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SPECTRA OF OPERATORS WITH BISHOP'S PROPERTY (β)

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ABSTRACT. Let X be a Banach space and let $\mathcal{A}(X)$ be the class that consists of all operators $T \in \mathcal{L}(X)$ such that for every $\lambda \in \mathbb{C}$, the range of $(T - \lambda I)$ has a finite-codimension when it is closed. For an integer $n \in \mathbb{N}$, we define the class $\mathcal{A}_n(X)$ as an extension of $\mathcal{A}(X)$. We then study spectral properties of such operators, and we extend some known results of multi-cyclic operators with (β).

INTRODUCTION

The concept of quasisimilarity has been studied by many authors, and it is well-known that this concept generally does not conserve the spectral structure of an operator (see for example [1, 15]). Additional references including certain relationships between spectra of specific quasisimilar operators include [4, 6, 9] and [11]. In this paper, for a Banach space X and an integer $n \in \mathbb{N}$, we define classes $\mathcal{A}_n(X)$ and $\mathcal{N}_n(X)$ as an extension of the classes $\mathcal{A}(X)$ and $\mathcal{N}(X)$, respectively, introduced in [6]. Among other results we compare these classes, and we show that if $T \in \mathcal{A}_n(X)$ and $S \in \mathcal{A}_m(Y)$ are quasisimilar operators with Bishop's property (β) on the Banach spaces X and Y for some tuple $(n, m) \in \mathbb{N}^2$, then T and S have the same approximate point spectrum, continuous spectrum and generalized spectrum and the same vein of other spectra.

1. PRELIMINARIES

Throughout this paper, X and Y are Banach spaces and $\mathcal{L}(X, Y)$ denotes the space of all bounded linear operators from X to Y. We set $\mathcal{L}(X) := \mathcal{L}(X, X)$ and for a bounded linear operator $A \in \mathcal{L}(X, Y)$, let A^*, $N(A)$, and $R(A)$ denote the adjoint operator, the null space, and the range of A. Also, we set $\alpha(A) := \dim N(A)$ and $\beta(A) := \dim N(A^*)$.

For an operator $T \in \mathcal{L}(X)$, we denote by $\mathrm{Lat}(T)$ the lattice of all closed T-invariant subspaces of X, and for $M \in \mathrm{Lat}(T)$, let $T|M \in \mathcal{L}(M)$ be the restriction of T to M.

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For X a Banach space, we define the class $\mathcal{A}(X)$ as the set of all operators $T \in \mathcal{L}(X)$ which satisfy the next relation

$$\rho_f(T) \subseteq \rho_{re}(T),$$

where $\rho_f(T) = \{\lambda \in \mathbb{C} : R(T - \lambda I) \text{ is closed } \}$ and $\rho_{re}(T)$ is the right essential spectrum. For $n \geq 1$ an integer, we denote by $\mathcal{A}_n(X)$ the set of all operators $T \in \mathcal{L}(X)$ such that there exists a decreasing family of closed subsets $(X_0, \ldots, X_n) \in \mathrm{Lat}(T)^{n+1}$ for which $X_0 = X$, $T|X_n \in \mathcal{A}(X_n)$ and $TX_i \subseteq X_{i+1}$ for $i = 0, 1, \ldots, n-1$. Also, we set $\mathcal{A}_0(X) := \mathcal{A}(X)$. Clearly, the class $\mathcal{A}_1(X)$ contains all compact operators, but finite rank operators are not in $\mathcal{A}(X)$ for any infinite dimensional Banach space X. More examples are given in the following:

(1) If X is finite dimensional, then every linear operator is in $\mathcal{A}(X)$.
(2) Every nilpotent operator T of order n is in $\mathcal{A}_n(X)$. Indeed, we can take $X_i = ker(T^{n-i})$. If $dim X = +\infty$, then $T \notin \mathcal{A}(X)$.
(3) If T is a normal operator such that every isolated point of $\sigma(T)$ is an eigenvalue of finite multiplicity, then T is in $\mathcal{A}(X)$; see J.B.Conway [5], XI Propositions 4.5 and 4.6.

