ASYMPTOTICS OF MATRIX INTEGRALS
AND TENSOR INVARIANTS OF COMPACT LIE GROUPS

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ABSTRACT. In this paper we give an asymptotic formula for a matrix integral which plays a crucial role in the approach of Diaconis et al. to random matrix eigenvalues. The choice of parameter for the asymptotic analysis is motivated by an invariant-theoretic interpretation of this type of integral. For arbitrary regular irreducible representations of arbitrary connected semisimple compact Lie groups, we obtain an asymptotic formula for the trace of permutation operators on the space of tensor invariants, thus extending a result of Biane on the dimension of these spaces.

1. INTRODUCTION

Let G be a compact connected Lie group, and let $(V_\lambda, \rho_\lambda)$ be an irreducible representation of G with highest weight λ. Consider the matrix integral

(1)
$$\int_G (\mathrm{Tr}\rho_\lambda(g))^{a_1} (\mathrm{Tr}\rho_\lambda(g^2))^{a_2} \ldots (\mathrm{Tr}\rho_\lambda(g^r))^{a_r} (\overline{\mathrm{Tr}\rho_\lambda(g)})^{b_1} \ldots (\overline{\mathrm{Tr}\rho_\lambda(g^r)})^{b_r} \, d\omega_G(g),$$

where $\{a_j\}_{j=1}^r$, $\{b_j\}_{j=1}^r$ are fixed sequences of nonnegative integers and ω_G denotes normalized Haar measure on G. In the case that $(V_\lambda, \rho_\lambda)$ is the standard representation of the unitary group $G = \mathrm{U}_n$, the integral (1) is nothing else than the $(a_1, \ldots, a_r, b_1, \ldots, b_r)$-moment of the random vector

(2)
$$(\mathrm{Tr}(g), \ldots, \mathrm{Tr}(g^r), \overline{\mathrm{Tr}(g)}, \ldots, \overline{\mathrm{Tr}(g^r)}),$$

where g is chosen from U_n according to the Haar measure. It has been proven by Diaconis and Shahshahani in [DS] (see also [DE]) that for n large enough, (1) coincides with the $(a_1, \ldots, a_r, b_1, \ldots, b_r)$-moment of a vector of independent complex Gaussian random variables of suitable variances, and consequently, the vector (2) converges in distribution to this Gaussian vector as the matrix size n tends to infinity. It has been observed ([BR], [St]) that the Diaconis-Shahshahani result is based on the fact that the integral (1) can be expressed as

(3)
$$\int_G \prod_{j=1}^r \mathrm{Tr}(\rho_\lambda(g)^j)^{a_j} \mathrm{Tr}(\rho_\lambda^*(g)^j)^{b_j} \, d\omega_G(g) = \mathrm{Tr}\left((\sigma_{k_a} \otimes \sigma_{k_b})(s,t)|_{[V_\lambda^{\otimes k_a} \otimes (V_\lambda^*)^{\otimes k_b}]^G} \right).$$

Here we write $k_a = \sum_{j=1}^{r} ja_j$ and define k_b analogously. σ_{k_a} denotes the obvious representation of the symmetric group $\mathfrak{S}_{k_a}$ on $V_\lambda^{\otimes k_a}$, and σ_{k_b} is its analogue on $(V_\lambda^*)^{\otimes k_b}$, where the representation $(V_\lambda^*, \rho_\lambda^*)$ is contragredient to $(V_\lambda, \rho_\lambda)$. $s \in \mathfrak{S}_{k_a}$ has cycle type $(1^{a_1} \ldots r^{a_r})$ and $t \in \mathfrak{S}_{k_b}$ has cycle type $(1^{b_1} \ldots r^{b_r})$.

$$[V_\lambda^{\otimes k_a} \otimes (V_\lambda^*)^{\otimes k_b}]^G$$
$$= \{T \in V_\lambda^{\otimes k_a} \otimes (V_\lambda^*)^{\otimes k_b} : (\rho_\lambda^{\otimes k_a} \otimes (\rho_\lambda^*)^{\otimes k_b})(g)T = T \text{ for all } g \in G\}$$

is the space of invariants of the G-action $\rho_\lambda^{\otimes k_a} \otimes (\rho_\lambda^*)^{\otimes k_b}$. Diaconis and Shahshahani study the integral (1) for a sequence (G_n) of classical groups of increasing rank, fixing the parameters $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_r)$. Since they consider standard representations of classical groups, substantial information about the right-hand side of (3) has become available since the epoch-making work of Weyl ([Wy]) and can be used to evaluate the left-hand side.

