

## UNIQUENESS OF UNCONDITIONAL BASIS IN LORENTZ SEQUENCE SPACES

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ABSTRACT. We show that the Lorentz sequence spaces  $d(\omega, p)$  with  $0 < p < 1$  and  $\inf \frac{\omega_1 + \dots + \omega_n}{n^p} > 0$  have unique unconditional basis. This completely settles the question of uniqueness of unconditional basis in Lorentz sequence spaces, and solves a problem raised by Popa in 1981 and Nawrocki and Ortyński in 1985.

### 1. INTRODUCTION

A quasi-Banach space  $(X, \|\cdot\|)$  is said to have *unique unconditional basis* if any two normalized unconditional bases of  $X$  are equivalent. In [8], [9], and [12] the authors showed that  $c_0$ ,  $\ell_1$ , and  $\ell_2$  are the only Banach spaces with unique unconditional basis. However, there are “many” nonlocally convex quasi-Banach spaces with that property: examples include a wide class of Orlicz and Lorentz sequence spaces, amongst which we find the spaces  $\ell_p$  for  $0 < p < 1$  ([5, 6]).

If  $\omega = (\omega_n)_{n \in \mathbb{N}} \in \ell_\infty \setminus \ell_1$  is a nonincreasing sequence of positive numbers and  $0 < p < \infty$ , the *Lorentz sequence space*  $d(\omega, p)$  is defined to be the space of all sequences of real numbers  $a = (a_n)_{n \in \mathbb{N}}$  such that the  $p$ -norm

$$\|a\|_{\omega, p} = \sup_{\pi \in \Pi} \left( \sum_{n=1}^{\infty} |a_{\pi(n)}|^p \omega_n \right)^{1/p}$$

is finite, where  $\Pi$  denotes the group of permutations of  $\mathbb{N}$ .

The canonical unit vector basis  $(e_n)_{n=1}^{\infty}$  of  $d(\omega, p)$  is 1-unconditional (in fact, it is 1-symmetric), and induces a  $p$ -convex lattice structure in  $d(\omega, p)$ .

In the sequel we will put  $(\omega_1 + \dots + \omega_n)^{1/p} = \sigma_n$  ( $n \in \mathbb{N}$ ).

If  $p \geq 1$ , the Lorentz space  $d(\omega, p)$  is a Banach space and its unconditional basis is unique if and only if  $\omega \notin c_0$  and  $p = 1$  or  $p = 2$  (that is, if and only if  $d(\omega, p)$  is isomorphic to  $\ell_1$  or  $\ell_2$ ). Furthermore, it is well known that for  $p \geq 1$ , the Lorentz space  $d(\omega, p)$  has a complemented subspace isomorphic to  $\ell_p$  ([3]).

In [13], Nawrocki and Ortyński investigated the quasi-Banach spaces  $d(\omega, p)$  for  $0 < p < 1$  and described their Banach envelopes. They also considered the question of uniqueness of the unconditional basis: they proved that all symmetric bases are equivalent (see also [14]) and that if  $\inf(\sigma_n/n) = 0$ , then  $d(\omega, p)$  has a complemented

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subspace isomorphic to  $\ell_p$  and hence has uncountably many nonequivalent unconditional bases. On the other hand, they also proved that if  $\lim_{n \rightarrow \infty} (\sigma_n/n) = \infty$  and  $0 < p < 1$ , all complemented subspaces of  $d(\omega, p)$  are isomorphic to  $d(\omega, p)$ . Later, it was further proved in [6] that in such cases all normalized unconditional bases of  $d(\omega, p)$  are equivalent.

