

## KATZNELSON-TZAFRIRI TYPE THEOREMS FOR INDIVIDUAL SOLUTIONS OF EVOLUTION EQUATIONS

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**ABSTRACT.** In this paper we present an extension of the Katznelson-Tzafriri Theorem to the asymptotic behavior of individual solutions of evolution equations  $u'(t) = Au(t) + f(t)$ . The obtained results do not require the uniform continuity of solutions as well as the well-posedness of the equations. The method of study is based on a recently developed approach to the spectral theory of functions that is direct and free of  $C_0$ -semigroups.

### 1. INTRODUCTION AND STATEMENT OF RESULTS

This is a companion paper of [10] in which the author developed a new approach to the spectral theory of functions on the line and on the half line to study the asymptotic behavior of solutions of evolution equations of the form

$$(1.1) \quad \dot{u}(t) = Au(t) + f(t),$$

where  $t$  is in  $J$  ( $J$  is either  $\mathbb{R}^+$  or  $\mathbb{R}$ ),  $A$  is a closed linear operator on a Banach space  $\mathbb{X}$ , and  $f \in BC(J, \mathbb{X})$ . This spectral theory is simple and free of  $C_0$ -semigroup theory, so it can apply to large classes of solutions and equations with general conditions.

In this paper we will discuss further applications of the theory by proving continuous analogs of the two well known Katznelson-Tzafriri Theorems in [8] to individual bounded solutions of evolution equations of the form (1.1) on the half line. Our main results are stated in the two theorems below (Theorems 1.1, 1.2). For this purpose we introduce classes of function spaces  $\mathcal{F}$  as subspaces of  $BC(\mathbb{R}^+, \mathbb{X})$  that satisfy so-called *Condition  $F^+$  or  $F^{++}$*  (see the definition in the next section). As an example of such function spaces  $\mathcal{F}$  one may take  $C_0(\mathbb{R}^+, \mathbb{X})$ .

**Theorem 1.1.** *Let  $\mathcal{F}$  be a function space that satisfies Condition  $F^+$ , let  $A$  be a linear operator on a Banach space  $\mathbb{X}$  such that  $\sigma(A) \cap i\mathbb{R} \subset \{0\}$ , and let  $u \in BC(\mathbb{R}^+, \mathbb{X})$  be a classical solution of (1.1) on  $\mathbb{R}^+$  with  $f \in \mathcal{F}$  such that*

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$Au(\cdot) \in BC(\mathbb{R}^+, \mathbb{X})$ . Then,

$$(1.2) \quad Au(\cdot) \in \mathcal{F}.$$

Recall that a function  $g \in L^1(\mathbb{R})$  is of spectral synthesis with respect to a closed subset  $E \subset \mathbb{R}$  if there exists a sequence  $(g_n)$  in  $L^1(\mathbb{R})$  such that

- i)  $\lim_{n \rightarrow \infty} \|g - g_n\|_1 = 0$ ;
- ii) each of the Fourier transforms  $\mathcal{F}g_n$  vanishes on a neighborhood of  $E$ .

**Theorem 1.2.** *Let  $\mathcal{F}$  be a function space that satisfies Condition  $\mathcal{F}^{++}$ , let  $u \in BC(\mathbb{R}^+, \mathbb{X})$  be a mild solution of (1.1) on  $\mathbb{R}^+$  with  $f \in \mathcal{F}$ , and let  $\phi \in L^1(\mathbb{R}^+)$  be of spectral synthesis with respect to  $i\sigma(A) \cap \mathbb{R}$ . Then,  $w \in \mathcal{F}$ , where*

