ASYMPTOTIC CONES AND ASSOUAD-NAGATA DIMENSION

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ABSTRACT. We prove that the dimension of any asymptotic cone over a metric space (X, ρ) does not exceed the asymptotic Assouad-Nagata dimension $\operatorname{asdim}_{AN}(X)$ of X. This improves a result of Dranishnikov and Smith (2007), who showed $\dim(Y) \leq \operatorname{asdim}_{AN}(X)$ for all separable subsets Y of special asymptotic cones $\operatorname{Cone}_\omega(X)$, where ω is an exponential ultrafilter on natural numbers.

We also show that the Assouad-Nagata dimension of the discrete Heisenberg group equals its asymptotic dimension.

1. INTRODUCTION

Given a metric space (X, ρ_X) and given a sequence $d = \{d_n\}_{n=1}^\infty$ of real numbers diverging to infinity, one constructs an *asymptotic cone* $\operatorname{Cone}_\omega(X, c, d)$ for every non-principal ultrafilter ω in natural numbers $\mathbf{N}$ as follows:

First one considers the pseudo-metric space of all functions $x \colon N \to X$ such that $\{\frac{\rho_X(x(n), c(n))}{d_n}\}$ is bounded for a given function $c \colon N \to X$ (that function serves as the base point of the asymptotic cone). The distance from x to y is the ω-limit $\lim_\omega \frac{\rho_X(x(n), y(n))}{d_n}$ of $\{\frac{\rho_X(x(n), y(n))}{d_n}\}$ (see [2] or [13] for details). By identifying x and y whose distance is 0 one gets a metric space $\operatorname{Cone}_\omega(X, c, d)$.

One can generalize the definition of asymptotic cones for non-principal ultrafilters ω over any infinite set I (see [18]). In that case one requires that $d = \{d_i\}_{i \in I}$ has the property that the ω-limit of $\{\frac{1}{d_i}\}_{i \in I}$ is 0. As all our proofs are valid for that case, we will work in that generality.

An important problem in geometric group theory is relating properties of finitely generated groups G to topological properties of their asymptotic cones (i.e., asymptotic cones of their Cayley graphs):

- A finitely generated group is virtually Abelian if and only if its asymptotic cones are isometric to the space $\mathbf{R}^n$ [14], [22].
- A finitely generated group is virtually nilpotent if and only if its asymptotic cones are locally compact [14], [24], [12].

- A finitely generated group is hyperbolic if and only if its asymptotic cones are **R**-trees [15].
- A finitely generated group G is relatively hyperbolic with respect to the finitely generated subgroups $H_1, \ldots, H_n$ if and only if every asymptotic cone of G is tree-graded with respect to ω-limits of sequences of cosets of the subgroups H_i [13].

It would be of interest to detect the property of a finitely generated group G equivalent to finite-dimensionality of its asymptotic cones.

The first time the dimension of asymptotic cones was discussed occurred in [14], where M. Gromov proved finite-dimensionality of asymptotic cones of a finitely generated group of polynomial growth by demonstrating that their Hausdorff dimension is finite.

J. Burillo [8] showed 1-dimensionality of asymptotic cones of Baumslag-Solitar groups and used it to demonstrate that the fundamental group of those cones is not free and uncountable.

More recently, J. Behrstock and Y. Minsky [3] proved the following:

Theorem 1.1 (Dimension Theorem of Behrstock-Minsky). *The maximal topological dimension of a locally compact subset of the asymptotic cone of a mapping class group is equal to the maximal rank of an abelian subgroup.*

The aim of this note is to relate asymptotic Assouad-Nagata dimension of a finitely generated group G to the dimension of its asymptotic cones. More generally, we prove $\dim(\mathrm{Cone}_\omega(X, c, d)) \leq \mathrm{asdim}_{AN}(X)$ for all metric spaces (X, ρ) and all non-principal ultrafilters ω. Its direct predecessor is the following result from [10]:

Theorem 1.2 (Dranishnikov-Smith). $\dim(Y) \leq \mathrm{asdim}_{AN}(X)$ *for every separable subset $Y \subset \mathrm{Cone}_\omega(X)$ if X is a proper metric space and ω is an exponential ultrafilter in* **N**.