In the present paper, we depart from the Diaconis-Shahshahani framework in two ways. Firstly, we consider arbitrary regular irreducible representations of arbitrary compact connected semisimple Lie groups. In this generality, one has *a priori* only poor control of the right-hand side of (3). So we will directly attack the left-hand side of this equation and thus obtain some asymptotic information about the spaces of invariants on the right-hand side. Secondly, we fix a group G and a representation V_λ and let the parameters k_a, k_b of the tensor powers (and hence the moment parameters a, b) tend to infinity. This may be thought of as a thermodynamic limit of a particle system, rather than a random matrix limit. For our asymptotic analysis we will use techniques which were developed by Biane ([B]), Klyachko and Kurtaran ([KK]), and Tate and Zelditch ([TZ]). Our results should be compared to those of Biane ([B]) and Kuperberg ([K]). An asymptotic result of a different kind for growing tensor powers of a fixed representation has recently been obtained by Collins and Śniady in [CS, Thm. 17].

Here is the setup for our main theorems: Assume that the compact connected Lie group G is semisimple, and that the highest weight λ of the fixed irreducible representation $(V_\lambda, \rho_\lambda)$ is regular, i.e., is in the interior of a Weyl chamber. Fix a maximal torus T in G. Denote by W the Weyl group. Write $\mathfrak{t}$ for the Lie algebra of T, and $\mathfrak{t}^*$ for its dual space. $I := \ker \exp \subset \mathfrak{t}$ is the integral lattice, and its dual $I^* := \{\varphi \in \mathfrak{t}^* : \varphi(I) \subseteq \mathbb{Z}\}$ is the weight lattice. Let $\Lambda^* \subset \mathfrak{t}^*$ be the abelian group generated by the roots, i.e. the root lattice, and write $\Lambda := (\Lambda^*)^*$ for its dual. It is well known that $\Lambda^* \subseteq I^*$, hence $I \subseteq \Lambda$, and that the group $\Pi(G) := \Lambda/I$ is a finite abelian group. It can be regarded as a subgroup of $T \cong \mathfrak{t}/I$. Write π for the canonical projection of T onto $T/\Pi(G)$. For any $\mu \in I^*$ write $\mathrm{m}_\lambda(\mu)$ for its multiplicity in V_λ. Then, the set of all weights of V_λ is $M_\lambda := \{\mu \in I^* : \mathrm{m}_\lambda(\mu) \neq 0\}$.

Fix a smooth function $f > 0$ on G and sequences $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_r)$ of nonnegative integers. For a positive integer N set $a_j := a_j(N) := N\alpha_j$, $b_j := N\beta_j$ $(j = 1, \ldots, r)$ and write $a = (a_1, \ldots, a_r), b = (b_1, \ldots, b_r)$. Furthermore, we set

$$(4) \qquad |\alpha| := \sum_{j=1}^{r} \alpha_j, \quad k_\alpha := \sum_{j=1}^{r} j\alpha_j, \quad l_\alpha := \sum_{j=1}^{r} j^2\alpha_j$$

and define $|\beta|, k_\beta, l_\beta, k_a$, etc. analogously. Throughout the paper we will assume that $k_\alpha = k_\beta$. We will study the integrals

(5) $$I_N = I_N(f, \alpha) := \int_G \prod_{j=1}^r (\mathrm{Tr}(\rho_\lambda(g^j)))^{N\alpha_j} f(g) \, d\omega_G(g) \quad \text{and}$$

(6) $$K_N = K_N(f, \alpha, \beta) := \int_G \prod_{j=1}^r (\mathrm{Tr}(\rho_\lambda(g^j)))^{N\alpha_j} \, \overline{(\mathrm{Tr}(\rho_\lambda(g^j)))}^{N\beta_j} \, f(g) \, d\omega_G(g).$$

Writing Φ_+ for the set of positive roots of (G, T), we define a polynomial κ on $\mathfrak{t}$ by

(7) $$\kappa(x) = \prod_{\alpha \in \Phi_+} \langle \alpha, x \rangle.$$

Finally, we define $A_\lambda \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathfrak{t}^*) \cong \mathfrak{t}^* \otimes \mathfrak{t}^*$ by

(8) $$A_\lambda = \frac{1}{\dim V_\lambda} \sum_{\mu \in M_\lambda} m_\lambda(\mu) \mu \otimes \mu.$$

Since G is assumed to be semisimple, and λ to be regular, A_λ is known to be positive definite (see [TZ]).

Now we are in a position to state our main results, to be proven in Sections 3 – 5 below.

