal exponential map sends $\bar{\mathcal{H}}_u$, the horizontal space for the normal connection in $\nu(S)$, onto the horizontal space $\mathcal{H}_p$ of π, we conclude that the tangent space of a dual leaf splits as an orthogonal direct sum

$$(2.1) \qquad T_p\mathcal{F}^{\#}(p) = \mathcal{H}_p \oplus \exp_{*u}\mathfrak{g}(\bar{p}) \cdot u.$$

2.2. **Identifying local holonomy with radial Jacobi fields.** Any $v \in \nu(S)$ is tangent to a normal geodesic $\gamma_v : \mathbb{R} \to M$, which always remains in a π-fiber as we know from Perelman's theorem [5]. Denote by $\bar{q} = \gamma_v(0) \in S$; given any pair of tangent vectors to the soul at $\bar{q}$, extend them locally in S to obtain fields $\bar{X}$ and $\bar{Y}$ with $[\bar{X}, \bar{Y}] = 0$ and to form the geodesic variations that result from parallel transport of v along the square obtained by following integral curves for $\bar{X}, \bar{Y}, -\bar{X}$ and $-\bar{Y}$ for time $\sqrt{s}$. It is well known that the Jacobi field J along γ_v associated to such variations is the one determined by the initial conditions $J(0) = 0$, $J'(0) = -\frac{1}{2}R(\bar{x}, \bar{y})v$ (see for instance [8]), and therefore collects all the local holonomy information at different points of the soul. Its interaction with the Sharafutdinov submersion π comes from the fact that if A denotes the O'Neill tensor of π, then $A_XY \circ \gamma_v = J$, where X and Y are the π-horizontal lifts of $\bar{x}$ and $\bar{y}$ along γ_v. Thus,

$$(2.2) \qquad A_XY(\exp v) = -\frac{1}{2}\exp_{*v}R(\bar{x}, \bar{y})v.$$

It follows from (2.1) that the tangent space to a dual leaf not only contains the space generated by the horizontal subspace and the image of the A-tensor (this was already implicit in [9]), but is in general strictly larger than it. We describe it in more detail below.

2.3. **Holonomy diffeomorphisms and the dual leaf.** For convenience, in what follows, we denote a π-fiber $\pi^{-1}(\bar{q})$ by $F_{\bar{q}}$. Any curve $\bar{\alpha} : [a, b] \to S$ induces a diffeomorphism $h_{\bar{\alpha}}$ between the π-fibers at its endpoints: given a point $q \in F_{\bar{\alpha}(a)}$, $h_{\bar{\alpha}}(q)$ is defined to be the endpoint of the horizontal lift of $\bar{\alpha}$ that starts at q. $h_{\bar{\alpha}}$ is called the *holonomy diffeomorphism* induced by $\bar{\alpha}$. Since horizontal curves are obtained by exponentiating parallel sections in the normal bundle, we have that

$$(2.3) \qquad h_{\bar{\alpha}} \circ \exp = \exp \circ P_{\bar{\alpha}}.$$

From the construction of $\mathcal{F}^{\#}$, it is clear that $h_{\bar{\alpha}}$ maps $\mathcal{F}^{\#} \cap F_{\bar{\alpha}(a)}$ to $\mathcal{F}^{\#} \cap F_{\bar{\alpha}(b)}$, and therefore its derivative preserves the vertical tangent space to the leaves of $\mathcal{F}^{\#}$. This enables us to give a version of the Ambrose-Singer theorem for leaves:

Lemma 2.1. *Let $p = \exp(u)$ for some $u \in \nu(S)$. For any curve $\bar{\alpha} : [a, b] \to S$ with $\bar{\alpha}(b) = \pi_S(u)$, denote by $h_{\bar{\alpha}}$ its holonomy diffeomorphism, and by α its horizontal lift with $\alpha(b) = p$. Then the set*

$$\mathcal{VF}_p^{\#} := \left\{ h_{\bar{\alpha}*}(A_XY(\alpha(a))) : \text{for all such } \bar{\alpha}\text{'s, all } \bar{x}, \bar{y} \in T_{\bar{\alpha}(a)}S \right\}$$

agrees with the π-vertical tangent subspace to $\mathcal{F}^{\#}(p)$ at p.