We say that $T \in \mathcal{L}(X)$ has the single-valued extension property, abbreviated SVEP, if, for every open subset $U \subseteq \mathbb{C}$, the only analytic solution $f : U \longrightarrow X$ of the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function on U . Furthermore, we say that T has Bishop's property (β), if, for each open subset U of $\mathbb{C}$ and every sequence of analytic functions $f_n : U \longrightarrow X$ for which $(T - \lambda I)f_n(\lambda) \to 0$ as $n \to \infty$, locally uniformly on U, it follows that $f_n(\lambda) \to 0$ as $n \to \infty$, again locally uniformly on U. For further details about these notions we refer to [2], [12] and [16].

Along this paper, we use $\sigma_p, \sigma_{ap}, \sigma_e,$ and σ_{SF} to denote the point, approximate point, Fredholm essential , and semi-Fredholm essential spectrum, respectively. We also denote by σ_{le} and σ_{re} the upper and lower semi-Fredholm essential spectra. Also, we use $\rho_{ap}, \rho_e, \rho_{SF}$, and ρ_{le} to denote the complement in $\mathbb{C}$ of $\sigma_{ap}, \sigma_e, \sigma_{SF}$ and σ_{le}, respectively. Recall, when T is a semi-Fredholm operator, the index of T is defined as $\mathrm{ind}(T) := \alpha(T) - \beta(T)$.

Consider an arbitrary operator $T \in \mathcal{L}(X)$ and let $\sigma_f(T)$ consist of all $\lambda \in \mathbb{C}$ for which $(T - \lambda I)$ fails to have closed range, and let $\rho_K(T)$ consist of all $\lambda \in \mathbb{C}$ for which $R(T - \lambda I)$ is closed and $N(T - \lambda I) \subseteq R((T - \lambda I)^n)$ for all $n \in \mathbb{N}$. The Kato spectrum $\sigma_K(T) := \mathbb{C} \setminus \rho_K(T)$ is sometimes referred to as the semi-regular spectrum or the Apostol spectrum. Finally, we set $\rho_f(T) := \mathbb{C} \setminus \sigma_f(T)$.

Next, for $T \in \mathcal{A}_n(X)$, we set $T_i := T|X_i$ for $i = 1, \ldots, n$, and recall that, for $\sigma_* \in \{\sigma_e, \sigma_{re}, \sigma_{le}, \sigma_{SF}, \sigma_{ap}, \sigma_f\}$, from [3, Theorems 3, 5, 6 and Example 11] we have

$$\sigma_*(T) \setminus \{0\} = \sigma_*(T_1) \setminus \{0\} \text{ and } \sigma_*(T_i) \setminus \{0\} = \sigma_*(T_{i+1}) \setminus \{0\} \text{ for } i = 1, \ldots, n-1.$$

Consequently,

$$\sigma_*(T) \setminus \{0\} = \sigma_*(T_n) \setminus \{0\} \quad \text{for} \quad \sigma_* \in \{\sigma_e, \sigma_{re}, \sigma_{le}, \sigma_{SF}, \sigma_{ap}, \sigma_f\}. \tag{1.1}$$

Proposition 1.1 ([6, Theorem 4.1])**.** *Let $T \in \mathcal{L}(X)$ and suppose that $T \in \mathcal{A}(X)$. Then the following statements hold:*

(1) $\rho_{SF}(T) = \rho_{re}(T) = \rho_f(T)$.
(2) $\rho_e(T) = \rho_{le}(T)$.
(3) *Moreover, if T has the SVEP, then*

$$\sigma_e(T) = \sigma_{SF}(T) = \sigma_f(T) = \sigma_{le}(T) = \sigma_{re}(T).$$

As an immediate consequence of Proposition 1.1 and (1.1) we get

Remark 1.1. For $T \in \mathcal{A}_n(X)$, we have

i) $\sigma_{re}(T) \setminus \{0\} = \sigma_f(T) \setminus \{0\}$;
ii) $\sigma_{le}(T) \setminus \{0\} = \sigma_f(T) \setminus \{0\}$ when T has the SVEP.

Recall that a linear bounded operator with closed range is called normally solvable, and for an operator $T \in \mathcal{L}(X)$, the minimum modulus of T, written $\gamma(T)$, is defined by

$$\gamma(T) := \inf\{\frac{\|Tx\|}{\mathrm{d}(x, N(T))} \quad : \quad x \in X \setminus N(T)\}$$

where d $(x, N(T))$ denotes the distance from x to $N(T)$ and with the convention that $\gamma(T) = \infty$ if $T = 0$. Notice here that T has closed range if and only if $\gamma(T) > 0$. We refer to [8] for further details and definitions.