$$(1.3) \quad w(t) := \int_0^\infty \phi(s)u(t+s)ds.$$

The proofs of these theorems will be given in Section 3. Notation and a short introduction into previous results in [10] will be given in the next section. For the works related to Theorem 1.1 we refer the reader to [1, 2]. Our Theorem 1.1 is an individual version of the part “(ii)  $\Rightarrow$  (i)” of [2, Theorem 3.10]. In fact, if  $A$  generates an eventually differentiable bounded  $C_0$ -semigroup  $T(t)$ ,  $\mathcal{F}$  is chosen to be  $C_0(\mathbb{R}^+, \mathbb{X})$  and  $f = 0$ , then since  $T(\tau)\mathbb{X} \subset D(A)$  for some  $\tau \geq 0$ , for each  $x \in \mathbb{X}$  the function  $u(t) := T(t)x$  is a classical solution (with  $t \geq \tau$ ). Therefore, our Theorem 1.1 applies. Notice that under the assumptions of Theorem 1.1, a version of “(i)  $\Rightarrow$  (ii)” of [2, Theorem 3.10] does not make sense in our context because there may not be bounded solutions to an ill-posed equation. As for Theorem 1.2, our result extends the individual version of Katznelson-Tzafriri Theorem given in [5, Theorem 5.1] that applies to homogeneous equations generating bounded  $C_0$ -semigroups. In fact, if  $A$  generates a bounded  $C_0$ -semigroup  $T(t)$ ,  $\mathcal{F}$  is chosen to be  $C_0(\mathbb{R}^+, \mathbb{X})$  and  $f = 0$ , then for each  $x \in \mathbb{X}$ , and  $\phi \in L^1(\mathbb{R}^+)$

$$\begin{aligned} T(t) \int_{-\infty}^\infty \phi(\xi)T(\xi)xd\xi &= \int_{-\infty}^\infty \phi(\xi)T(t+\xi)xd\xi \\ &= \int_{-\infty}^\infty \phi(\xi)u(t+\xi)d\xi := w(t), \end{aligned}$$

where  $u(t) := T(t)x$ . Our result shows in particular that  $\lim_{t \rightarrow \infty} w(t) = 0$ .

For more complete accounts of related works we refer the reader to the monographs [1, 12] and their references, as well as the papers [2, 4, 5, 6, 7, 11, 13].

## 2. PRELIMINARIES

Throughout the paper we denote by  $\mathbb{R}$  the real line, by  $\mathbb{R}^+$  the positive half line  $[0, \infty)$ , by  $\mathbb{R}^-$  the negative half line  $(-\infty, 0]$ , and by  $\mathbb{X}$  a Banach space over the complex plane  $\mathbb{C}$ . If  $A$  is a linear operator on a Banach space  $\mathbb{X}$ ,  $D(A)$  stands for its domain;  $\sigma(A)$  stands for its spectrum.  $L(\mathbb{X})$  stands for the Banach space of all bounded linear operators in  $\mathbb{X}$  with the usual norm  $\|\cdot\|$ . If  $\lambda \in \rho(A)$ , then  $R(\lambda, A)$  denotes the resolvent  $(\lambda - A)^{-1}$ . In this paper we also use the following notation:

- i)  $BC(J, \mathbb{X})$  is the space of all  $\mathbb{X}$ -valued bounded and continuous functions on  $J$ , where  $J$  is either  $\mathbb{R}$ , or  $\mathbb{R}^+$ ;
- ii)  $L^\infty(J, \mathbb{X})$  and  $L^1(J, \mathbb{X})$  are the space of all  $\mathbb{X}$ -valued measurable and essentially bounded functions on  $J$ , and the space of all  $\mathbb{X}$ -valued measurable and integrable functions on  $J$ , respectively;

- iii)  $C_0(\mathbb{R}^+, \mathbb{X}) := \{f \in BC(\mathbb{R}^+, \mathbb{X}) : \lim_{t \rightarrow \infty} f(t) = 0\}$ ;
- iv) If  $A$  is a linear operator on  $\mathbb{X}$ , then the operator of multiplication by  $A$  on  $BC(J, \mathbb{X})$ , denoted by  $\mathcal{A}$ , is defined on  $D(\mathcal{A}) := \{g \in BC(J, \mathbb{X}) : g(t) \in D(A), \text{ for all } t \in J, Ag(\cdot) \in BC(J, \mathbb{X})\}$ , by  $\mathcal{A}g = Ag(\cdot)$  for each  $g \in D(\mathcal{A})$ .

**Definition 2.1.** An  $\mathbb{X}$ -valued continuous function  $u$  on  $\mathbb{R}^+$  is said to be a mild solution of (1.1) on  $\mathbb{R}^+$  (with  $f \in L^\infty(\mathbb{R}^+, \mathbb{X})$ ) if for every  $t \in \mathbb{R}^+$ ,  $\int_0^t u(s)ds \in D(A)$ , and

$$(2.1) \quad u(t) - u(0) = A \int_0^t u(s)ds + \int_0^t f(s)ds, \quad \text{for all } t \in \mathbb{R}^+.$$