Theorem 1. *Assume that* $\gcd\{j : \alpha_j \neq 0\} = 1$. *Then*
(9)
$$I_N(f, \alpha) = \frac{(2\pi)^d (\dim V_\lambda)^{N|\alpha|} \kappa(A_\lambda^{-1}\rho)}{(2\pi l_\alpha N)^{(\dim G)/2} \sqrt{\det A_\lambda}} \left(\sum_{h \in \Pi(G)} \nu_{Nk_\alpha \lambda}(h) f^G(h) + O(N^{-1/2}) \right),$$

where d is the number of positive roots, ρ is half the sum of the positive roots, f^G is the class function $f^G(g) = \int f(x^{-1}gx) d\omega_G(x)$, and $\nu_{Nk_\alpha\lambda}$ is the character on $\Pi(G)$ determined by the weight $Nk_\alpha\lambda$:

(10) $$\nu_{Nk_\alpha\lambda}(h) = e^{2\pi\sqrt{-1}Nk_\alpha\langle \lambda, \psi_h \rangle},$$

$\psi_h \in \Lambda$ being a coset representative of $h \in \Lambda/I = \Pi(G)$.

Theorem 2. *Suppose that* $\gcd\{j : \alpha_j \neq 0 \text{ or } \beta_j \neq 0\} = 1$. *Then*
(11)
$$K_N(f, \alpha, \beta) = \frac{(2\pi)^d (\dim V_\lambda)^{N(|\alpha|+|\beta|)} \kappa(A_\lambda^{-1}\rho)}{(2\pi N(l_\alpha + l_\beta))^{(\dim G)/2} \sqrt{\det A_\lambda}} \left(\sum_{h \in \Pi(G)} f^G(h) + O(N^{-1/2}) \right).$$

Remark 1.1. Note that in Theorem 1 the leading term vanishes if $Nk_\alpha\lambda = k_a\lambda$ is not contained in the lattice Λ^* and if $f \equiv 1$. In fact, in this case one has $[V_\lambda^{\otimes k_a}]^G = \{0\}$; hence $I_N(1, \alpha) = 0$ by (3). This is because the existence of a nontrivial invariant implies that there is a sequence $\mu_1, \ldots, \mu_{k_a}$ of weights for the irreducible representation V_λ such that $\mu_1 + \cdots + \mu_{k_a} = 0$. Recall the well-known fact (see [TZ]) that, if the highest weight λ is regular, then the root lattice Λ^* is spanned by the differences $\mu - \mu'$ between two weights μ, μ' for V_λ. This implies that

$$1 = e^{2\pi\sqrt{-1}\langle \mu_1 + \cdots + \mu_{k_a}, \psi \rangle} = e^{2\pi\sqrt{-1}k_a\langle \lambda, \psi \rangle}$$

for any ψ in the dual Λ of the root lattice Λ^*, and hence $k_a\lambda \in \Lambda^*$.

Remark 1.2. In view of (3), Theorems 1 and 2 give asymptotic formulae for the trace of permutations on the space of tensor invariants. Specializing to the identity permutation, one obtains the asymptotics of the dimension of these spaces. Specifically, taking in Theorem 1 $\alpha_j = 0$ for $j \geq 2$ and $\alpha_1 = 1$, $f \equiv 1$, and assuming that λ is in the root lattice Λ^*, we obtain an asymptotic formula for the dimension of the space of tensor invariants, namely

$$\dim[V_\lambda^{\otimes N}]^G = \frac{|\Pi(G)|(\dim V_\lambda)^N \kappa(A_\lambda^{-1}\rho)}{(2\pi)^{(\mathrm{rk}\,G)/2} N^{(\dim G)/2}\sqrt{\det A_\lambda}}(1 + O(N^{-1/2})).$$

This formula has been obtained by Biane in [B] (see also [TZ]).

2. MATRIX INTEGRALS AND TENSOR INVARIANTS

Before turning to the proof of Theorems 1 and 2, we provide a short and self-contained proof of equation (3), on which the invariant-theoretic interpretation of the integrals I_N and K_N is based. Write $H := [V_\lambda^{\otimes k_a} \otimes (V_\lambda^*)^{\otimes k_b}]^G$. Then, the orthogonal projection onto H is given by

$$(12) \qquad \pi_H(T) = \int_G (\rho_\lambda^{\otimes k_a} \otimes (\rho_\lambda^*)^{\otimes k_b})(g)T \, d\omega_G(g), \quad T \in V_\lambda^{\otimes k_a} \otimes (V_\lambda^*)^{\otimes k_b}.$$

