

ON THE SYMPLECTIC PHASE SPACE OF KdV

T. KAPPELER, F. SERIER, AND P. TOPALOV

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ABSTRACT. We prove that the Birkhoff map Ω for KdV constructed on $H_0^{-1}(\mathbb{T})$ can be interpolated between $H_0^{-1}(\mathbb{T})$ and $L_0^2(\mathbb{T})$. In particular, the symplectic phase space $H_0^{1/2}(\mathbb{T})$ can be described in terms of Birkhoff coordinates.

1. INTRODUCTION

In [12] it is shown that the Birkhoff map for the Korteweg - de Vries equation (KdV), on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, introduced and studied in detail in [9, 6], can be analytically extended to a real analytic diffeomorphism

$$\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$$

from the Sobolev space of distributions $H_0^{-1}(\mathbb{T})$ (dual of $H_0^1(\mathbb{T})$) to the Hilbert space of sequences $\mathfrak{h}^{-1/2}$ where for any $\alpha \in \mathbb{R}$,

$$\mathfrak{h}^\alpha := \{z = (x_k, y_k)_{k \geq 1} \mid \|z\|_\alpha < \infty\},$$

with

$$\|z\|_\alpha := \left(\sum_{k \geq 1} k^{2\alpha} (x_k^2 + y_k^2) \right)^{1/2}.$$

In this paper we show that Ω can be interpolated between $H_0^{-1}(\mathbb{T})$ and $L_0^2(\mathbb{T})$.

Theorem 1. *For any $-1 \leq \alpha \leq 0$,*

$$\Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$$

is a real analytic diffeomorphism.

As an application of Theorem 1 we characterize the regularity of a potential $q \in H^{-1}(\mathbb{T})$ in terms of the decay of the gap lengths $(\gamma_k)_{k \geq 1}$ of the periodic spectrum of Hill's operator $-\frac{d^2}{dx^2} + q$ on the interval $[0, 2]$. More precisely, recall that the periodic spectrum of $-\frac{d^2}{dx^2} + q$ on the interval $[0, 2]$ is discrete. When listed in increasing order (with multiplicities) the eigenvalues $(\lambda_k)_{k \geq 0}$ satisfy

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots$$

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The gap lengths $\gamma_k = \gamma_k(q)$ are then defined by

$$\gamma_k := \lambda_{2k} - \lambda_{2k-1} \quad (k \geq 1).$$

Theorem 2. *For any $q \in H^{-1}(\mathbb{T})$ and any $-1 \leq \alpha \leq 0$, the potential q is in $H^\alpha(\mathbb{T})$ if and only if $(\gamma_k(q))_{k \geq 1} \in \mathfrak{h}^\alpha$.*

In a subsequent paper we will use Theorem 1 to study the solutions of the KdV equation (see [2, 3], [14], [21]) in the symplectic phase space $H_0^{-1/2}(\mathbb{T})$ introduced by Kuksin [16].

Method of proof: Theorem 2 can be shown to be a consequence of Theorem 1 and formulas relating the n 'th action variable I_n with the n 'th gap length γ_n and their asymptotics as $n \rightarrow \infty$. In view of results established in [12] the proof of Theorem 1 consists in showing that for any $-1 < \alpha < 0$ the restriction of Ω to $H_0^\alpha(\mathbb{T})$, $\Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$ is onto. Our method of proof combines a study of the Birkhoff map at the origin together with a strikingly simple deformation argument to show that the map $\Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$ is onto. More precisely it uses that (1) $d_0\Omega_\alpha : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$ is a linear isomorphism, (2) that the map $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$ is a canonical bi-analytic diffeomorphism, and (3) that the Hamiltonian vector field defining the deformation is actually in L^2 . The same method could also be used for the proof of analogous results for more general weighted Sobolev spaces. In a subsequent work we plan to apply our technique to the defocusing nonlinear Schrödinger equation.