Proposition 1.2. *If $T \in \mathcal{A}_n(X)$, then the following assertions hold:*
(i) *If $R(T)$ is not closed, then $T \in \mathcal{A}(X)$.*
(ii) *If $T \in \mathcal{A}_n(X) \setminus \mathcal{A}(X)$, then $R(T)$ is closed and $\alpha(T) = \beta(T) = +\infty$.*

Proof. (i) Since $T \in \mathcal{A}_n(X)$, it follows from (1.1) that

$$\sigma_{re}(T) \setminus \{0\} = \sigma_f(T) \setminus \{0\}. \tag{1.2}$$

It is clear that, if $R(T)$ is not closed, then $\sigma_{re}(T) = \sigma_f(T)$, which implies $T \in \mathcal{A}(X)$.

(ii) If $T \in \mathcal{A}_n(X) \setminus \mathcal{A}(X)$ using (i) $R(T)$ is closed. It is clear that $\beta(T) = +\infty$. Suppose that $\alpha(T)$ is finite; it will come that T is normally solvable and has an index. Let $\lambda \in \mathbb{C}$ be given such that $0 < |\lambda| < \gamma(T)$. From Theorem V.1.6 of [8], we get

$T - \lambda I$ is normally solvable ; $\alpha(T - \lambda I) \leq \alpha(T)$, and ind $(T - \lambda I) =$ ind (T).

Hence $\lambda \in \rho_f(T)$ and (1.2) implies that $\lambda \in \rho_{re}(T)$. Consequently, $\alpha(T - \lambda I)$ and $\beta(T - \lambda I)$ are both finite. Finally it follows from the fact that $\alpha(T)$ is finite and ind $(T) =$ ind $(T - \lambda I)$ that $\beta(T)$ is finite. Contradiction, which completes the argument. □

A straightforward application of Proposition 1.1 and Proposition 1.2, gives the following result

Theorem 1.2. *Suppose that $T \in \mathcal{A}_n(X) \setminus \mathcal{A}(X)$. Then the following statements hold:*
(1) $\sigma_{SF}(T) = \sigma_{re}(T) = \sigma_f(T) \cup \{0\}$.
(2) $\sigma_e(T) = \sigma_{le}(T)$.
(3) *Moreover, if T has the SVEP, then*

$$\sigma_e(T) = \sigma_{SF}(T) = \sigma_{le}(T) = \sigma_{re}(T) = \sigma_f(T) \cup \{0\}.$$

2. The main results

In the following, for a subset E of a Banach space, we set span$\{E\}$ for the closure of the linear space generated by E and $\overline{E}$ for the closure of E. We also denote $\mathbb{A}_n(X)$ for the set of all operators in $\mathcal{A}_n(X)$ such that $X_i := \overline{R(T^i)}$ for $i = 1, \ldots, n$.

2.1. **Quasisimilar operators in $\mathcal{A}_n(X)$.** In this subsection, let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be quasisimilar operators, equivalently, there exist $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$ one to one and with dense ranges, such that $AT = SA$ and $TB = BS$. It is clear that, for each integer $n \geq 1$, T^n and S^n are quasisimilar operators. We set $\bar{T}_n := T|\overline{R(T^n)}$ and $\bar{S}_n := S|\overline{R(S^n)}$, and so, from Lemma 5.2 of [6], we obtain the following

Lemma 2.1. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be quasisimilar operators; then $\bar{T}_n$ and $\bar{S}_n$ are quasisimilar operators.*

Here we study the spectral picture of quasisimilar operators with property (β).

Proposition 2.1. *Suppose that $(T,S) \in \mathbb{A}_n(X) \times \mathbb{A}_n(Y)$ are quasisimilar operators with property (β); then*

$$\sigma_{ap}(T) = \sigma_{ap}(S).$$

Proof. Since $\bar{T}_n$ and $\bar{S}_n$ are quasisimilar operators by Lemma 2.1, and since $\bar{T}_n$ and $\bar{S}_n$ satisfy property (β), it follows from Theorem 3.7.15 of [12] together with (3) of Proposition 1.1, that

$$\sigma_f(\bar{T}_n) = \sigma_f(\bar{S}_n)$$

as $\sigma_p(\bar{T}_n) = \sigma_p(\bar{S}_n)$; thus

$$\sigma_{ap}(\bar{T}_n) = \sigma_{ap}(\bar{S}_n).$$

From this fact and (1.1), we deduce that

$$\sigma_{ap}(T) \setminus \{0\} = \sigma_{ap}(S) \setminus \{0\}.$$

Let us prove that $0 \in \sigma_{ap}(T)$ if and only if $0 \in \sigma_{ap}(S)$. Without loss of generality we can assume that T and S are not nilpotent operators (indeed, in this case, we have $\sigma_{ap}(T) = \sigma_{ap}(S) = \{0\}$).