For $A \in \mathrm{End}_G(V_\lambda^{\otimes k_a} \otimes (V_\lambda^*)^{\otimes k_b})$, then,

$$\mathrm{Tr}(A|_H) = \mathrm{Tr}(A\pi_H) = \mathrm{Tr}(\pi_H A).$$

We thus obtain

Lemma 2.1. *For any $A \in \mathrm{End}_G(V_\lambda^{\otimes k_a} \otimes (V_\lambda^*)^{\otimes k_b})$, we have*

$$\mathrm{Tr}(A|_H) = \int_G \mathrm{Tr}\left((\rho_\lambda^{\otimes k_a} \otimes (\rho_\lambda^*)^{\otimes k_b})(g)\ A\right) d\omega_G(g)$$

$$= \int_G \mathrm{Tr}\left(A\ (\rho_\lambda^{\otimes k_a} \otimes (\rho_\lambda^*)^{\otimes k_b})(g)\right) d\omega_G(g).$$

In view of this, (3) is implied by the following well-known lemma (see [DS], [Ra]):

Lemma 2.2. *Let V be a d-dimensional complex vector space, $B \in \mathrm{End}_{\mathbb{C}}(V)$, and $s \in \mathfrak{S}_k$ a permutation of type $(1^{a_1} 2^{a_2} \ldots r^{a_r})$; hence $k = k_a = \sum_{j=1}^r j a_j$. Let $\mathfrak{S}_k$ act on $V^{\otimes k}$ via $\sigma_k(s)(\bigotimes_{i=1}^k v_i) := \bigotimes_{i=1}^k v_{is^{-1}}$. Then the trace of $B^{\otimes k}\sigma_k(s) \in \mathrm{End}_{\mathbb{C}}(V^{\otimes k})$ is*

$$\mathrm{Tr}\left(B^{\otimes k}\sigma_k(s)\right) = \prod_{j=1}^r \mathrm{Tr}(B^j)^{a_j}.$$

Proof. Fix an inner product $\langle\,,\,\rangle$ on V. This induces an inner product on $V^{\otimes k}$, which is also denoted by $\langle\,,\,\rangle$. Let $e_1, \ldots, e_d$ be an orthonormal basis for V. Write $\mathcal{F}$ for the set of maps from $\{1, \ldots, k\}$ to $\{1, \ldots, d\}$, $\mathcal{F}_S$ for the restrictions to a subset S of $\{1, \ldots, k\}$, $e_\varphi := \bigotimes_{i=1}^k e_{\varphi(i)}$. Then $\{e_\varphi : \varphi \in \mathcal{F}\}$ is an orthonormal basis of $V^{\otimes k}$. Write

$$s = \prod_{j=1}^r \prod_{i=1}^{a_j} \zeta_i^j,$$

where $\{\zeta_i^j : i = 1, \ldots, a_j\}$ are the cycles of length j in s. Furthermore, for any $t \in \mathfrak{S}_k$, write $[t] := \{\nu = 1, \ldots, k : \nu t \neq \nu\}$. Then

$$\mathrm{Tr}(B^{\otimes k}\sigma_k(s)) = \sum_{\varphi \in \mathcal{F}} \left\langle B^{\otimes k}\sigma_k(s)e_\varphi, e_\varphi \right\rangle = \sum_{\varphi \in \mathcal{F}} \prod_{j=1}^{k} \left\langle Be_{js^{-1}\varphi}, e_{j\varphi} \right\rangle$$

$$= \sum_{\varphi \in \mathcal{F}} \prod_{j=1}^{r} \prod_{i=1}^{a_j} \prod_{l \in [\zeta_i^j]} \left\langle Be_{l(\zeta_i^j)^{-1}\varphi}, e_{l\varphi} \right\rangle = \prod_{j=1}^{r} \prod_{i=1}^{a_j} \sum_{\varphi \in \mathcal{F}_{[\zeta_i^j]}} \prod_{l \in [\zeta_i^j]} \left\langle Be_{l(\zeta_i^j)^{-1}\varphi}, e_{l\varphi} \right\rangle$$

$$= \prod_{j=1}^{r} \prod_{i=1}^{a_j} \mathrm{Tr}(B^j) = \prod_{j=1}^{r} \mathrm{Tr}(B^j)^{a_j}.$$