Proof. By (2.1) the vertical component of the tangent space of $\mathcal{F}^{\#}(p)$ at p is $\exp_{*u}\mathfrak{g}(p) \cdot u$, which consists of all vectors of the form $\exp(P_{\bar{\alpha}}R(\bar{x}, \bar{y}))u$, with $\bar{\alpha}$,

$\bar{x}$, and $\bar{y}$ as in the statement. Under the identification of a vector space with its tangent space, $P_{\bar{\alpha}*} = P_{\bar{\alpha}}$ since the latter is linear, and by (2.2) and (2.3),

$$\exp_{*u}(P_{\bar{\alpha}}R(\bar{x}, \bar{y}))u = \exp_{*u} P_{\bar{\alpha}}\left(R(\bar{x}, \bar{y})(P_{\bar{\alpha}}^{-1}u)\right) = \exp_{*u} \circ P_{\bar{\alpha}*}\left(R(\bar{x}, \bar{y})(P_{\bar{\alpha}}^{-1}u)\right)$$

$$= h_{\bar{\alpha}*} \circ \exp_* R(\bar{x}, \bar{y})(P_{\bar{\alpha}}^{-1}u) = -2h_{\bar{\alpha}*}A_X Y(\alpha(a)). \qquad \square$$

It is also clear that the derivative of a holonomy diffeomorphism h preserves the normal bundle of leaves in $\mathcal{F}^\#$, on which it is in fact isometric. The reason for this is given in [9], where it is shown that the normal space of $\mathcal{F}^\#$ along a horizontal geodesic is spanned by parallel Jacobi fields.

2.4. **The pullback bundle.** For $u \in \nu(S)$, consider the holonomy bundle $H(u)$ through u. The restriction of the projection π_S to $H(u)$ (which will be denoted by Π) allows us to construct the pullback bundle $\Pi^*\nu(S)$. Observe that this bundle always splits off a line bundle $\mathbb{R}$, namely the one generated by the base point in $H(u)$ (more precisely, the total space of the pullback bundle is the set of pairs (v, w), where $v \in H(u)$ and $\pi_S(v) = \pi_S(w)$; the line bundle has as total space $\{(v, sv) \mid v \in H(u), s \in \mathbb{R}\}$. It follows that whenever the dual leaf through $p = \exp(u)$ does not fill the whole boundary of a tubular neighbourhood of the soul, then $\Pi^*\nu(S)$ is reducible even further:

Lemma 2.2. *In the above situation, $\Pi^*\nu(S) = \mathbb{R} \oplus \mathfrak{g} \cdot \mathbb{R} \oplus N$, where*
- $\mathbb{R}$ *is the line bundle generated by v at each point $v \in H(u)$;*
- $\mathfrak{g} \cdot \mathbb{R}$ *is the bundle whose fiber at each $v \in H(u)$ is $\mathfrak{g}(\pi_S(v)) \cdot \mathbb{R}v$;*
- N *is the orthogonal complement of the first two summands.*

The bundle N could of course also be replaced by any other transversal complement to the first two summands.

3. Applications

In this section we present some consequences of the above discussion.

3.1. **Quasipositive curvature.** Let $B_r(S)$ be a tubular neighborhood of radius r about the soul. For small $r > 0$ this set is convex, and therefore its boundary S_r inherits a metric of nonnegative curvature. Since any horizontal geodesic remains at constant distance from S, S_r is composed entirely of dual leaves to the Sharafutdinov foliation.