A continuously differentiable function  $u$  on  $\mathbb{R}^+$  is said to be a classical solution of (1.1) on  $\mathbb{R}^+$  (with  $f \in BC(\mathbb{R}^+, \mathbb{X})$ ) if for every  $t \in \mathbb{R}^+$ ,  $u(t) \in D(A)$ , and

$$(2.2) \quad u'(t) = Au(t) + f(t), \quad \text{for all } t \in \mathbb{R}^+.$$

Let us introduce the so-called *Conditions  $F^+$  and  $F^{++}$*  for a function space  $\mathcal{F} \subset BC(\mathbb{R}^+, \mathbb{X})$ . Let  $\psi$  be any function in  $L^\infty(\mathbb{R}^+, \mathbb{X})$  (or  $L^\infty(\mathbb{R}^-, \mathbb{X})$ , respectively). Throughout the paper we will identify it with its natural extension to  $\psi \in L^\infty(\mathbb{R}, \mathbb{X})$  by setting  $\psi(t) = 0$  for all  $t < 0$  (or  $t > 0$ , respectively).

**Definition 2.2.** A function space  $\mathcal{F} \subset BC(\mathbb{R}^+, \mathbb{X})$  is said to satisfy *Condition  $F^+$*  if

- i) It is closed, and contains  $C_0(\mathbb{R}^+, \mathbb{X})$ .
- ii) If  $g \in \mathcal{F}$ , then the function  $\mathbb{R}^+ \ni t \mapsto e^{i\xi t}g(t) \in \mathbb{X}$  is in  $\mathcal{F}$  for all  $\xi \in \mathbb{R}$ .
- iii) For each  $h \in \mathcal{F}$ ,  $Re\lambda > 0$ ,  $Re\eta < 0$ , the function  $y(\cdot), z(\cdot)$ , defined as

$$(2.3) \quad y(t) = \int_t^\infty e^{\lambda(t-s)}h(s)ds, \quad z(t) = \int_0^t e^{\eta(t-s)}h(s)ds, \quad t \in \mathbb{R}^+,$$

are in  $\mathcal{F}$ .

- iv) For each  $B \in L(\mathbb{X})$  and  $f \in \mathcal{F}$ , the function  $Bf(\cdot)$  is in  $\mathcal{F}$ .

If in addition to the above conditions,  $\mathcal{F}$  satisfies the following:

- v) For each function  $\psi \in L^1(\mathbb{R}^-)$ ,  $\psi * g \in \mathcal{F}$  for each  $g \in \mathcal{F}$ ,

then  $\mathcal{F}$  is said to satisfy *Condition  $F^{++}$* .

As an example of a function space that satisfies *Conditions  $F^+$  and  $F^{++}$* , we can take  $\mathcal{F} = C_0(\mathbb{R}^+, \mathbb{X})$ . Another function space that satisfies *Condition  $F^{++}$*  is  $C_0(\mathbb{R}^+, \mathbb{X}) + AA(\mathbb{R}^+, \mathbb{X})$ , the space of all restrictions to  $\mathbb{R}^+$  of the  $\mathbb{X}$ -valued almost automorphic functions. Note that  $AA(\mathbb{R}^+, \mathbb{X})$  contains non-uniformly continuous functions, so it is not a subspace of  $BUC(\mathbb{R}^+, \mathbb{X})$ .

Consider the quotient space  $\mathbb{Y} := BC(\mathbb{R}^+, \mathbb{X})/\mathcal{F}$  whose elements are denoted by  $\tilde{g}$  with  $g \in BC(\mathbb{R}^+, \mathbb{X})$ . We will use  $\tilde{\mathcal{D}}$  to denote the operator induced by the differential operator  $\mathcal{D}$  on  $\mathbb{Y}$  which is defined as follows: The domain  $D(\tilde{\mathcal{D}})$  is the set of all classes that contains a differentiable function  $g \in BC(\mathbb{R}^+, \mathbb{X})$  such that  $g' \in BC(\mathbb{R}^+, \mathbb{X})$ ;  $\tilde{\mathcal{D}}\tilde{g} := \tilde{g}'$  for each  $\tilde{g} \in D(\tilde{\mathcal{D}})$ .