$\square$

3. Phase function for the matrix integral

We start by rewriting the integral (5) using Weyl's integration formula, assuming for simplicity that f is a class function:

$$(13) \qquad I_N(f, \alpha) = \frac{1}{|W|} \int_T \prod_{j=1}^{r} \mathrm{Tr}(\rho_\lambda(t^j))^{N\alpha_j} f(t)|\Delta(t)|^2 \, dt,$$

where $T \subset G$ is a maximal torus, dt is Haar measure on T, normalized as a probability measure, and $\Delta(t)$ is the Weyl denominator. We define the following function on the complexified Lie algebra $\mathfrak{t}^{\mathbb{C}}$:

$$(14) \qquad k(w) = \sum_{\mu \in M_\lambda} m_\lambda(\mu)e^{2\pi\langle \mu, w \rangle},$$

where $w = \tau + \sqrt{-1}\varphi \in \mathfrak{t} \oplus \sqrt{-1}\mathfrak{t} = \mathfrak{t}^{\mathbb{C}}$, and linear forms in $\mathfrak{t}^*$ are extended complex linearly to $\mathfrak{t}^{\mathbb{C}}$. Note that the restriction of k to $\sqrt{-1}\mathfrak{t}$ is essentially the character of V_λ.

Let $d\varphi$ denote Lebesgue measure on $\mathfrak{t}$, normalized so that the fundamental domain T_o of the integral lattice I has volume 1. Then the integral I_N can be written in the form:

$$(15) \quad I_N = \frac{1}{|W|} \int_{T_o} F(\sqrt{-1}\varphi)^N f(\varphi)|\Delta(\varphi)|^2 \, d\varphi, \quad F(\sqrt{-1}\varphi) = \prod_{j=1}^{r} k(\sqrt{-1}j\varphi)^{\alpha_j},$$

where the class function f is, through the exponential map, regarded as a function on $\mathfrak{t}$.

Lemma 3.1. *We have the inequality*

$$(16) \qquad \left| F(\sqrt{-1}\varphi) \right| \leq k(0)^{|\alpha|} = (\dim V_\lambda)^{|\alpha|}.$$

Equality holds if and only if φ is in the dual lattice Λ of the root lattice Λ^*.

Proof. The proof is similar to the proof of Lemma 1.4 in [TZ]. Note that the assumption about the gcd is used here. $\square$

By Lemma 3.1, the integral (13) or (15) is localized on $\ker \pi = \Pi(G) = \Lambda/I$. Let g be a smooth cut-off function on T around the unit such that the support of g does not contain any element in $\ker \pi$ other than the unit. Then, the translate

$g_h(t) = g(h^{-1}t)$ is a cut-off function around $h \in \ker \pi$. Let $\varphi \in \Lambda$. Then it is easy to see that

$$(17) \qquad k(\sqrt{-1}j\varphi) = e^{2\pi\sqrt{-1}j\langle \lambda, \varphi \rangle} k(0),$$

which is not zero. Thus, around each $\varphi \in \Lambda$, we can take a branch of the logarithm to define the following function Φ:

$$(18) \qquad \Phi(w) := \sum_{j=1}^{r} \alpha_j \log k(jw), \quad w = \tau + \sqrt{-1}\varphi \in \mathfrak{t}^{\mathbb{C}},$$

where φ varies in a neighborhood of a point in Λ. Then, by Lemma 3.1, we can write the integral I_N as follows:

$$(19) \qquad I_N = \frac{1}{|W|} e^{N\Phi(0)} \sum_{h \in \ker \pi} \int_{\mathfrak{t}} e^{N(\Phi(\sqrt{-1}\varphi) - \Phi(0))} g_h(\varphi) f(\varphi) |\Delta(\varphi)|^2 \, d\varphi$$

plus a term of order $O(e^{-cN})$ for some $c > 0$. To compute each of the integrals in the sum in (19), we note that

$$\Delta(\varphi + \psi_h) = e^{2\pi\sqrt{-1}\langle \rho, \psi_h \rangle} \Delta(\varphi)$$

for each $h \in \ker \pi \cong \Lambda/I$, where $\psi_h \in \Lambda$ satisfies $\exp(\psi_h) = h$ and ρ is half the sum of the positive roots. Furthermore, by (17), we have

$$\Phi(\sqrt{-1}(\varphi + \psi_h)) = \Phi(\sqrt{-1}\varphi) + 2\pi\sqrt{-1} \, k_\alpha \langle \lambda, \psi_h \rangle.$$

Therefore, we obtain:

Lemma 3.2. *We have*

$$(20) \qquad I_N = \frac{1}{|W|} \sum_{h \in \ker \pi} \nu_{Nk_\alpha \lambda}(h) \int e^{N\Phi(\sqrt{-1}\varphi)} g(\varphi) f_h(\varphi) |\Delta(\varphi)|^2 \, d\varphi$$

plus a term of order $O(e^{-cN})$, where $g(\varphi)$ is a cut-off function around $\varphi = 0$, and $f_h(\varphi) = f(\varphi + \psi_h)$ with a representative $\psi_h \in \Lambda$ for $h \in \ker \pi$.