Notice that the asymptotic cones in Theorem 1.2 are based on the sequence $d_n = n$.

The way Theorem 1.2 was proved in [10] was by constructing an injective map $\mathrm{Cone}_\omega(X) \setminus [x_0] \to \nu_L(X)$, where $\nu_L(X)$ is the sublinear Higson corona of X (see Theorem 5.1 of [10]). Since $\mathrm{asdim}_{AN}(X) \geq \dim(\nu_L(X))$ for every proper metric space (X, d) (see Theorem 3.10 in [10]), Theorem 1.2 follows.

It is not clear if the methods of this paper allow us to improve Theorem 1.1 and show that the dimension of asymptotic cones of mapping class groups equals the maximal rank of an abelian subgroup. We learned from Jason Behrstock that in order to remove the 'locally compact' hypothesis in Theorem 1.1 all one needs to show is that the dimension of asymptotic cones of mapping class groups is finite. That leads to the question of Assouad-Nagata dimension of mapping class groups being finite. That question remains open. Even the finiteness of the asymptotic dimension of mapping class groups remains open (except in genus less than 3) (see [5] for an overview of asymptotic dimension).

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2. Assouad-Nagata dimension

A metric space (X, ρ) is of *Assouad-Nagata dimension* (see [19] and [1]) at most n (notation: $\dim_{AN}(X) \leq n$) if there is a constant $c > 0$ such that for any $r > 0$,

there is a cover $\mathcal{U}_r$ of X whose elements have diameter at most $c \cdot r$ and every r-ball $B(x, r)$ intersects at most $n + 1$ elements of $\mathcal{U}_r$. Equivalently (see [7]), there is a constant $K > 0$ such that for any $r > 0$, X can be expressed as $X = X_0 \cup \ldots \cup X_n$ so that all r-components of X_i are $K \cdot r$-bounded (two points are in the same r-component of a metric space if they can be connected by a finite chain of points so that each link is of size less than r).

If the existence of $\mathcal{U}_r$ is guaranteed only for sufficiently large r, then we say *the asymptotic Assouad-Nagata dimension* of X is at most n (notation: $\mathrm{asdim}_{AN}(X) \leq n$).

For connections of Assouad-Nagata dimension to Lipschitz extensions, see [17] and [6].

Of basic importance to us is the following result:

Theorem 2.1 (Lang-Schlichenmaier [17])**.** *If* $f \colon (X, \rho_X) \to (Y, \rho_Y)$ *is a quasi-symmetric embedding of metric spaces, then* $\dim_{AN}(X) \leq \dim_{AN}(Y)$.

We will apply Theorem 2.1 in the case of bi-Lipschitz homeomorphisms and snowflaking. Recall that a *snowflaked version* of a metric space (X, ρ) is the space (X, ρ^p) for some $0 < p < 1$ (see [16], p. 98).

(X, ρ) has the *n-dimensional Nagata property* if for every configuration $y_1, \ldots, y_{n+2}$ of $n + 2$ points in X the conditions $\rho(y_i, B(x, r/2)) < r$ for some $x \in X$ and all $i \leq n + 2$ imply the existence of $i \neq j$ such that $\rho(y_i, y_j) < r$.

Theorem 2.2. *Let* (X, ρ_X) *be a metric space and* $n \geq 0$. $\dim_{AN}(X) \leq n$ *if and only if* (X, ρ_X) *is bi-Lipschitz homeomorphic to a metric space* (Y, ρ_Y) *such that some snowflaked version of* (Y, ρ_Y) *has the n-dimensional Nagata property.*

Proof. In one direction it follows from Theorem 2.1 and the fact that $\dim_{AN}(Y) \leq n$ if Y has the *n*-dimensional Nagata property. Indeed, if $r > 0$ and A is a maximal subset of Y with respect to the property $\rho(x, y) \geq r$ for all $x \neq y \in A$, then the family of balls $\mathcal{U}_r = \{B(y, r)\}_{y \in A}$ covers Y and any ball $B(x, r/2)$ intersects at most $n + 1$ elements of $\mathcal{U}_r$ (see the proof of Theorem 4.2 in [11] along the same lines).