It is easy to see that for  $f \in BC(\mathbb{R}^+, \mathbb{X})$  (see [10, page 13])

$$\begin{aligned}
 (2.4) \quad R(\lambda, \tilde{\mathcal{D}})\tilde{f}(t) &= \begin{cases} \int_t^\infty e^{\lambda(t-s)} \tilde{f}(s) ds, & Re\lambda > 0, \quad t \in \mathbb{R}^+, \\ -\int_0^t e^{\lambda(t-s)} \tilde{f}(s) ds, & Re\lambda < 0, \quad t \in \mathbb{R}^+ \end{cases} \\
 &= \begin{cases} \int_0^\infty e^{-\lambda\eta} \tilde{f}(t+\eta) d\eta, & Re\lambda > 0, \quad t \in \mathbb{R}^+, \\ -\int_0^t e^{\lambda\eta} \tilde{f}(t-\eta) d\eta, & Re\lambda < 0, \quad t \in \mathbb{R}^+. \end{cases}
 \end{aligned}$$

Under the above notation, the operator  $\tilde{\mathcal{D}}$  is a closed operator with  $\sigma(\tilde{\mathcal{D}}) \subset i\mathbb{R}$ .

**Definition 2.3.** Let  $\mathcal{F}$  be a function space that satisfies Condition  $F^{++}$ , and let  $f \in BC(\mathbb{R}^+, \mathbb{X})$ . Then the *reduced spectrum* of  $f$  with respect to  $\mathcal{F}$ , denoted by  $sp_{\mathcal{F}}^+(f)$ , is defined to be the set of all reals  $\xi \in \mathbb{R}$  such that  $R(\lambda, \tilde{\mathcal{D}})\tilde{f}$ , as a complex function of  $\lambda$  in  $\mathbb{C} \setminus i\mathbb{R}$ , has no holomorphic extension to any neighborhood of  $i\xi$  in the complex plane.

Since  $sp_{\mathcal{F}}^+(f)$  is the same for all elements  $f$  in a class  $\tilde{g}$ , the use of the notation  $sp_{\mathcal{F}}^+(\tilde{g})$  makes sense. The following theorem was proved in [10]:

**Theorem 2.4.** *Let  $\mathcal{F}$  be a function space of  $BC(\mathbb{R}^+, \mathbb{X})$  that satisfies Condition  $F^+$ , and let  $f$  be in  $BC(\mathbb{R}^+, \mathbb{X})$  such that  $sp_{\mathcal{F}}^+(f)$  is countable. Moreover, assume that*

$$(2.5) \quad \lim_{\alpha \downarrow 0} \alpha R(\alpha + i\xi, \tilde{\mathcal{D}})\tilde{f} = 0$$

for all  $\xi \in sp_{\mathcal{F}}^+(f)$ . Then,  $f \in \mathcal{F}$ .

For each function  $f \in L^\infty(\mathbb{R}, \mathbb{X})$ ,  $\xi \in \mathbb{R}$ ,  $Re\lambda \neq 0$ , let us recall that

$$(2.6) \quad R(\lambda, \mathcal{D})f(\xi) = \begin{cases} \int_0^\infty e^{-\lambda\eta} f(\xi + \eta) d\eta & (\text{if } Re\lambda > 0), \\ -\int_{-\infty}^0 e^{-\lambda\eta} f(\xi + \eta) d\eta & (\text{if } Re\lambda < 0). \end{cases}$$

Recall also that the Carleman transform of  $f \in L^\infty(\mathbb{R}, \mathbb{X})$  is defined as

$$(2.7) \quad \hat{f}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda\eta} f(\eta) d\eta & (\text{if } Re\lambda > 0), \\ -\int_{-\infty}^0 e^{-\lambda\eta} f(\eta) d\eta & (\text{if } Re\lambda < 0). \end{cases}$$

It can be shown in the same manner as in the proof of [9, Proposition 2.3] that the set of  $\zeta \in \mathbb{R}$  such that  $\hat{f}(\lambda)$  has no holomorphic extension to any neighborhood of  $i\zeta$  (Carleman spectrum of  $f$ ) coincides with the set of all  $\zeta \in \mathbb{R}$  such that  $R(\lambda, \mathcal{D})f$  has no holomorphic extension to any neighborhood of  $i\zeta$ . We will denote this set by  $sp(f)$  and call it the spectrum of  $f$ . The reader is referred to [1, 9] for more information on this concept of spectrum.

### 3. PROOF OF THE MAIN RESULTS

The main idea of proving Theorems 1.1, and 1.2 is to apply Theorem 2.4. We will fix  $\mathcal{F}$  as a function space that satisfies *Condition  $F^{++}$* . Below we denote  $\sigma_i(A) := \{\xi \in \mathbb{R} \mid i\xi \in \sigma(A)\}$ .