Theorem 3.1. *If the induced metric on some S_r has positive sectional curvature at one point, then the normal holonomy of the soul acts transitively on the sphere bundle; namely, for any two vectors u, $v \in \nu(S)$ with $\|u\| = \|v\|$, there is a curve $c : I \to S$ in the soul such that $P_c(u) = v$. In particular, the normal exponential map $\exp : \nu(S) \to M$ is a diffeomorphism.*

Proof. S_r can contain one or more fibers of the dual foliation. In the first case, the holonomy acts transitively by the discussion in the previous section, and we are done. So assume S_r contains more than one leaf. Let $\mathcal{F}^\#(p)$ be the leaf passing through the positive curvature point p. The normal subspace to $\mathcal{F}^\#(p)$ at p splits into the direct sum of a radial direction orthogonal to S_r and a nonempty subspace V_p tangent to S_r that generates parallel Jacobi vector fields along horizontal geodesics by Theorem 6.1 in [9]. Thus if $x \in \mathcal{H}_p$ and $v \in V_p$, then the extrinsic curvature of the plane spanned by x and v is zero. It is clear from the Gauss equation

that the same is true for the sectional curvature of the induced metric in S_r, since the normal vector to S_r is parallel along the horizontal geodesic tangent to x. This of course contradicts our assumption on the curvature of S_r. The reason why the normal exponential map is a diffeomorphism is because the parallel translate of a ray direction (i.e., of a unit vector u such that the geodesic $t \mapsto \exp(tu)$ is minimal on $[0, s]$ for all $s > 0$) is again a ray direction. Since there is always at least one ray direction, every direction must then be such. $\square$

Another standard consequence of having transitive holonomy is that the ideal boundary of M is a point (see for instance [4]). The novelty in our situation is that the assumption involves the sectional curvature at only one point, suggesting a previously unknown strong rigidity in nonnegative curvature.

3.2. The normal cut and conjugate locii for the soul and the dual foliation.
Dual leaves are constructed by means of horizontal geodesics, and the latter are easily described by parallel translation and the normal exponential map. One would therefore expect the singularities of the exponential map to be reflected in the dual leaves:

Theorem 3.2. *If $p \in \mathcal{F}^{\#}(p)$ is a conjugate or cut point for S, then the same is true of any other point in $\mathcal{F}^{\#}(p)$.*

Proof. Given any $q \in \mathcal{F}^{\#}(p)$, there exists some piecewise geodesic $\bar{\alpha} : I \to S$ whose horizontal lift α to p ends at q. Let $u \in \nu(S)$ with $\exp(u) = p$ such that the parallel transport of u along $\bar{\alpha}$ (which we denote by U) exponentiates to α. If p is a conjugate point of $\exp : \nu(S) \to M$, then there is some $v \in \nu(S)$ with $\exp_{u*}(v) = 0$. By (2.3), $\exp_{P_{\bar{\alpha}(u)*}}(P_{\bar{\alpha}}v) = h_* \circ \exp_{u*}(v) = 0$, so that q is a conjugate point of S along the geodesic tangent to $P_{\alpha}(u)$.

To show the second part of the theorem, recall that p is a a cut point of the soul if it is a cut point of $\pi(p)$ along some geodesic γ. This means that if $p = \gamma(t_0)$, then for any $\varepsilon > 0$ there exists a shorter geodesic c^ε connecting $\pi(p)$ to $\gamma(t_0 + \varepsilon)$. Obviously these geodesics biangles are preserved under the holonomy diffeomorphisms h, and therefore q is a cut point of $\pi(q)$ along $h \circ \gamma$. $\square$

3.3. The Cheeger-Gromoll convex function.
The main tool in the proof of the Soul Theorem is the construction of a convex function with compact sublevel sets [1]. This can also be applied to any nonnegatively curved Alexandrov space. If the latter is taken to be the "quotient" space obtained from the leaf closures of the dual foliation, then there is a direct relation between the two functions. These facts are stated in a more precise manner in section 7 of [9], but since we will need them later, we include a version that suffices to our needs here:

Theorem 3.3 (Wilking, 06)**.** *Let M be an open manifold with nonnegative sectional curvature. If S is a soul of M, then there exists a noncompact Alexandrov space A and a submetry $\sigma : M \to S \times A$, where the target is endowed with the product metric. If $\pi_A : S \times A \to A$ denotes projection onto the second factor, then the fibers of $\pi_A \circ \sigma$ agree with the closures of the dual leaves.*

We can now explicitly describe the relation between the convex functions.