Now, suppose that $0 \notin \sigma_{ap}(T)$, hence $0 \notin \sigma_{ap}(\bar{T}_n)$ and we get $0 \notin \sigma_{ap}(\bar{S}_n)$. It follows that $R(\bar{S}_n)$ is closed. Thus $\bar{S}_n$ is a Fredholm operator because $\bar{S}_n$ is injective and $\bar{S}_n \in \mathcal{A}(\overline{R(S^n)})$, and so, $\bar{S}_n^n$ is Fredholm operator in $\overline{R(S^n)}$. Consequently, there exists a finite-dimensional subspace F of $\overline{R(S^n)}$ such that

$$\overline{R(S^n)} = R(\bar{S}_n^n) \oplus F.$$

Since $R(\bar{S}_n^n) \subseteq R(S^n)$, it is easy to verify that

$$R(S^n) = R(\bar{S}_n^n) \oplus (F \cap R(S^n)).$$

Therefore, $R(S^n)$ is closed, which implies that S^n has a bounded inverse because S^n is injective. Thus, S also has a bounded inverse, and so $0 \notin \sigma_{ap}(S)$. Finally, the reverse implication is obtained by symmetry. $\square$

In what follows, let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$ be quasisimilar operators, and let $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$ be the quasi-affinities for which $AT = SA$ and $TB = BS$. In an easy way we refine Proposition 2.1 as follows

Theorem 2.1. *Let T and S be quasisimilar operators with property (β). Suppose that $T \in \mathcal{A}_n(X)$ and $S \in \mathcal{A}_m(Y)$ for some $(n,m) \in \mathbb{N}^2$. Then*

(1) $\sigma_*(T) = \sigma_*(S)$ *for* $\sigma_* \in \{\sigma_e, \sigma_{re}, \sigma_{le}, \sigma_{SF}, \sigma_{ap}\}$.
(2) $\sigma_f(T) \cup \{0\} = \sigma_f(S) \cup \{0\}$.

Furthermore, if $\alpha(T)$ is finite, then $\sigma_f(T) = \sigma_f(S)$.

Proof. Since T and S are with property (β), Theorem 3.7.15 of [12] together with the quasisimilarity of T and S imply that

$$\sigma_e(T) = \sigma_e(S), \tag{2.1}$$

and since Bishop's property implies the SVEP for an arbitrary bounded operator, Theorem 1.2 and Proposition 1.1 together with (1.2) implies that

$$\sigma_*(T) = \sigma_*(S) \qquad \text{for} \qquad \sigma_* \in \{\sigma_e, \sigma_{re}, \sigma_{le}, \sigma_{SF}\}$$

and

$$\sigma_f(T) \cup \{0\} = \sigma_f(S) \cup \{0\}.$$

If $\alpha(T)$ is finite, then $\alpha(S)$ is finite, and so, $(T, S) \in \mathcal{A}(X) \times \mathcal{A}(Y)$ by Proposition 1.2. Thus, $\sigma_f(T) = \sigma_f(S)$ by (2.1) and Proposition 1.1.

Now, let us prove $\sigma_{ap}(T) = \sigma_{ap}(S)$. It is known that

$$\sigma_{ap}(T) = \sigma_p(T) \cup \sigma_f(T) \quad \text{and} \quad \sigma_{ap}(S) = \sigma_p(S) \cup \sigma_f(S), \tag{2.2}$$

and it is clear that $\sigma_p(T) = \sigma_p(S)$. If $0 \in \sigma_p(T)$, then

$$\begin{aligned} \sigma_{ap}(T) &= \sigma_p(T) \cup \sigma_f(T) \cup \{0\} \\ &= \sigma_p(S) \cup \sigma_f(S) \cup \{0\} \\ &= \sigma_{ap}(S). \end{aligned}$$