Next, we compute the first and second derivatives of the phase function Φ at points in $\ker \pi$.

Lemma 3.3. *Any $\varphi \in \Lambda$ is a critical point of Φ. Furthermore, the negative of the Hessian of Φ, $H(\varphi) : \mathfrak{t} \to \mathfrak{t}^*$, is given by $H(\varphi) = (2\pi)^2 \, l_\alpha A_\lambda$, which is independent of $\varphi \in \Lambda$ and is positive definite, where l_α and A_λ are defined in (4) and (8), respectively.*

Proof. That each $\varphi \in \Lambda$ is a critical point of Φ is proven by the fact that

$$\sum_{\mu \in M_\lambda} m_\lambda(\mu)\mu = 0,$$

because the left-hand side of the above is a W-invariant vector and G is assumed to be semisimple. The fact that the linear map $A_\lambda : \mathfrak{t} \to \mathfrak{t}^*$ is positive definite is proven in [TZ]. $\qquad \square$

Therefore, what we need to do is to find asymptotics of each of the integrals in (20), each of which is an integral of functions supported around the origin. However, there is a difficulty: for each integral, the origin is the unique critical point for the complex phase function Φ, but the Weyl denominator Δ vanishes at the origin. In the next section, we will use ideas of Biane ([B]) and Klyachko-Kurtaran ([KK]) to circumvent this problem.

4. The method of Biane and Klyachko-Kurtaran

In this section we study the following integral, which is a slight generalization of the integral on the right-hand side of (20):

$$(21) \qquad J_N := \int_T e^{N\Phi(t)} g(t) |\Delta(t)|^2 \, dt,$$

where dt is normalized Haar measure on the maximal torus T and Δ denotes the Weyl denominator. g is a compactly supported smooth function that does not vanish in the unit element of T. We fix a W-invariant inner product on $\mathfrak{t}$ such that the volume of the parallelotope determined by an orthonormal basis equals 1. We assume that the complex-valued smooth phase function Φ, which we view as a function on the complexified Lie algebra $\mathfrak{t}^{\mathbb{C}}$, satisfies the following conditions:

(i) $\operatorname{Re}(\Phi(\sqrt{-1}\varphi)) \le \Phi(0)$, with equality if and only if $\varphi = 0$;
(ii) the origin is a critical point of Φ;
(iii) $H = -\partial^2 \Phi(0) : \mathfrak{t} \to \mathfrak{t}^*$, the negative of the Hessian of Φ at the origin, is positive definite;
(iv) in the Taylor expansion

$$(22) \qquad \Phi(\sqrt{-1}\varphi) - \Phi(0) = -\langle H\varphi, \varphi \rangle/2 - \sqrt{-1}\,\Theta(\varphi) + R_4(\varphi)$$

up to fourth order, Θ is a real-valued homogeneous polynomial of degree 3;
(v) the linear map H commutes with the action of the Weyl group W.

Note that the phase functions of the integrals I_N and K_N (see Sections 3 and 5) satisfy these conditions. We are now in a position to state the main result of this section:

Theorem 4.1. *Under the above conditions on Φ and g one has*

$$(23) \qquad J_N = \left(\frac{2\pi}{N}\right)^{(\dim G)/2} \frac{(2\pi)^d g(0) e^{N\Phi(0)} |W|}{\sqrt{\det H}}\, \kappa(H^{-1}\rho)\,(1 + O(N^{-1/2})),$$

where $|W|$ is the order of the Weyl group W, d is the number of positive roots, ρ is half the sum of the positive roots, and the polynomial κ on $\mathfrak{t}$ is defined in (7).

Proof. First of all, we normalize Lebesgue measure on $\mathfrak{t}$ so that the volume of the fundamental domain T_o of the lattice I equals 1, and write

$$(24) \qquad J_N = e^{N\Phi(0)} \int_{T_o} e^{N[\Phi(\sqrt{-1}\varphi) - \Phi(0)]} g(\varphi) |\Delta(\varphi)|^2 \, d\varphi.$$