In this paper we will prove that if  $(u_n)_{n=1}^\infty$  is a normalized, unconditional, and complemented basic sequence in  $d(\omega, p)$  with  $0 < p < 1$  and  $\inf(\sigma_n/n) > 0$ , then  $(u_n)_{n=1}^\infty$  is equivalent to the canonical unit vector basis of the space. This completely settles the question of uniqueness of unconditional basis in Lorentz sequence spaces, and answers the question raised by Popa in [14] and Nawrocki and Ortyński in [13] about whether or not  $d(\omega, p)$  has a complemented subspace isomorphic to  $\ell_p$  when  $0 < p < 1$ ,  $\inf(\sigma_n/n) > 0$ , and  $\lim(\sigma_n/n) \neq \infty$ .

We use standard Banach space theory terminology and notation throughout, as may be found in [2] and [10, 11]. For the necessary background in the general theory of quasi-Banach spaces we refer the reader to [7].

2. COMPLEMENTED BASIC SEQUENCES IN  $d(\omega, p)$ ,  $0 < p < 1$

An unconditional basic sequence  $(u_n)_{n=1}^\infty$  in a quasi-Banach space  $X$  is *complemented* if there exists a sequence  $(u_n^*)_{n=1}^\infty$  in  $X^*$  such that the map  $P$  defined on  $X$  by  $P(x) = \sum_{n=1}^\infty u_n^*(x)u_n$  is a bounded linear projection onto the closed linear span  $[u_n]$  of  $(u_n)_{n=1}^\infty$ .

Our first lemma summarizes several results and ideas contained in [1], mainly Corollary 3.6 (see also [4]).

**Lemma 2.1.** *Let  $X$  be a quasi-Banach space such that:*

- (i)  *$X$  has a 1-unconditional basis  $(e_k)_{k=1}^\infty$  that induces a  $p$ -convex lattice structure in  $X$  for some  $p > 0$ ,*
- (ii) *the Banach envelope of  $X$  is isomorphic to  $\ell_1$ , and*
- (iii)  *$X$  is lattice isomorphic to  $X \oplus X$ .*

*If  $(u_n)_{n=1}^\infty$  is a normalized, unconditional, and complemented basic sequence in  $X$ , then  $(u_n)_{n=1}^\infty$  is equivalent to a normalized, unconditional, and complemented basic sequence  $(v_n)_{n=1}^\infty$  in  $X$  such that the sets  $S_n := \{k \in \mathbb{N} : e_k^*(v_n) \neq 0\}$  are disjoint and finite,  $S_n$  coincides with  $\{k \in \mathbb{N} : v_n^*(e_k) \neq 0\}$ , and there exists a constant  $\nu > 0$  such that  $e_k^*(v_n) > \nu$  and  $v_n^*(e_k) > \nu$  for all  $k \in S_n$  and all  $n \in \mathbb{N}$ .*

The following is Lemma 3.1 of [14]:

**Lemma 2.2.** *Let  $(e_n)_{n=1}^\infty$  be the canonical basis of  $d(\omega, p)$ ,  $0 < p < 1$ . If*

$$u_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i, \quad n = 1, 2, \dots,$$

*is a bounded block basic sequence of  $(e_n)_{n=1}^\infty$  such that*

$$\lim_{n \rightarrow \infty} a_n = 0,$$

*then there is a subsequence of  $(u_n)_{n=1}^\infty$  that is equivalent to the canonical basis of  $\ell_p$ .*

The proof of our next lemma can be found in [5] or [6].

**Lemma 2.3.** *Let  $X$  be a  $p$ -convex quasi-Banach lattice for some  $p > 0$ , and let  $(e_n)_{n \in \mathbb{N}}$  and  $(x_n)_{n \in \mathbb{N}}$  be two normalized unconditional basic sequences in  $X$ . Let  $(e_n^*)_{n \in \mathbb{N}} \subset X^*$  and  $(x_n^*)_{n \in \mathbb{N}} \subset X^*$  be the sequences of biorthogonal linear functionals associated to  $(e_n)_{n \in \mathbb{N}}$  and  $(x_n)_{n \in \mathbb{N}}$ , respectively. Suppose that there is a constant  $\beta > 0$  and an injective map  $\sigma : S \subset \mathbb{N} \rightarrow \mathbb{N}$  so that*

$$|e_{\sigma(n)}^*(x_n)| \geq \beta \quad \text{and} \quad |x_n^*(e_{\sigma(n)})| \geq \beta$$

for all  $n \in S$ . Then the unconditional basic sequences  $(x_n)_{n \in S}$  and  $(e_{\sigma(n)})_{n \in S}$  are equivalent.