If, $0 \notin \sigma_p(T)$, then $\alpha(T) = 0$ and so $\sigma_f(T) = \sigma_f(S)$. Hence by (2.2) we get $\sigma_{ap}(T) = \sigma_{ap}(S)$, which completes the proof. $\square$

Recall that, for $T \in \mathcal{L}(X)$, the continuous spectrum $\sigma_c(T)$ is defined to be the set of all $\lambda \in \mathbb{C}$ such that $(T - \lambda I)^{-1}$ exists but is not continuous and $R(T - \lambda I)$ is dense in X. Equivalently, T is one to one and with nontrivial dense range (see for example [8]). It is then not hard to see that

$$\sigma_c(T) = \sigma_p(T^*)^c \cap \sigma_p(T)^c \cap \sigma_f(T) \tag{2.3}$$

where K^c is the complement of the subset K of $\mathbb{C}$. Also, it is known that $(T - \lambda I)$ is regular if $\lambda \in \rho_K(T)$ and there exist two closed subspaces E_λ and F_λ of X such that

$$X = N(T - \lambda I) \oplus E_\lambda = R(T - \lambda I) \oplus F_\lambda.$$

Let, as usual, reg (T) denote the regular set of T, and let $\sigma_g(T)$ denote to the generalized spectrum of T, which is the complement of reg (T). We now extend the result of Theorem 2.1 to the continuous and the generalized spectrum.

Outlining the proof of [6, Lemma 5.2] we obtain the following result that will be needed in the next theorem.

Lemma 2.2. *Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$ and $A \in \mathcal{L}(X, Y)$ be operators such that $AT = SA$ and A has dense range. Then*

$$\overline{A\overline{R(T - \lambda I)}} = \overline{R(S - \lambda I)} \qquad \textit{for all} \qquad \lambda \in \mathbb{C}.$$

Theorem 2.2. *Suppose that, $T \in \mathcal{A}_n(X)$ and $S \in \mathcal{A}_m(Y)$ for some $(n, m) \in \mathbb{N}^2$ are quasisimilar operators with property (β). Then the following assertions hold:*

(*a*) $\sigma_c(T) = \sigma_c(S)$,
(*b*) $\sigma_g(T) = \sigma_g(S)$.

Proof. (a) First, let us prove that

$$\sigma_p(T)^c \cap \sigma_f(T) = \sigma_p(S)^c \cap \sigma_f(S). \tag{2.4}$$

In fact, it is obvious that $\sigma_p(T)^c = \sigma_p(S)^c$. If $0 \notin \sigma_p(T)$, Theorem 2.1 implies $\sigma_f(T) = \sigma_f(S)$, and so, (2.4) holds. In the other case, when $0 \in \sigma_p(T)$, we have

$$\begin{aligned} \sigma_p(T)^c \cap \sigma_f(T) &= \sigma_p(T)^c \cap (\sigma_f(T) \cup \{0\}) \\ &= \sigma_p(S)^c \cap (\sigma_f(S) \cup \{0\}) \\ &= \sigma_p(S)^c \cap \sigma_f(S). \end{aligned}$$

On the other hand, since T^* and S^* are quasisimilar operators, we get that $\sigma_p(T^*)^c = \sigma_p(S^*)^c$. This, together with (2.3) and (2.4) entail the desired result.

(b) Since T and S have the SVEP, then from Corollary 3.1.7 of [12], we have $\rho_K(T) = \rho_{ap}(T)$ and $\rho_K(S) = \rho_{ap}(S)$, and from Theorem 2.1 we obtain

$$\rho_K(T) = \rho_K(S). \tag{2.5}$$

Now, let $\lambda \in \text{reg }(T)$ be given. Since $\lambda \in \rho_K(T)$ and from (2.5) we get $\lambda \in \rho_K(S)$; hence $R(S - \lambda I)$ is closed and $N(S - \lambda I) = \{0\}$. On the other hand, there exist two closed subspaces $E_\lambda = X$ and F_λ of X such that

$$X = N(T - \lambda I) \oplus E_\lambda = R(T - \lambda I) \oplus F_\lambda.$$

Suppose that $\lambda \neq 0$; Remark 1.1 i) together with $\lambda \in \rho_f(T)$ and $T \in \mathcal{A}_n(X)$ imply that $\lambda \in \rho_{re}(T)$. Hence F_λ is finite-dimensional, and thus AF_λ is finite-dimensional. Hence,