**3.1. Proof of Theorem 1.1.** Since  $u$  is a mild solution of (1.1), by [10, Lemma 4.7] we have

$$(3.1) \quad sp_{\mathcal{F}}^+u \subset \sigma_i(A) \subset \{0\}.$$

Moreover, for  $Re\lambda \neq 0$ ,

$$(3.2) \quad R(\lambda, \tilde{\mathcal{D}})\tilde{u} = R(\lambda, \tilde{\mathcal{A}})\tilde{u}.$$

Let  $w = \mathcal{A}u$ . Then, by the assumption,  $w \in BC(\mathbb{R}^+, \mathbb{X})$ , so by the identity  $R(\lambda, \tilde{\mathcal{D}})\tilde{\mathcal{D}}\tilde{u} = \lambda R(\lambda, \tilde{\mathcal{D}})\tilde{u} - \tilde{u}$ , we have

$$(3.3) \quad \begin{aligned} R(\lambda, \tilde{\mathcal{D}})\tilde{w} &= R(\lambda, \tilde{\mathcal{D}})\tilde{\mathcal{A}}\tilde{u} \\ &= R(\lambda, \tilde{\mathcal{D}})\tilde{\mathcal{D}}\tilde{u} \\ &= \lambda R(\lambda, \tilde{\mathcal{D}})\tilde{u} - \tilde{u}. \end{aligned}$$

Therefore, by (3.1),

$$(3.4) \quad sp_{\mathcal{F}}^+w \subset sp_{\mathcal{F}}^+u \subset \{0\}.$$

Moreover, we have

$$(3.5) \quad \begin{aligned} \lim_{\alpha \downarrow 0} \|\alpha R(\alpha, \tilde{\mathcal{D}})\tilde{w}\| &= \lim_{\alpha \downarrow 0} \|\alpha(\alpha R(\alpha, \tilde{\mathcal{D}})\tilde{u} - \tilde{u})\| \\ &\leq \lim_{\alpha \downarrow 0} \|\alpha^2 R(\alpha, \tilde{\mathcal{D}})\tilde{u}\| + \lim_{\alpha \downarrow 0} \|\alpha\tilde{u}\| \\ &\leq \lim_{\alpha \downarrow 0} \alpha^2 \|\frac{\tilde{u}}{\alpha}\| \\ &= 0. \end{aligned}$$

Therefore, the assertion of Theorem 1.1 follows from Theorem 2.4.

**3.2. Proof of Theorem 1.2.** Let  $\psi$  be any function in  $L^\infty(\mathbb{R}^+, \mathbb{X})$  (or  $L^\infty(\mathbb{R}^-, \mathbb{X})$ , respectively); recall that it can be identified with its extension to  $\psi \in L^\infty(\mathbb{R}, \mathbb{X})$  by setting  $\psi(t) = 0$  for all  $t < 0$  (or  $t > 0$ , respectively).

Theorem 1.2 follows immediately from Theorem 2.4 and the following lemma

**Lemma 3.1.** *Under the assumption of Theorem 1.2,*

$$(3.6) \quad sp_{\mathcal{F}}^+(w) = \emptyset,$$

where

$$(3.7) \quad w(t) := \int_0^\infty \phi(s)u(t+s)ds, \text{ for all } t \in \mathbb{R}^+.$$

*Proof.* To prove Lemma 3.1, we need several technical steps. Let  $\varphi(\theta) := \phi(-\theta)$  for all  $\theta \in \mathbb{R}$ . We re-write the function  $w$  as

$$\begin{aligned} w(t) &= \int_{-\infty}^\infty \phi(s)u(t+s)ds \\ &= \int_{-\infty}^\infty \varphi(t-s)u(s)ds \\ &= \varphi * u. \end{aligned}$$

First, we assume that the Fourier transform of  $\phi$  (denoted by  $\mathcal{F}\phi$ ) vanishes in a neighborhood of  $i\sigma(A) \cap \mathbb{R}$ . As shown below this assumption does not restrict the generality of the proof. Since the Fourier transform  $\mathcal{F}\varphi$  of  $\varphi$  satisfies  $\mathcal{F}\phi(-\xi) = \mathcal{F}\varphi(\xi)$  it vanishes in a neighborhood of  $-(i\sigma(A) \cap \mathbb{R}) = \mathbb{R} \setminus \sigma_i(A)$ , so, as is well known