Related work: Theorem 1 improves on earlier results in [12] where it was shown that $\Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$ is a bianalytic diffeomorphism onto its image for any $-1 < \alpha < 0$. For partial results in this direction see also [18]. The statement of Theorem 2 adds to numerous results characterizing the regularity of a potential by the decay of the corresponding gap lengths; see e.g. [4], [7], [15], [17], [19] and the references therein. However only a few results concern potentials in spaces of distributions; see [8], [15] (cf. also [12] and the references therein). In a first attempt we have tried to apply the most beautiful and most simple approach among all the papers cited, due to Pöschel [19], to our case. However his methods seem to fail if $\alpha \leq -3/4$.

The idea of using flows to prove that a map is onto is not new in this subject. It has been used e.g. by Pöschel and Trubowitz in their book [20] or, to give a more recent example, in work of Chelkak and Korotyaev [1]. More precisely, in [20, Theorem 2, p. 115], the authors use flows to characterize sequences coming up as sequences of Dirichlet eigenvalues of Schrödinger operators $-\frac{d^2}{dx^2} + q$ on $[0, 1]$ with an even L^2 -potential q . Note however, that in this paper the use of flows is of a different nature, best explained by the fact that they are regularizing - in other words, the vector fields describing the deformations are in a higher Sobolev space than the underlying phase space.

2. PROOF OF THEOREM 1

Let Ω be the Birkhoff map $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$ constructed in [12]; see also the Appendix for a brief summary of the results in [12]. By Theorem 3 in the Appendix, the Birkhoff map Ω is onto and for any given $\alpha > -1$ its restriction to

$H_0^\alpha(\mathbb{T})$ is a map

$$(1) \quad \Omega_\alpha := \Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$$

which is a bianalytic diffeomorphism onto its image. Hence, in order to prove Theorem 1 we need to prove that (1) is onto.

Assume that there exists $-1 \leq \alpha \leq 0$ such that $\Omega_\alpha : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$ is *not* onto. As $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$ is onto it then follows that there exists

$$(2) \quad q_0 \in H_0^{-1}(\mathbb{T}) \setminus H_0^\alpha(\mathbb{T})$$

such that $\Omega(q_0) \in \mathfrak{h}^{\alpha+1/2}$.

As $\Omega(0) = 0$ and as by Corollary 1 in the Appendix below the differential

$$d_0\Omega_\alpha : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$$

of (1) at $q = 0$ is a linear isomorphism, one gets from the inverse function theorem that there exist an open neighborhood U_α of zero in $H_0^\alpha(\mathbb{T})$ and an open neighborhood V_α of zero in $\mathfrak{h}^{\alpha+1/2}$ such that

$$(3) \quad \Omega|_{U_\alpha} : U_\alpha \rightarrow V_\alpha$$

is a diffeomorphism.

Recall that for any $k \geq 1$ the angle variable θ_k constructed in [12] is a real-analytic function on $H_0^{-1}(\mathbb{T}) \setminus D_k$ with values in $\mathbb{R}/2\pi\mathbb{Z}$ where $D_k := \{q \in H_0^{-1}(\mathbb{T}) \mid \gamma_k(q) = 0\}$ is a real-analytic sub-variety in $H_0^{-1}(\mathbb{T})$ (cf. the Appendix). As θ_k is real-analytic, the mapping $H_0^{-1}(\mathbb{T}) \setminus D_k \rightarrow H_0^1(\mathbb{T})$, $q \mapsto \frac{\partial\theta_k}{\partial q}(q)$, is real-analytic¹ and therefore,

$$(4) \quad H_0^{-1}(\mathbb{T}) \setminus D_k \rightarrow L_0^2(\mathbb{T}), \quad q \mapsto Y_k(q) := \frac{d}{dx} \frac{\partial\theta_k}{\partial q}(q),$$

is real-analytic as well. Then Y_k is a Hamiltonian vector field on $H_0^{-1}(\mathbb{T}) \setminus D_k$, which defines a dynamical system

$$(5) \quad \dot{q} = Y_k(q), \quad q(0) = q_0 \in H_0^{-1}(\mathbb{T}) \setminus D_k.$$