The other direction is proved in Proposition 2.2 of [1]. $\qquad \square$

3. Maps with metrically parallel fibers

J. Burillo [8] defined a very interesting class of maps, maps with metrically parallel fibers. Recall $f \colon (X, \rho_X) \to (Y, \rho_Y)$ is a map *with metrically parallel fibers* if for every $x \in X$ and $y \in f(X)$ there is $x' \in f^{-1}(y)$ such that $\rho_X(x, x') = \rho_Y(f(x), f(x')) = \rho_Y(f(x), y)$ (see Definition 6 in [8]). It seems maps with metrically parallel fibers provide a natural ground for applying Hurewicz type theorems from [7] (see the proof of Theorem 3.4).

Examples of maps with metrically parallel fibers.
   (1) The projection $X \times Y \to Y$ if the metric on $X \times Y$ is either Euclidean or the sum of corresponding metrics on X and Y.
   (2) $Im \colon \mathbf{H}^2 \to \mathbf{R}$, where $\mathbf{H}^2$ is the hyperbolic plane in the upper half plane model (see [8]). Notice $\mathbf{R}$ is identified with $(0, \infty)$ equipped with the metric $d(x, y) = |\ln(x) - \ln(y)|$.

We will weaken the condition of the existence of an x' such that $\rho_X(x, x') = \rho_Y(f(x), f(x')) = \rho_Y(f(x), y)$ to simply $\rho_X(x, f^{-1}(y)) \leq \rho_Y(f(x), y)$ for all $x \in X$ and $y \in f(X)$. One reason for the change is that the following natural functions easily satisfy the new condition.

(1) The function $x \to d(x, x_0)$ from any **R**-tree T to $\mathbf{R}_+$, where x_0 is a base point of T. We assume every point of T lies inside an infinite geodesic.
(2) Any epimorphism $f\colon (G, d_G) \to (H, d_H)$ of finitely generated groups equipped with word metrics induced by finite sets of generators S in G and $f(S)$ in H.

The second reason is the following.

Proposition 3.1. *If $f\colon (X, \rho_X) \to (Y, \rho_Y)$ is a Lipschitz map, then the following conditions are equivalent:*

a. *The natural function $f^{-1}(y) \to y$ establishes a bi-Lipschitz equivalence between $f(X)$ and the space of fibers of f equipped with the Hausdorff distance.*
b. *There is $\mu > 0$ such that $\rho_X(x, f^{-1}(y)) \leq \mu \cdot \rho_Y(f(x), y)$ for all $x \in X$ and $y \in f(X)$.*

Proof. a) $\implies$ b). Recall the Hausdorff distance $d_H(A, B)$ between two subsets A and B of X is the supremum of $\rho_X(a, B)$ and $\rho_X(b, A)$ over all $a \in A$ and $b \in B$. A surjective map $g\colon S \to T$ of metric spaces is a bi-Lipschitz equivalence if there are constants $\lambda, \mu > 0$ such that $\lambda \cdot d_T(g(a), g(b)) \leq d_S(a, b) \leq \mu \cdot d_T(g(a), g(b))$ for all $a, b \in S$. Thus b) follows easily from a) as $\rho_X(x, f^{-1}(y)) \leq d_H(f^{-1}(f(x)), f^{-1}(y)) \leq \mu \cdot \rho(f(x), y)$ for all $x \in X$ and $y \in f(X)$.