Proposition 3.4. *Let $b : M \to \mathbb{R}$ be the Cheeger-Gromoll function constructed at the initial point p, and $b_A : A \to \mathbb{R}$ constructed at $\sigma(p)$. Then $b = b_A \circ \pi_A \circ \sigma$.*

Denote by $\sigma_A = \pi_A \circ \sigma$. We prove the above claim together with:

Proposition 3.5. *If $\gamma : [0, \infty) \to M$ is a ray, then $\sigma_A \circ \gamma$ is a ray in A.*

Proof. By Remark 8.3.b in [9], σ_A takes level sets of b to level sets of b_A. Since $b(p) = b_A(\sigma_A(p)) = 0$, Proposition 3.4 follows. On the other hand, it is well known that γ is a ray if and only if $b(\gamma(t)) = b(\gamma(0)) + t$ for all $t > 0$ and similarly for A. Since $b_A(\sigma_A(\gamma(t))) = b(\gamma(t)) = t + b(\gamma(0)) = t + b_A(\sigma_A(\gamma(0)))$, we have that $\sigma_A \circ \gamma$ is a ray in A. $\qquad\square$

The submetry σ_A and the space A are interesting in their own right, since they contain much of the large scale information of M. We point out a few of their properties:

- If $p = \exp(u)$, Wilking's definition implies that $\sigma_A^{-1}(\sigma_A(p)) = \overline{\mathcal{F}^{\#}(p)}$; its tangent space splits as $\mathcal{H}_p \oplus \exp_{*u} \overline{\mathfrak{g}(\bar{p})} \cdot u$; therefore, the horizontal tangent space for σ_A coincides with the π-vertical directions inducing parallel Jacobi fields along π-horizontal geodesics. Thus any geodesic in A lifts to geodesics entirely contained in Sharafutdinov fibers by [9].
- As Wilking points out in [9], Remark 8.3(a), the space A is bilipschitz equivalent to the quotient $\nu_p(S)/\overline{H}$ of the normal space $\nu_p(S)$ to S at $p \in S$ by the closure of the holonomy group H of the normal bundle of S. In fact, if we denote by $F_p/\overline{H}$ the image of this set by the exponential map, then A is isometric to any quotient $F_{\pi(p)}/\overline{H}$, for $p \in M$. This is hardly surprising, since the holonomy diffeomorphisms in M are isometries on the σ_A-horizontal subspaces.
- The ideal boundaries of A and M are isometric. To see this, notice that for any positive λ, σ_A induces a submetry $\sigma_A^\lambda : \lambda M \to \lambda A$. Now choose a point $p \in S$ and $\sigma_A(p) \in A$. Usual Gromov-Hausdorff arguments for pointed metric spaces show that there is a limiting submetry $\sigma_A^\infty : C_0(M(\infty)) \to C_0(A(\infty))$, where C_0 denotes the Euclidean cone. If $\gamma : [0, \infty) \to M$ is a ray starting at p, it is easy to see that the fibers of σ_A along the points $\gamma(t)$ all share a common upper bound for their diameter, since they are obtained by exponentiating the closure of the holonomy orbit of the vectors $\dot\gamma(t)$. Therefore, in the limit the submetry σ_A^∞ becomes bijective, hence an isometry.

3.4. The Sharafutdinov retraction preserves dual leaves.

As mentioned earlier, Sharafutdinov constructed in [7] a retraction of M onto S mapping each $p \in M$ to $\pi(p)$. This map was used later by Perelman in his proof of the soul conjecture of Cheeger and Gromoll. The retraction is given by a map $H : [0, 1] \times M \to M$, and each curve $H(\cdot, p) : [0, 1] \to M$ is called a Sharafutdinov line. It is often more convenient to use *outgoing Sharafutdinov lines (OSL)* instead (see [3] for a definition and properties needed for this section). Specifically, a curve $t \mapsto H(t, p)$, $t \in [0, 1]$, is nothing more than an OSL $c : [0, a] \to M$ with $c(0) = H(1, p)$, $c(a) = p$.