Let $q_0 \in H_0^{-1}(\mathbb{T}) \setminus D_k$ and assume that

$$\Omega(q_0) = (z_1^0, z_2^0, \dots) \in \mathfrak{h}^{\alpha+1/2}$$

where for any $n \geq 1$, $z_n^0 = (x_n^0, y_n^0)$. Take $\varepsilon > 0$ such that the ball

$$B(2\varepsilon) := \{z \in \mathfrak{h}^{\alpha+1/2} \mid \|z\|_{\alpha+1/2} < 2\varepsilon\}$$

is contained in the neighborhood V_α of zero in $\mathfrak{h}^{\alpha+1/2}$ chosen in (3). Denote by $I_n = I_n(q)$ the n 'th action variable of a potential q ; see the Appendix. Note that for any q in $H_0^{-1}(\mathbb{T})$

$$(6) \quad 2I_n(q) = \|z_n(q)\|^2 = x_n(q)^2 + y_n(q)^2$$

where $\Omega(q) = (z_n(q))_{n \geq 1}$ and $z_n(q) = (x_n(q), y_n(q))$. Consider the sequence of potentials $(q_n)_{n \geq 1}$ in $H_0^{-1}(\mathbb{T})$ defined recursively for $n \geq 1$ by

$$q_n := \begin{cases} q_{n-1} & \text{if } 2I_n(q_{n-1}) < \varepsilon / (n^{1+2\alpha} 2^n), \\ (q_{n-1})_n & \text{otherwise,} \end{cases}$$

¹ $\frac{\partial\theta_k}{\partial q}$ denotes the L^2 -gradient of θ_k .

where $(q_{n-1})_n$ is obtained by shifting q_{n-1} along the flow of the vector field Y_n such that

$$2I_n((q_{n-1})_n) < \varepsilon / (n^{1+2\alpha} 2^n).$$

The existence of $(q_{n-1})_n$ follows from Lemma 1 (a) below. Moreover, by the commutator relations (19) in the Appendix,

$$Y_n(I_m) = \{I_m, \theta_n\} = 0 \quad (n \neq m),$$

the vector field Y_n preserves the values of the action variables I_m for any $m \neq n$. In particular, we get

$$(7) \quad 2I_j(q_n) \leq \varepsilon / (j^{1+2\alpha} 2^j), \quad \forall 1 \leq j \leq n,$$

and

$$(8) \quad 2I_j(q_n) = \|z_j^0\|^2, \quad \forall j > n.$$

One obtains from (7), (8), and $\|z_j\|^2 = 2I_j$ (cf. (17)) that

$$(9) \quad \|\Omega(q_n)\|_{\alpha+1/2}^2 = \sum_{j=1}^{\infty} j^{1+2\alpha} \|z_j(q_n)\|^2 \leq \varepsilon \sum_{1 \leq j \leq n} \frac{1}{2^j} + \sum_{j \geq n+1} j^{1+2\alpha} \|z_j^0\|^2.$$

As $\sum_{j \geq 1} j^{1+2\alpha} \|z_j^0\|^2 = \|\Omega(q_0)\|_{\alpha+1/2}^2 < \infty$, one gets from (9) that there exists $N \geq 1$ such that for any $n \geq N$, $\|\Omega(q_n)\|_{\alpha+1/2} < 2\varepsilon$. In particular, $\Omega(q_N) \in V_\alpha$ and, as $\Omega|_{U_\alpha} : U_\alpha \rightarrow V_\alpha$ is a diffeomorphism, the bijectivity of the Birkhoff map $\Omega : H_0^{-1} \rightarrow \mathfrak{h}^{-1/2}$ implies that

$$(10) \quad q_N \in U_\alpha \subseteq H_0^\alpha(\mathbb{T}).$$

On the other side, it follows from (2) and Lemma 1 (b) that

$$(q_n)_{n \geq 1} \subseteq H_0^{-1}(\mathbb{T}) \setminus H_0^\alpha(\mathbb{T})$$

which implies $q_N \in H_0^{-1}(\mathbb{T}) \setminus H_0^\alpha(\mathbb{T})$, contradicting (10). This completes the proof of Theorem 1. □

The following lemma was used in the proof of Theorem 1.