$$\begin{aligned} Y = \overline{AX} &= \overline{AR(T - \lambda I) \oplus AF_\lambda} \qquad (A \text{ is injective}) \\ &= \overline{\overline{AR(T - \lambda I)} + AF_\lambda} \\ &= \overline{R(S - \lambda I) + AF_\lambda} \\ &= R(S - \lambda I) + AF_\lambda, \end{aligned}$$

because AF_λ is finite dimensional and $R(S-\lambda I)$ is closed; see [8]. Since $N(S-\lambda I) = \{0\}$, then

$$Y = N(S - \lambda I) \oplus Y.$$

Consequently, $\lambda \in \text{reg }(S)$.

Now, suppose that $\lambda = 0$, hence $0 \in \rho_K(T)$, and from $SVEP$, $\rho_K(T) = \rho_{ap}(T)$. It follows that $N(T) = \{0\}$, which implies $T \in \mathcal{A}(X)$ by Proposition 2.2. Thus F_0 is finite-dimensional, and therefore, for $\lambda \neq 0$, we have

$$Y = R(S) + AF_0 = N(S) \oplus Y.$$

Consequently, $0 \in \text{reg }(S)$. The reverse implication is obtained by symmetry. □

We notice that this result is true for multi-cyclic operators and hence extends Theorem 5.6 of [11].

3. Examples and applications

Next, we shall apply the previous results to some classical classes of operators.

3.1. **Unilateral weighted shifts.** Let H be a Hilbert space, let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis and let $(\omega_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative numbers. The unilateral weighted shift S_ω associated with the sequence $(\omega_n)_{n\in\mathbb{N}}$ is the operator defined on the basis by $S_\omega(e_n) = \omega_n e_{n+1}$ for $n \geq 0$. It is well known that S_ω is bounded precisely when w_n is a bounded sequence and that the spectrum is always a disc. Setting $r_1(S_\omega) = \lim_{n\to\infty} \inf_k(\omega_n \cdots \omega_{n+k})^{1/n}$, we have $\sigma_f(S_\omega) = \{z \in \mathbb{C}, \text{ such that } r_1(S_\omega) \leq |z| \leq r(S_\omega)\}$; here $r(S_\omega)$ stands for the usual spectral radius. For further information we refer to [14]. Since from [14] all eigenvectors of S_ω^* are simple, it follows that $\sigma_f(S_\omega) = \sigma_e(S_\omega) = \sigma_{re}(S_\omega)$ and hence that all shifts are in $\mathcal{A}$.

3.2. **Rationally cyclic operators.** Let T be a bounded operator on an infinite dimensional Hilbert space H. T is said to be rationally cyclic if there exists $x \in H$ such that

$$H = \{R(T)x : \ R \text{ is a rational function with poles off } \sigma(T)\}.$$

It is not difficult to see in this case that $dimker(T-\lambda)^* \leq 1$ for any complex number λ; it follows that $T \in \mathcal{A}$. Now if we consider any power of T, it may fail to be rationally cyclic. However all results here apply to this setting.

3.3. **Multi-cyclic operators.** An operator $T \in \mathcal{L}(X)$ is called a multi-cyclic operator of order n for some integer $n \geq 1$, abbreviated n-multi-cyclic, if there exist n vectors $x_1, \ldots, x_n \in X$ such that $X = \operatorname{span}\{T^k x_i; i = 1, \ldots, n \ ; \ k \geq 0\}$ and if for every $n-1$ vector $z_1, z_2, \ldots, z_{n-1}$ in X, the subspace $\operatorname{span}\{T^k z_i; i = 1, \ldots, n-1 \ ; \ k \geq 0\}$ is proper (see [10] and [7]).