b) $\implies$ a). Assume f is λ-Lipschitz (which means $\rho(f(x), f(y)) \leq \lambda \cdot \rho_X(x, y)$ for all $x, y \in X$) and $y_1, y_2 \in f(X)$. Since $\rho_X(x, f^{-1}(y_2)) \leq \mu \cdot \rho(y_1, y_2)$ for all $x \in f^{-1}(y_1)$ and $\rho_X(x, f^{-1}(y_1)) \leq \mu \cdot \rho(y_1, y_2)$ for all $x \in f^{-1}(y_2)$, we get $d_H(f^{-1}(y_1), f^{-1}(y_2)) \leq \mu \cdot \rho_Y(y_1, y_2)$. Similarly, as f is λ-Lipschitz, one has $\lambda \cdot \rho_X(x, f^{-1}(y_2)) \geq \rho_Y(y_1, y_2)$ for all $x \in f^{-1}(y_1)$ resulting in $\frac{\rho_Y(y_1, y_2)}{\lambda} \leq d_H(f^{-1}(y_1), f^{-1}(y_2))$ for all $y_1, y_2 \in f(X)$. $\square$

Proposition 3.1 provides a global explanation for the term 'metrically parallel fibers'. Also, after rescaling Y one can simply achieve $\mu = 1$ in Proposition 3.1, so that is going to be our assumption.

It was suggested to us by K. Brodskiy that maps as in Proposition 3.1(b) ought to be called **metrically open** due to the following:

Proposition 3.2 (K. Brodskiy). *Let $f\colon X \to Y$ be a surjective map of metric spaces. The following properties are equivalent:*

1. *There exists $\mu > 0$ such that for any $x \in X$ and $y \in Y$,*
$$d_X(x, f^{-1}(y)) \leq \mu d_Y(f(x), y).$$
2. *There exists $\lambda > 0$ such that for any $x \in X$ and any $R > 0$,*
$$B(f(x), \lambda R) \subset f(B(x, R)).$$

Proof. In both directions assume the relation $\mu = 1/\lambda$.

1) $\implies$ 2). If $d_Y(f(x), f(z)) < \lambda R$ and $f(z) = y$, then $d_X(x, f^{-1}(y)) < R$, so there is $t \in f^{-1}(y)$ satisfying $d_X(x, t) < R$. Thus $y = f(t) \in f(B(x, R))$.

2) $\implies$ 1). Suppose $d_Y(f(x), y) < \lambda R$. That means $y \in B(f(x), \lambda R)$ and there is $z \in B(x, R)$ so that $y = f(z)$. Thus $d_X(x, f^{-1}(y)) < R$, which proves $d_X(x, f^{-1}(y)) \leq \mu d_Y(f(x), y)$. $\square$

J. Burillo proved in [8] (Theorem 16) that $\dim(X) \leq \dim(Y)$ if there is a map $f\colon X \to Y$ with parallel fibers such that $f^{-1}(y)$ is an ultrametric space for all $y \in Y$. Recall (A, ρ) is *ultrametric* if $\rho(x, z) \leq \max(\rho(x, y), \rho(y, z))$ for all triples x, y, z in A. Our analog of that result for Assouad-Nagata dimension has a much simpler proof (if Lipschitz maps are involved):

Theorem 3.3. *Suppose $f\colon (X, \rho_X) \to (Y, \rho_Y)$ is a Lipschitz map with ultrametric fibers. If*

$$\rho_X(x, f^{-1}(y)) \leq \rho_Y(f(x), y)$$

for all $x \in X$ and $y \in f(X)$, then $\dim_{AN}(X) \leq \dim_{AN}(Y)$.

Proof. Let f be λ-Lipschitz. Let us show that if $B \subset Y$ and all $\lambda \cdot r$-components of B are K-bounded, then all r-components of $f^{-1}(B)$ are $(4K+r)$-bounded. Indeed, if $x_1, \ldots, x_n$ is a chain of points such that $f(x_i) \in B$ and $\rho_X(x_i, x_{i+1}) < r$ for all $i < n$, then the $f(x_i)$ belong to the same $\lambda \cdot r$-component of B. Hence there is a fiber $F \subset f^{-1}(B)$ such that $\rho_X(x_i, F) < K$ for all $i \leq n$ and we can pick $y_i \in F$ satisfying $\rho_X(x_i, y_i) < K$ for all $i \leq n$. Thus all y_i belong to the same $(2K + r)$-component of F. As F is ultrametric, that component is of size less than $2K + r$. Therefore $\rho_X(x_1, x_n) \leq 2K + \rho_X(y_1, y_n) < 2K + 2K + r = 4K + r$.