We may regard g as a function on $\mathfrak{t}$ with arbitrarily small compact support around the origin, since by (i) the integrand in (24) is bounded by e^{-cN} (with a constant $c > 0$) outside a compact neighborhood of the origin. (i) and (iv) imply that for fixed $a > 0$ one can choose $b > 0$ such that $\operatorname{Re}\Phi(\sqrt{-1}\varphi) - \Phi(0) \le -b\langle H\varphi, \varphi\rangle$ for $|\varphi| \le a$. Substituting (22) into J_N and changing the variable φ to $N^{-1/2}\varphi$, we have

$$(25)$$
$$J_N = N^{-(\operatorname{rk}G)/2} e^{N\Phi(0)}$$

$$\times \int e^{-\langle H\varphi, \varphi \rangle/2 \,-\sqrt{-1}N\,\Theta(N^{-1/2}\varphi)\, +NR_4(N^{-1/2}\varphi)} g(N^{-1/2}\varphi) |\Delta(N^{-1/2}\varphi)|^2 \, d\varphi.$$

As in [B], [TZ], using the identity $\Delta = \prod_{\alpha \in \Phi_+} (e^{\pi\sqrt{-1}\alpha} - e^{-\pi\sqrt{-1}\alpha})$, it is easy to see that

$$\Delta(N^{-1/2}\varphi) = (2\pi\sqrt{-1})^d N^{-d/2} \kappa(\varphi)(1 + O(N^{-1}|\varphi|^2)).$$

We note that $g(N^{-1/2}\varphi) = g(0)(1 + O(N^{-1/2}|\varphi|))$. Substituting these formulas into (25) and introducing a smooth cut-off function χ such that $\chi = 1$ around the support of g, we obtain

(26)
$$J_N = \frac{(2\pi)^{2d}g(0)e^{N\Phi(0)}}{N^{d+(\mathrm{rk}G)/2}}$$
$$\times \int e^{-\langle H\varphi,\varphi\rangle/2-\sqrt{-1}N^{-1/2}\Theta(\varphi)+NR_4(N^{-1/2}\varphi)}\chi(N^{-1/2}\varphi)|\kappa(\varphi)|^2 \, d\varphi \, (1+O(N^{-1/2})).$$

Here, we note that g and its derivatives are bounded on $\mathfrak{t}$, and that the exponential in the integrand is bounded by $e^{-b\langle H\varphi,\varphi\rangle}$ if $|\varphi|/N^{1/2} \leq a$. As indicated above, we may assume that $g(N^{-1/2}\varphi) = 0$ for $|\varphi|/N^{1/2} \geq a$.

Next, as in [TZ], we divide the integral in (26) into several parts as follows. We set $E_N(\varphi) := e^{-\langle H\varphi,\varphi\rangle/2-\sqrt{-1}N^{-1/2}\Theta(\varphi)}$ and write

(27)
$$J_N = \frac{(2\pi)^{2d}g(0)e^{N\Phi(0)}}{N^{d+(\mathrm{rk}G)/2}}\left(\sum_{j=1}^{4} I_j(N)\right)(1+O(N^{-1/2})),$$

where the integrals $I_j(N)$, $j = 1, 2, 3, 4$ are given by

$$I_1(N) = \int E_N(\varphi)|\kappa(\varphi)|^2 \, d\varphi,$$

$$I_2(N) = \int E_N(\varphi)(e^{NR_4(N^{-1/2}\varphi)} - 1)\chi(N^{-1/4}\varphi)|\kappa(\varphi)|^2 \, d\varphi,$$

(28)
$$I_3(N) = \int E_N(\varphi)e^{NR_4(N^{-1/2}\varphi)}(1 - \chi(N^{-1/4}\varphi))\chi(N^{-1/2}\varphi)|\kappa(\varphi)|^2 \, d\varphi,$$

$$I_4(N) = \int E_N(\varphi)(\chi(N^{-1/4}\varphi) - 1)|\kappa(\varphi)|^2 \, d\varphi,$$

where we used the relation $\chi(N^{-1/4}\varphi)\chi(N^{-1/2}\varphi) = \chi(N^{-1/4}\varphi)$ for sufficiently large N.

Now, the integrand in $I_2(N)$ vanishes for $|\varphi| \geq cN^{1/4}$, and thus we have $e^{NR_4(N^{-1/2}\varphi)} = O(1)$ and $NR_4(N^{-1/2}\varphi) = |\varphi|^4O(1/N)$. Hence, $(e^{NR_4(N^{-1/2}\varphi)} - 1)$ is bounded by $c|\varphi|^4O(\frac{1}{N})$. But we have $|E_N(\varphi)| = e^{-\langle H\varphi,\varphi\rangle/2}$, and hence $|\varphi|^4E_N(\varphi)$ is integrable uniformly in N. Thus we have $I_2(N) = O(1/N)$. As to $I_3(N)$, the function $E_N(\varphi)e^{NR_4(N^{-1/2}\varphi)}$ is dominated by $e^{-b\langle H\varphi,\varphi\rangle}$ wherever the integrand does not vanish. Since $\chi(N^{-1/4}\varphi) = 1$ for $|\varphi| \leq cN^{1/4}$ for some $c > 0$, we easily have $I_3(N) = O(N^{c_1}e^{-c_2N^{1/2}})$ for some $c_1, c_2 > 0$. Similarly, we have $I_4(N) = O(N^{c_1}e^{-c_2N^{1/2}})$.