It turns out that dual leaves are preserved at each stage of this retraction:

Theorem 3.6. *Suppose $p, q \in \mathcal{F}^{\#}(p)$. If $t \in [0, 1]$, then $H(t, p)$ and $H(t, q)$ belong to the same dual leaf.*

Proof. In [3], it is shown that if h is a holonomy diffeomorphism and $c : [0, \infty) \to M$ is an OSL starting at p, then $h \circ c$ is an OSL at $h(p)$. Therefore, $H(t, h(p)) = h \circ H(t, p)$ by the property mentioned in the above paragraph, and the claim follows from the fact that any two points in a dual leaf are joined by a piecewise-smooth horizontal geodesic. $\square$

Perelman constructed a Sharafutdinov-like retraction for open Alexandrov spaces with nonnegative curvature [6]. It turns out that if A is the Alexandrov space from Theorem 3.3, then the submetry $\sigma_A : M \to A$ preserves that construction:

Theorem 3.7. *Let $c : [0, 1] \to M$ be the Sharafutdinov line with $c(0) = p$, $c(1) \in S$. Then $\bar{c} := \sigma_A \circ c : [0, 1] \to A$ is the unique Sharafutdinov line with $\bar{c}(0) = \sigma_A(p)$, $\bar{c}(1) \in \sigma_A(S)$.*

Proof. According to [6], the Sharafutdinov lines are left tangent to a "gradient" vector field of b: namely, for $p \in M$ with $b(p) > 0$, the set $C = b^{-1}(b(p))$ is totally convex and thus an Alexandrov space with nonnegative curvature. Its space of directions at p is an Alexandrov space with curvature greater than or equal to 1, and hence contains a unique "soul"; i.e, a direction at maximal distance from its boundary in the tangent space. Renormalizing this field suitably, it can be shown that it has left continuous integral curves tracing the Sharafutdinov retraction.

In our case, and from the arguments in [6], it remains to show that the Sharafutdinov fields of M and A are related by means of the derivative of σ_A. However, this is clear from the description of the tangent spaces of the C_t's given in [10], together with the fact that the σ_A-images of the C_t's form precisely the convex exhaustion of A. $\square$

References

1. Jeff Cheeger and Detlef Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972), 413–443. MR0309010 (46:8121)
2. Luis Guijarro, *On the metric structure of open manifolds with nonnegative curvature*, Pacific J. Math. **196** (2000), no. 2, 429–444. MR1800586 (2002b:53042)
3. Luis Guijarro and Gerard Walschap, *The metric projection onto the soul*, Trans. Amer. Math. Soc. **352** (2000), no. 1, 55–69. MR1487617 (2000c:53034)
4. ———, *Transitive holonomy group and rigidity in nonnegative curvature*, Math. Z. **237** (2001), no. 2, 251–257. MR1838310 (2002j:53055)
5. G. Perelman, *Proof of the soul conjecture of Cheeger and Gromoll*, J. Differential Geom. **40** (1994), no. 1, 209–212. MR1285534 (95d:53037)
6. Grisha Perelman, *Aleksandrov spaces with curvature bounded below II*, Unpublished.
7. V. A. Šarafutdinov, *The Pogorelov-Klingenberg theorem for manifolds that are homeomorphic to $\mathbf{R}^n$*, Sibirsk. Mat. Ž. **18** (1977), no. 4, 915–925, 958. MR0487896 (58:7488)
8. Gerard Walschap, *Nonnegatively curved manifolds with souls of codimension* 2, J. Differential Geom. **27** (1988), no. 3, 525–537. MR940117 (89g:53067)
9. Burkhard Wilking, *A duality theorem for riemannian foliations in nonnegative sectional curvature*, Geometric and Functional Analysis (to appear).
10. Jin-Whan Yim, *Distance nonincreasing retraction on a complete open manifold of nonnegative sectional curvature*, Ann. Global Anal. Geom. **6** (1988), no. 2, 191–206. MR982765 (90a:53049)

Department of Mathematics, Universidad Autónoma de Madrid, Spain
E-mail address: `luis.guijarro@uam.es`

Department of Mathematics, University of Oklahoma, Norman, Oklahoma
E-mail address: `gerard@ou.edu`