This is our main theorem:

**Theorem 2.4.** *Suppose  $0 < p < 1$  and  $\inf(\sigma_n/n) > 0$ . If  $(u_n)_{n=1}^\infty$  is a normalized,  $K$ -unconditional, and complemented basic sequence in  $d(\omega, p)$ , then  $(u_n)_{n=1}^\infty$  is equivalent to the canonical unit vector basis  $(e_k)_{k=1}^\infty$  of the space. In particular, all complemented subspaces of  $d(\omega, p)$  with unconditional basis are isomorphic to  $d(\omega, p)$ , and  $d(\omega, p)$  has unique unconditional basis.*

*Proof.* The space  $d(\omega, p)$  satisfies all the conditions of Lemma 2.1 since  $d(\omega, p)$  is isomorphic to  $d(\omega, p) \oplus d(\omega, p)$  and, by [13, Theorem 1],  $\inf(\sigma_n/n) > 0$  if and only if the Banach envelope of  $d(\omega, p)$  is isomorphic to  $\ell_1$ . Consequently, we can assume that the sets

$$S_n = \{k \in \mathbb{N} : e_k^*(u_n) \neq 0\} = \{k \in \mathbb{N} : u_n^*(e_k) \neq 0\}$$

are disjoint and finite, and that there exists a constant  $\nu > 0$  such that  $e_k^*(u_n) > 0$  and  $u_n^*(e_k) > \nu$  for all  $k \in S_n$  and all  $n \in \mathbb{N}$ .

Suppose that

$$\inf_{n \in \mathbb{N}} \sup_{k \in S_n} e_k^*(u_n) = 0.$$

Then, by Lemma 2.2 there exists a subsequence  $(u_{n_m})_{m=1}^\infty$  of  $(u_n)_{n=1}^\infty$  that is  $C$ -equivalent to the canonical  $\ell_p$ -basis. For each  $m \in \mathbb{N}$ , pick  $k_m \in S_{n_m}$ ; hence by the disjointness of the supports,

$$P \left( \sum_{m=1}^N e_{k_m} \right) = \sum_{m=1}^N u_{n_m}^*(e_{k_m}) u_{n_m}$$

for all  $N \in \mathbb{N}$ . Therefore,

$$\begin{aligned} CN^{1/p} &\leq \left\| \sum_{m=1}^N u_{n_m} \right\|_{\omega, p} \leq \frac{K}{\nu} \left\| \sum_{m=1}^N u_{n_m}^*(e_{k_m}) u_{n_m} \right\|_{\omega, p} \\ &= \frac{K}{\nu} \left\| P \left( \sum_{m=1}^N e_{k_m} \right) \right\|_{\omega, p} \\ &\leq \frac{K \|P\|}{\nu} \left\| \sum_{m=1}^N e_{k_m} \right\|_{\omega, p} \\ &= \frac{K \|P\|}{\nu} \sigma_N \end{aligned}$$

for all  $N \in \mathbb{N}$ , and so

$$\inf_N \frac{\omega_1 + \cdots + \omega_N}{N} \geq \frac{\nu^p C^p}{K^p \|P\|^p}.$$

That implies that  $\omega = (\omega_n)_{n \in \mathbb{N}} \notin c_0$ , and hence that  $d(\omega, p)$  is isomorphic to  $\ell_p$ . It is well known that, for  $0 < p < 1$ , all complemented subspaces of  $\ell_p$  are isomorphic to  $\ell_p$  and that  $\ell_p$  has unique unconditional basis.