Finally, we consider the integral $I_1(N)$. Note that $e^{-\sqrt{-1}\,\Theta(\varphi)N^{-1/2}} = 1 + O(|\varphi|^3/N^{1/2})$ since $\Theta(\varphi)$ is real. Thus, invoking the identity

$$\int_{\mathfrak{t}} e^{-\langle Hx,x\rangle/2}|\kappa(x)|^2 \, dx = \frac{(2\pi)^{(\mathrm{rk}G)/2}\,|W|\,\kappa(H^{-1}\rho)}{\sqrt{\det H}},$$

known as Mehta's conjecture and proven in [Mc] and [Op], we have

$$
(29) \quad
\begin{aligned}
I_1(N) &= \int e^{-\langle H\varphi, \varphi \rangle} |\kappa(\varphi)|^2 \, d\varphi (1 + O(N^{-1/2})) \\
&= \frac{(2\pi)^{(\mathrm{rk}G)/2} \, |W| \, \kappa(H^{-1}\rho)}{\sqrt{\det H}} (1 + O(N^{-1/2})),
\end{aligned}
$$

which completes the proof. □

5. Proof of the main results

Theorem 1 is a direct application of Theorem 4.1 with Lemma 3.3 to the integral on the right-hand side of (20). To prove Theorem 2, we proceed as in Section 3 and use the Weyl integration formula to obtain

$$
(30) \quad K_N = \frac{1}{|W|} \int_T J(t)^N f(t) |\Delta(t)|^2 \, dt, \quad J(t) = \prod_{j=1}^{r} \mathrm{Tr}(\rho_\lambda(t^j))^{\alpha_j} \overline{\mathrm{Tr}(\rho_\lambda(t^j))}^{\beta_j}.
$$

We also apply Theorem 4.1 to find asymptotics of the integral K_N as follows. As in Lemma 3.1, we have

$$
(31) \quad |J(\sqrt{-1}\varphi)| \le k(0)^{|\alpha|+|\beta|} = (\dim V_\lambda)^{|\alpha|+|\beta|},
$$

with equality if and only if $\varphi \in \Lambda$. We define

$$
(32) \quad \Psi(w) = \sum_{j=1}^{s} [\alpha_j \log k(jw) + \beta_j \log k(j\overline{w})], \quad w = \tau + \sqrt{-1}\varphi \in \mathfrak{t}^\mathbb{C},
$$

around each $\varphi \in \Lambda$. Next, we need to compute the Hessian of Ψ at $\varphi \in \Lambda$. We note that, for $\varphi \in \Lambda$, we have

$$
k(\sqrt{-1}j\varphi) = e^{2\pi\sqrt{-1}j\langle \lambda, \varphi \rangle} k(0), \quad (\partial k)(\sqrt{-1}j\varphi) = 0,
$$

$$
\partial^2 (\log k(\sqrt{-1}j\varphi)) = -4\pi^2 j^2 A_\lambda.
$$

Therefore, we have the following lemma:

Lemma 5.1. *Each $\varphi \in \Lambda$ is a critical point of Ψ. The negative of the Hessian of Ψ, denoted $D(\varphi)$, is given by $D(\varphi) = (2\pi)^2 (l_\alpha + l_\beta) A_\lambda$, and hence $D = D(\varphi)$ does not depend on $\varphi \in \Lambda$, and is positive definite.*

This time, the term involving $2\pi\sqrt{-1}j\langle \lambda, \psi_h \rangle$ disappears because of the assumption $\sum j\alpha_j = \sum j\beta_j$. We thus have

$$
K_N = \frac{1}{|W|} \sum_{h \in \ker \pi} \int_\mathfrak{t} e^{N\Psi(\sqrt{-1}\varphi)} g(\varphi) f_h(\varphi) |\Delta(\varphi)|^2 \, d\varphi,
$$

where f_h is defined in Lemma 3.2. The assumption of Theorem 4.1 is satisfied by the phase function Ψ, and applying it, we obtain Theorem 2.

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