(see e.g. [1, Chap. 4, and 5]), the Carleman transform  $\widehat{\varphi}(\lambda)$  of  $\varphi$  is holomorphic in a neighborhood of  $\mathbb{R} \setminus \sigma_i(A)$ . In turn, by the proof of [9, Proposition 2.3],  $R(\lambda, \mathcal{D})\varphi$  should be holomorphic in a neighborhood of  $\mathbb{R} \setminus \sigma_i(A)$  as well. Next, by a simple computation (by considering  $\varphi$  and  $u$  as functions on  $\mathbb{R}$ ), we can show that

$$(3.8) \quad R(\lambda, \mathcal{D})(\varphi * u) = R(\lambda, \mathcal{D})\varphi * u.$$

By *Condition F<sup>++</sup>*, for each  $g \in \mathcal{F}$ ,  $\psi \in L^1(\mathbb{R}^-)$ ,  $\psi * g \in \mathcal{F}$ . Therefore,  $\psi$  induces a map  $BC(\mathbb{R}^+, \mathbb{X})/\mathcal{F} \ni \tilde{g} \mapsto \tilde{\psi} * \tilde{g} \in BC(\mathbb{R}^+, \mathbb{X})/\mathcal{F}$ . Notice also that if  $\psi_n \rightarrow \psi$  in  $L^1(\mathbb{R})$ , then the induced maps  $\tilde{\psi}_n \rightarrow \tilde{\psi}$ . Next, consider the canonical projection  $p : BC(\mathbb{R}^+, \mathbb{X}) \rightarrow BC(\mathbb{R}^+, \mathbb{X})/\mathcal{F}$  and the restriction  $r$  to the half line of elements of  $BC(\mathbb{R}^+, \mathbb{X})$ . Hence, from (3.8),

$$(3.9) \quad R(\lambda, \tilde{\mathcal{D}})(\widetilde{\varphi * u}) = p \circ r(R(\lambda, \mathcal{D})(\varphi * u)) = p \circ r(R(\lambda, \mathcal{D})\varphi * u).$$

This shows that  $R(\lambda, \tilde{\mathcal{D}})(\widetilde{\varphi * u})$  is holomorphic at any  $\xi \in \mathbb{C}$  whenever  $R(\lambda, \mathcal{D})\varphi$  is also. Now, by the above remark that  $R(\lambda, \mathcal{D})\varphi$  is holomorphic in a neighborhood of  $\mathbb{R} \setminus \sigma_i(A)$ , (3.9) yields in particular that

$$(3.10) \quad sp_{\mathcal{F}}^{\perp}(\varphi * u) \subset \mathbb{R} \setminus \sigma_i(A).$$

On the other hand, a simple computation shows that

$$(3.11) \quad R(\lambda, \mathcal{D})(\varphi * u) = \varphi * R(\lambda, \mathcal{D})u.$$

Recall that we consider  $u$  as a function on  $\mathbb{R}$  with  $u(t) = 0$  for all  $t < 0$ , so for  $t \geq 0$  and  $Re\lambda < 0$ ,

$$R(\lambda, \mathcal{D})(\varphi * u)(t) = - \int_{-\infty}^{\infty} \varphi(t - \xi) \int_0^{\xi} e^{-\lambda(\zeta - \xi)} u(\zeta) d\zeta.$$

Hence,

$$(3.12) \quad R(\lambda, \tilde{\mathcal{D}})(\widetilde{\varphi * u}) = \tilde{\varphi} * R(\lambda, \tilde{\mathcal{D}})\tilde{u}.$$

By [10, Lemma 4.7],  $sp_{\mathcal{F}}^{\perp}(u) \subset \sigma_i(A)$ ,  $R(\lambda, \tilde{\mathcal{D}})\tilde{u}$  is holomorphic in a neighborhood of  $\sigma_i(A)$ , so  $R(\lambda, \tilde{\mathcal{D}})(\widetilde{\varphi * u})$  is also. In turn, this yields that

$$(3.13) \quad sp_{\mathcal{F}}^{\perp}(\varphi * u) \subset \sigma_i(A).$$

Now (3.10) and (3.13) prove (3.6).

In the general case, if  $\phi$  is approximated by  $\phi_n$  (in  $L^1(\mathbb{R})$ ) whose Fourier transform  $\mathcal{F}\phi_n$  vanishes in a neighborhood of  $i\sigma(A) \cap \mathbb{R}$ , we note that (3.8) and (3.12) allows us to “pass to the limits”. Therefore, (3.6) is proved, so the proof of Theorem 1.2 is complete. □

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