Suppose $\dim_{AN}(Y) = m < \infty$ (the case $\dim_{AN}(Y) = \infty$ is trivial) and let $c > 0$ be the corresponding constant for Y. Given $r > 0$ pick a cover $\{Y_1, \ldots, Y_{m+1}\}$ of Y such that any $\lambda \cdot r$-component of each Y_i has diameter less than $c \cdot \lambda \cdot r$. Now r-components of each set $f^{-1}(Y_i)$ are bounded by $(4c\lambda + 1) \cdot r$, which proves $\dim_{AN}(X) \leq m$. □

Notice Theorem 3.3 implies the well-known fact (see [17]) $\dim_{AN}(T) \leq 1$ for any **R**-tree T using $f\colon T \to \mathbf{R}_+$, $f(x) = d(x, x_0)$.

Bell and Dranishnikov [4] introduced the concept of $\mathrm{asdim}\{A_s\}_{s \in S} \leq n$ uniformly with respect to $s \in S$. The analogous concept of $\dim_{AN}(A_s) \leq n$ *uniformly with respect to $s \in S$* means that the constant c in the definition of Assouad-Nagata dimension can be chosen the same for all A_s, $s \in S$. Geometrically, the most common occurrence is when all A_s are isometric.

Here is a generalization of Theorem 3.3:

Theorem 3.4. *Suppose that $f\colon (X, \rho_X) \to (Y, \rho_Y)$ satisfies $\rho_X(x, f^{-1}(y)) \leq \rho_Y(f(x), y)$ for all $x \in X$ and $y \in Y$. If f is Lipschitz and*

$$\dim_{AN}(f^{-1}(y)) \leq k$$

uniformly with respect to $y \in f(X)$, then $\dim_{AN}(X) \leq k + \dim_{AN}(Y)$.

Proof. The proof consists of applying Theorem 7.1 in [7]. That result says that $\dim_{AN}(X) \leq \dim_{AN}(f) + \dim_{AN}(Y)$ if f is Lipschitz. We need to show $\dim_{AN}(f) \leq k$, which means the existence of constants a and b such that for any $r_X, R_Y > 0$ and any subset A of X so that $f(A)$ is R_Y-bounded, A can be expressed as $A = A_0 \cup \ldots \cup A_k$ with r_X-components of each A_i being $(a \cdot r_X + b \cdot R_Y)$-bounded.

Assume f is λ-Lipschitz. Pick $y \in f(A)$ and put $F = f^{-1}(y)$. Cover F with sets $F_0, \ldots, F_k$ such that $(2R_Y + r_X)$-components of F_i are $c \cdot (2R_Y + r_X)$-bounded. Look at $A_i = B(F_i, R_Y)$ and notice its r_X-components are $(c(2R_Y + r_X) + 2R_Y)$-bounded. As the A_i cover A (use $\rho_X(x, F) \leq \rho_Y(f(x), f(F)) < R_Y$ for all $x \in A$), $a = c$ and $b = 2c + 2$ work. □

4. DIMENSION OF ASYMPTOTIC CONES

Proposition 4.1. *Let* (X, ρ_X) *be a metric space and* $n \geq 0$*. If* (X, ρ_X) *has the* n*-dimensional Nagata property, then every asymptotic cone over* X *has the* n*-dimensional Nagata property.*

Proof. Let's rephrase the n-dimensional Nagata property as follows: given $r > 0$ and points y_i, z_i, $i \leq n + 2$, such that $\rho_X(y_i, z_i) < r$ and $\rho_X(x, z_i) < r/2$ for some $x \in X$ and all $i \leq n + 2$, there is $i \neq j$ such that $\rho_X(y_i, y_j) < r$.