Lemma 1. *For any $k \geq 1$ and for any initial data $q_0 \in H_0^{-1}(\mathbb{T}) \setminus D_k$ the initial value problem (5) has a unique solution in $C^1((-I_k^0, \infty), H_0^{-1}(\mathbb{T}))$ where $I_k^0 \geq 0$ is the value of the action variable I_k at q_0 . The solution has the following additional properties:*

- (a) $\lim_{t \rightarrow -I_k^0+0} I_k(q(t)) = 0;$
- (b) $q(t) - q_0 \in L_0^2(\mathbb{T}).$

Proof of Lemma 1. By Theorem 3 in the Appendix, the Birkhoff map $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$,

$$q \mapsto \Omega(q) = (z_1, z_2, \dots), \quad z_n = (x_n, y_n),$$

is a bianalytic diffeomorphism that transforms the Poisson structure $\frac{d}{dx}$ on $H_0^{-1}(\mathbb{T})$ (cf. the Appendix) into the canonical Poisson structure on $\mathfrak{h}^{-1/2}$ defined by the relations $\{x_m, x_n\} = \{y_m, y_n\} = 0$ and $\{x_m, y_n\} = \delta_{mn}$ that hold for any $m, n \geq 1$.² Moreover, it follows from the construction of the Birkhoff map Ω that θ_k is the

²Here δ_{mn} denotes the Kronecker delta.

argument of the complex number $x_k + iy_k$. In particular, in Birkhoff coordinates $(z_1, z_2, \dots) \in \mathfrak{h}^{-1/2}$, one has for any $q \in H_0^{-1}(\mathbb{T}) \setminus D_k$

$$(11) \quad d\Omega(Y_k) = \frac{x_k}{x_k^2 + y_k^2} \frac{\partial}{\partial x_k} + \frac{y_k}{x_k^2 + y_k^2} \frac{\partial}{\partial y_k}.$$

The dynamical system corresponding to the vector field (11) in $\mathfrak{h}^{-1/2}$ has a unique solution for any initial data $(x_n^0, y_n^0)_{n \geq 1}$ that is defined on the time interval $(-((x_k^0)^2 + (y_k^0)^2)/2, \infty)$. Hence, as $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$ is a diffeomorphism, the dynamical system (5) has a unique solution $q(t)$ on $H_0^{-1}(\mathbb{T}) \setminus D_k$ defined for $t \in (-I_k(q_0), \infty)$. Moreover, one gets from (11) and (6) that

$$\lim_{t \rightarrow -I_k(q_0)+0} I_k(q(t)) = 0.$$

This completes the proof of (a). In order to prove (b) we integrate both sides of (5) in $H_0^{-1}(\mathbb{T})$ and get that for any $t \in (-\infty, I_k(q_0))$,

$$(12) \quad q(t) = q_0 + \int_0^t Y_k(q(s)) ds.$$

As the mapping (4) is real-analytic (and hence, continuous) and as the solution $q(t)$ of (5) is a C^1 -curve $(-\infty, I_k(q_0)) \rightarrow H_0^{-1}(\mathbb{T})$, the integrand in (12) is in $C^0((-\infty, I_k(q_0)), L_0^2(\mathbb{T}))$. In particular, the integral in (12) converges with respect to the L^2 -norm, and hence represents an element in $L_0^2(\mathbb{T})$. This proves (b). \square

3. PROOF OF THEOREM 2

Note that for any constant $c \in \mathbb{R}$, the potentials q and $q + c$ have the same sequence of gap lengths $(\gamma_k)_{k \geq 1}$. Hence it is enough to prove the statement of the theorem for $q \in H_0^{-1}(\mathbb{T})$.