Now, set $\mathcal{N}_0(X) = \mathcal{M}(X)$ and for an integer $n \geq 1$, we define the class $\mathcal{N}_n(X)$ as the set of all operators $T \in \mathcal{L}(X)$ such that there exist an integer m and $(x_1, x_2, \ldots, x_m) \in X^m$ for which

$$R(T^n) \subseteq \operatorname{span}\{T^{n-1+k} x_i; i = 1, 2, \ldots, m; k \geq 0\}.$$

It is not hard to see that for $T \in \mathcal{N}_n(X)$ there exist $(z_1, z_2, \ldots, z_m) \in X^m$ such that

$$\overline{R(T^n)} = \operatorname{span}\{T^k z_i; i = 1, 2, \ldots, m; k \geq 0\}.$$

So, the restriction of T to $\overline{R(T^n)}$ is an s-multi-cyclic operator for some integer s, $1 \leq s \leq m$. The class $\mathcal{M}(X)$ consists of all operators $T \in \mathcal{L}(X)$ for which there is an integer n such that T is an n-multi-cyclic operator. Since $\mathcal{A}(X)$ contains $\mathcal{M}(X)$ (see [6]), it is clear that

$$\mathcal{N}_n(X) \subseteq \mathbb{A}_n(X) \subseteq \mathcal{A}_n(X).$$

We give some examples to show that the inclusions are strict for these classes of operators.

Let $X = H$ be a Hilbert space and $(e_n)_{n\geq 0}$ be an orthonormal basis of H. Let $(w_n)_{n\geq 0}$ be a bounded sequence of complex numbers. Recall that the unilateral weighted shift with weight $(w_n)_{n\geq 0}$ is given by

$$\mathrm{T}e_n = w_n e_{n+1} \qquad \text{for all } n \in \mathbb{N}$$

and its adjoint operator is given by $\mathrm{T}^* e_0 = 0$ and $\mathrm{T}^* e_n = \overline{w}_{n-1} e_{n-1}$ for all $n \geq 1$. It is known that, if T is injective, we have that $R(\mathrm{T})$ is closed if and only if the sequence $(\frac{1}{w_n})_{n\geq 0}$ is bounded. For more details about weighted shifts see [14].

In the following examples, we consider $\mathcal{H}$ to be the Hilbert space given by $\mathcal{H} := H \oplus H$.

Example 1. Let T be the usual unilateral unweighted shift ($w_n \equiv 1$) and N be the unilateral weighted shift with $(w_n)_{n\geq 0}$ such that $w_0 = 1$ and $w_n = 0$ for all $n \geq 1$. Set $L := \mathrm{T} \oplus N$ as a bounded operator on $\mathcal{H}$, and so, $L^* = \mathrm{T}^* \oplus N^*$. Then, it is easy to verify that $L \in \mathcal{N}_n(\mathcal{H}) \setminus \mathcal{M}(\mathcal{H})$. On the other hand, $R(\mathrm{T})$ is closed. Hence, $R(L) = R(\mathrm{T}) \oplus \mathrm{span}\{e_0\}$ is closed. This together with the fact that $\alpha(L)$ and $\beta(L)$ are both infinite, implies that

$$L \in \mathcal{N}_n(\mathcal{H}) \setminus \mathcal{A}(\mathcal{H}) \subseteq \mathcal{A}_n(\mathcal{H}) \setminus \mathcal{A}(\mathcal{H}).$$

Example 2. Consider L defined as in Example 1 but T is the unilateral weighted shift with weight $(\frac{1}{n+1})_{n\geq 0}$. Thus, $R(\mathrm{T})$ is not closed and so $R(L)$ is not closed. This together with $L \in \mathcal{N}_n(\mathcal{H})$ and Remark 2.1 i), imply that

$$L \in \mathcal{A}(\mathcal{H}) \setminus \mathcal{M}(\mathcal{H}).$$

Now, let X be a Banach space, and with a similar proof of Proposition 5.1 of [6], we have the following proposition

Proposition 3.1. *Let T and S be quasisimilar bounded operators; then $T \in \mathcal{N}_n(X)$ if and only if $S \in \mathcal{N}_n(Y)$.*

As an immediate consequence of Proposition 1.2, Theorems 1.2, 2.1, 2.2, and $\mathcal{N}_n(X) \subseteq \mathcal{A}_n(X)$, we get the following corollaries.