We conclude that if  $\omega = (\omega_n)_{n \in \mathbb{N}} \in c_0$ , then there exists  $\delta > 0$  such that

$$\sup_{k \in S_n} e_k^*(u_n) > \delta$$

for all  $n \in \mathbb{N}$ . By Lemma 2.3, with  $\beta = \min\{\nu, \delta\}$ ,  $(x_n)_{n \in \mathbb{N}}$  is equivalent to a reordered subsequence of the canonical unit vector basis of  $d(\omega, p)$ . Since such basis is symmetric, the result follows.  $\square$

There is a “maximal” Lorentz sequence space  $d(\Omega, p)$  with unique unconditional basis:

**Theorem 2.5.** *If  $0 < p \leq 1$  and  $(\omega_n)_{n \in \mathbb{N}}$  is a positive and nonincreasing sequence such that*

$$\inf \frac{\sigma_n}{n} = \delta > 0,$$

then

$$\delta^{-1} \left( \sum_{n=1}^N a_n^p \omega_n \right)^{1/p} \geq \left( \sum_{n=1}^N a_n^p (n^p - (n-1)^p) \right)^{1/p} \geq \sum_{n=1}^N a_n$$

for all positive and nonincreasing sequences  $(a_n)_{n \in \mathbb{N}}$  and every  $N \in \mathbb{N}$ . In particular, for every  $0 < p \leq 1$  there is a Lorentz sequence space  $d(\Omega, p)$  that contains all Lorentz sequence spaces  $d(\omega, p)$  with unique unconditional basis.

*Proof.* If  $0 < p \leq 1$  and  $(\omega_n)_{n \in \mathbb{N}}$  is a positive and nonincreasing sequence such that

$$\inf \frac{\sigma_n}{n} = \delta > 0,$$

then

$$\begin{aligned} \left( \sum_{n=1}^N a_n^p \omega_n \right)^{1/p} &= \left( \sum_{n=1}^N a_n^p (\sigma_n^p - \sigma_{n-1}^p) \right)^{1/p} \\ &= \left( a_N^p \sigma_N^p + \sum_{n=1}^{N-1} (a_n^p - a_{n+1}^p) \sigma_n^p \right)^{1/p} \\ &\geq \left( a_N^p \delta^p N^p + \delta^p \sum_{n=1}^{N-1} (a_n^p - a_{n+1}^p) n^p \right)^{1/p} \\ &= \delta \left( \sum_{n=1}^N a_n^p (n^p - (n-1)^p) \right)^{1/p} \end{aligned}$$

for all positive and nonincreasing sequences  $(a_n)_{n \in \mathbb{N}}$  and every  $N \in \mathbb{N}$ . Furthermore, since the function  $(x + 1)^p - x^p$  is decreasing in  $(0, \infty)$ ,

$$\begin{aligned} \sum_{n=1}^{N+1} a_n^p (n^p - (n-1)^p) &= \sum_{n=1}^N a_n^p (n^p - (n-1)^p) + a_{N+1}^p ((N+1)^p - N^p) \\ &\geq \sum_{n=1}^N a_n^p (n^p) p - (n-1)^p - \left( \sum_{n=1}^N a_n \right)^p + \left( \sum_{n=1}^{N+1} a_n \right)^p \end{aligned}$$

for every  $a_1, \dots, a_N \geq a_{N+1}$  and  $N \in \mathbb{N}$ . By induction on  $N$ ,

$$\sum_{n=1}^N a_n^p (n^p - (n-1)^p) \geq \left( \sum_{n=1}^N a_n \right)^p$$

for all positive and nonincreasing sequences  $(a_n)_{n \in \mathbb{N}}$  and every  $N \in \mathbb{N}$ . The above inequality is strict unless  $a_1 = \dots = a_n$  and  $a_{n+1} = \dots = a_N = 0$  for some  $n = 1, \dots, N$ .  $\square$

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