For $q \in H_0^{-1}(\mathbb{T})$ given, let

$$z = (z_1, z_2, \dots) = \Omega(q),$$

where for any $n \geq 1$, $z_n = (x_n, y_n)$. By Proposition 1 in the Appendix, there exist constants $0 < C_1 < C_2 < \infty$ and $n_0 \geq 1$ depending on q such that for any $n \geq n_0$,

$$(13) \quad C_1 \frac{\gamma_n^2}{n} \leq I_n \leq C_2 \frac{\gamma_n^2}{n}$$

where I_n is the n 'th action variable of the given potential q . Using that

$$I_n = (x_n^2 + y_n^2)/2$$

we get from (13) that for any given $\alpha \geq -1$,

$$(14) \quad (z_n)_{n \geq 1} \in \mathfrak{h}^{\alpha+1/2} \iff (\gamma_n)_{n \geq 1} \in \mathfrak{h}^\alpha.$$

On the other side, it follows from Theorem 1 and the injectivity of $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$ that

$$(15) \quad (z_n)_{n \geq 1} \in \mathfrak{h}^{\alpha+1/2} \iff q \in H_0^\alpha(\mathbb{T}).$$

Theorem 2 now follows from (14) and (15). \square

4. APPENDIX

In this Appendix we collect the properties of the Birkhoff map $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$ constructed in [12] that were used in the proofs of Theorem 1 and Theorem 2.

The Korteweg - de Vries equation (KdV)

$$\begin{aligned} q_t - 6qq_x + q_{xxx} &= 0, \\ q|_{t=0} &= q_0 \end{aligned}$$

on the circle can be viewed as an integrable PDE, i.e. an integrable Hamiltonian system of infinite dimension. As a phase space we consider the Sobolev space $H^\alpha(\mathbb{T})$ ($\alpha \geq -1$) of real valued distributions on the circle. The Poisson bracket is the one proposed by Gardner,

$$(16) \quad \{F, G\} := \int_{\mathbb{T}} \frac{\partial F}{\partial q} \frac{d}{dx} \left(\frac{\partial G}{\partial q} \right) dx$$

where F, G are C^1 -functions on $H^\alpha(\mathbb{T})$ and $\frac{\partial F}{\partial q}, \frac{\partial G}{\partial q}$ denote the L^2 -gradients of F and G , respectively, which are assumed to be sufficiently smooth so that the Poisson bracket is well defined. For q sufficiently smooth, i.e. $q \in H_0^1(\mathbb{T})$, the Hamiltonian \mathcal{H} corresponding to KdV is given by

$$\mathcal{H}(q) = \int_{\mathbb{T}} ((\partial_x q)^2/2 + q^3) dx$$

and the KdV equation can be written in Hamiltonian form

$$q_t = \frac{d}{dx} \frac{\partial \mathcal{H}}{\partial q}.$$

Note that the Poisson structure is degenerate and admits the average $[q] := \int_{\mathbb{T}} q(x) dx$ as a Casimir function. Moreover, the Poisson structure is regular and induces a trivial foliation whose leaves are given by

$$H_c^\alpha(\mathbb{T}) = \{q \in H^\alpha(\mathbb{T}) \mid [q] = c\}.$$

Introduce the set

$$D_k := \{q \in H_0^{-1}(\mathbb{T}) \mid \gamma_k(q) = 0\}.$$

For any $q \in H_0^{-1}(\mathbb{T}) \setminus D_k$ define

$$(17) \quad z_k(q) := \sqrt{2I_k(q)} \begin{pmatrix} \cos(\theta_k(q)) \\ \sin(\theta_k(q)) \end{pmatrix},$$

where $I_k(q)$ is the k 'th action variable and $\theta_k(q)$ is the k 'th angle variable of the KdV equation (cf. § 3, 4 in [12]). It is shown in [12, § 5] that the mapping $H_0^{-1} \setminus D_k \rightarrow \mathbb{R}^2$, $q \mapsto z_k(q)$, extends analytically to $H_0^{-1}(\mathbb{T})$. For any $q \in H_0^{-1}(\mathbb{T})$ the action variables $(I_k)_{k \geq 1}$ of KdV are defined in terms of the periodic spectrum of the Schrödinger operator $-\frac{d^2}{dx^2} + q$ using the same formulas as in [5] (cf. also [9]). For any given $\alpha \geq -1$ and for any $k \geq 1$ the action I_k is a real analytic function on $H_0^\alpha(\mathbb{T})$ (cf. Proposition 3.3 in [12]). The angle θ_k is defined modulo 2π and is a real analytic function on $H_0^\alpha(\mathbb{T}) \setminus (D_k \cap H_0^\alpha)$, where $D_k \cap H_0^\alpha = \{q \in H_0^\alpha(\mathbb{T}) \mid \gamma_k(q) = 0\}$ is a real analytic sub-variety in $H_0^\alpha(\mathbb{T})$ of co-dimension two (cf. Proposition 4.3 in [12]). By § 6 in [12] we have the following commutator relations:

$$(18) \quad \{I_m, I_n\} = 0 \quad \text{on } H_0^{-1}(\mathbb{T}),$$

$$(19) \quad \{I_m, \theta_n\} = \delta_{nm} \quad \text{on } H_0^{-1}(\mathbb{T}) \setminus D_n$$

and

$$(20) \quad \{\theta_m, \theta_n\} = 0 \text{ on } H_0^{-1}(\mathbb{T}) \setminus (D_m \cup D_n)$$

for any $m, n \geq 1$. For any $q \in H_0^{-1}(\mathbb{T})$ define

$$\Omega(q) := (z_1(q), z_2(q), \dots)$$

where $z_k = z_k(q)$ is given by (17). It is shown in [12] that $\Omega(q) \in \mathfrak{h}^{-1/2}$. Recall that, for any $\alpha \in \mathbb{R}$, \mathfrak{h}^α denotes the Hilbert space

$$\mathfrak{h}^\alpha = \{z = (x_k, y_k)_{k \geq 1} \mid \|z\|_\alpha < \infty\},$$

with the norm

$$\|z\|_\alpha := \left(\sum_{k \geq 1} k^{2\alpha} (x_k^2 + y_k^2) \right)^{1/2}.$$

We supply $\mathfrak{h}^{-1/2}$ with a Poisson structure defined by the relations $\{x_m, x_n\} = \{y_m, y_n\} = 0$ and $\{x_m, y_n\} = \delta_{mn}$ valid for any $m, n \geq 1$. The following result is proved in [12].

Theorem 3. *The mapping $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$ satisfies the following properties:*

- (i) Ω is a bianalytic diffeomorphism that preserves the Poisson bracket;
- (ii) for any $\alpha > -1$, the restriction $\Omega_\alpha \equiv \Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$ which is one-to-one and bianalytic onto its image. In particular, the image is an open subset in $\mathfrak{h}^{\alpha+1/2}$.

Corollary 1. *For any $\alpha > -1$,*

$$d_0 \Omega_\alpha : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$$

is a linear isomorphism.

We will also need the following proposition (cf. [12, § 3]).

Proposition 1. *There exists a complex neighborhood \mathcal{W} of $H_0^{-1}(\mathbb{T})$ in the complex space $H_0^{-1}(\mathbb{T}, \mathbb{C})$ such that the quotient I_n/γ_n^2 , defined on $H_0^{-1}(\mathbb{T}) \setminus D_n$, extends analytically to \mathcal{W} for all n . Moreover, for any $\varepsilon > 0$ and any $p \in \mathcal{W}$ there exists $n_0 \geq 1$ and an open neighborhood $U(p)$ of p in \mathcal{W} so that*

$$\left| 8\pi n \frac{I_n}{\gamma_n^2} - 1 \right| \leq \varepsilon$$

for any $n \geq n_0$ and for any $q \in U(p)$.

Further we recall that for any $q \in H_0^{-1}(\mathbb{T})$ one has that $I_n(q) = 0$ if and only if $\gamma_n(q) = 0$. In particular, one concludes from (17) and the fact $\gamma_n(0) = 0, \forall n \geq 1$, that $\Omega(0) = 0$.

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MATHEMATICS INSTITUTE, UNIVERSITY OF ZURICH, WINTERTHURERSTR. 190, 8057 ZURICH, SWITZERLAND

E-mail address: tk@math.unizh.ch

ECOLE CENTRALE DE LYON, INSTITUT C. JORDAN, UMR CNRS 5208, 36 AVENUE GUY DE COLLONGUE, 69134 ECULLY, CEDEX, FRANCE

E-mail address: frederic.serier@ec-lyon.fr

DEPARTMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, 360 HUNTINGTON AVE., BOSTON, MASSACHUSETTS 02115

E-mail address: p.topalov@neu.edu