Corollary 3.1. *Let $T \in \mathcal{N}_n(X)$ with $\alpha(T)$ is finite; then $T \in \mathcal{A}(X)$. In particular, $T \in \mathcal{N}_n(X)\setminus\mathcal{A}(X)$ implies that $R(T)$ is closed, and $\alpha(T)$ and $\beta(T)$ are both infinite.*

Corollary 3.2. *Suppose that $T \in \mathcal{N}_n(X) \setminus \mathcal{A}(X)$. Then the following statements hold:*

(1) $\sigma_{SF}(T) = \sigma_{re}(T) = \sigma_f(T) \cup \{0\}$.
(2) $\sigma_e(T) = \sigma_{le}(T)$.
(3) *Moreover, if T has the SVEP, then*

$$\sigma_e(T) = \sigma_{SF}(T) = \sigma_{le}(T) = \sigma_{re}(T) = \sigma_f(T) \cup \{0\}.$$

Here $T \in \mathcal{N}_n(X)$ if and only if $S \in \mathcal{N}_n(Y)$ for all $n \in \mathbb{N}$. So, from the previous results we get

Corollary 3.3. *Suppose that $T \in \mathcal{N}_n(X)$ and $S \in \mathcal{N}_n(Y)$ are quasisimilar operators. Then*

$$\sigma_*(T) = \sigma_*(S) \qquad \text{for} \qquad \sigma_* \in \{\sigma_e, \sigma_{re}, \sigma_{le}, \sigma_{SF}, \sigma_{ap}, \sigma_c, \sigma_g\}$$

and $\sigma_f(T) \cup \{0\} = \sigma_f(S) \cup \{0\}$.

Furthermore, if $\alpha(T)$ is finite, then $\sigma_f(T) = \sigma_f(S)$.

References

[1] J. Agler, E. Franks and D. A. Herrero, *Spectral pictures of operators quasisimilar to the unilateral shift;* J. Reine Angew. Math. 422 (1991) 1 – 20. MR1133315 (92m:47007)

[2] E. Albrecht and J. Eschmeier, *Analytic functional models and local spectral theory;* Proc. London Math. Soc. 75 (1997) 323 – 348. MR1455859 (98f:47043)

[3] B. A. Barnes, *Common operator properties of the linear operators RS and SR;* Proc. Amer. Math. Soc. 126 (1998) 1055 – 1061. MR1443814 (98f:47003)

[4] L. Chen and Y. Zukin, *Bishop's property* (β) *and essential spectra of quasisimilar operators;* Proc. Amer. Math. Soc. vol 128 (1999)$485-493$.
[5] J.B.Conway, A course in functional analysis; Graduate texts in Mathematics, 96, second edition, 1990. MR1070713 (91e:46001)
[6] M. Drissi and M. El Hodaibi, *Spectra of quasisimilar operators;* Submitted.
[7] M. EL Guendafi, M. Mbekhta and E. H. Zerouali, *Bounded point evaluations for multicyclic operators,* Banach Center Publ. 67 (2005) $199-217$. MR2143926 (2006e:47012)
[8] S. Goldberg, Unbounded linear operators; Mc-Graw-Hill, New York, 1966. MR0200692 (34:580)
[9] D. A. Herrero, *On the essential spectra of quasisimilar operators;* Can. J. Math. 40 (1988)$1436-1457$. MR990108 (90b:47006)
[10] D. A. Herrero and L. Rodman, *The multicyclic n-tuples of an n-multicyclic operator and analytic structures on its Spectrum;* Indiana. Univ. Math. J. 34 (1985)$619-629$. MR794579 (87c:47025)
[11] M. Mbekhta and E. H. Zerouali, *Point d'évaluation pour les opérateurs cycliques ayant la propriété de Bishop β;* Journal of Functional Anaysis 206 (2004)$69-86$. MR2024346 (2004i:47005)
[12] K. B. Laursen and M. M. Neumann, An introduction to Local spectral theory; London Mathematical Society Monograph, New series, Vol. 20, Clarendon Press, Oxford, 2000. MR1747914 (2001k:47002)
[13] M.Schechter, *On perturbations of essential spectra;* J. London Math. Soc. 1, (1969)$343-347$. MR0248557 (40:1809)
[14] A. L. Shields, *Weighted shift operators and analytic function theory;* Math. Surveys. Amer. Math. Soc. Providence 13(1974), $49-128$. MR0361899 (50:14341)
[15] B. Sz-Nagy and C. Foias, Harmonic Analysis of Operators on Hilbert space, North Holland-American Elsevier, New York, 1970 (English edition). MR0275190 (43:947)
[16] F. H. Vasilescu, Analytic Functional Calculus and Spectral Decomposition, Editura Academiei, Bucharest, 1982. MR690957 (85b:47016)

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