Suppose $(Y, \rho_Y) = \mathrm{Cone}_\omega(X, c, d)$ does not have the n-dimensional Nagata property and there exist points $x \in Y$, $y_i, z_i \in Y$ for $i \leq n + 2$ such that for some $r > \epsilon > 0$ the following inequalities hold:

    a. $\rho_Y(y_i, z_i) < r - \epsilon$ and $\rho_Y(x, z_i) < r/2 - \epsilon/2$ for all $i \leq n + 2$,
    b. $\rho_Y(y_i, y_j) \geq r$ for all $i \neq j$.

For each i pick $P_i \in \omega$ such that $\rho_X(y_i(n), z_i(n)) < (r - \epsilon) \cdot d_n$ and $\rho_X(z_i(n), x(n)) < (r/2 - \epsilon/2) \cdot d_n$ for all $n \in P_i$. Similarly, for all $i \neq j$, pick $P_{ij} \in \omega$ such that $\rho_X(y_i(n), y_j(n)) \geq (r - \epsilon) \cdot d_n$ for all $n \in P_{ij}$. Let P be the intersection of all P_i and all P_{ij}. Since $P \neq \emptyset$, we arrive at a contradiction: (X, ρ_X) cannot have the n-dimensional Nagata property. $\qquad\square$

Corollary 4.2. *If* (X, ρ_X) *is ultrametric, then every asymptotic cone* $\mathrm{Cone}_\omega(X, c, d)$ *is ultrametric.*

Proof. It follows from the well-known fact that being ultrametric is the same as having the 0-dimensional Nagata property (easy exercise left to the reader). $\qquad\square$

Corollary 4.3. *For any metric space* (X, ρ_X)*,*

$$\dim(\mathrm{Cone}_\omega(X, c, d)) \leq \dim_{AN}(\mathrm{Cone}_\omega(X, c, d)) \leq \mathrm{asdim}_{AN}(X, \rho_X).$$

Proof. Suppose $\mathrm{asdim}_{AN}(X, \rho_X) = n$. Notice $\mathrm{Cone}_\omega(X, \rho_X, d)$ is isometric to $\mathrm{Cone}_\omega(X, \rho_2, d)$, where $\rho_2(x, y) = \rho_X(x, y)$ if $\rho_X(x, y) \geq 1$ and $\rho_2(x, y) = 1$ if $0 < \rho_X(x, y) < 1$. Also, $\dim_{AN}(X, \rho_2) = n$ as $\mathrm{asdim}_{AN}(X, \rho_2) = n$ and the microscopic Assouad-Nagata dimension of (X, ρ_2) is 0 (see [6]). By Theorem 2.2 there is a metric space (Y, ρ_Y) bi-Lipschitz equivalent to (X, ρ_2) such that (Y, ρ_Y^p) has the n-dimensional Nagata property for some $0 < p < 1$. Therefore $\mathrm{Cone}_\omega(Y, \rho_Y, d)$ is bi-Lipschitz equivalent to $\mathrm{Cone}_\omega(X, \rho_X, d)$. Since the snowflaked version $\mathrm{Cone}_\omega(Y, \rho_Y^p, d^p)$ of $\mathrm{Cone}_\omega(Y, \rho_Y, d)$ has the n-dimensional Nagata property by Proposition 4.1, we conclude

$$\dim_{AN}(\mathrm{Cone}_\omega(X, \rho_X, d)) \leq n.$$

As $\dim(Y) \leq \dim_{AN}(Y)$ for all metric spaces (see [17], Theorem 2.2), the proof is completed. $\qquad\square$

J. Burillo [8] showed that asymptotic cones of Baumslag-Solitar groups $BS_{p,q}$ (groups with generators t, x and the sole relation $t^{-1}x^p t = x^q$ where $p \neq q$) have dimension 1 by displaying a map f from the Cayley graph of $BS_{p,q}$ to an **R**-tree T with metrically parallel fibers such that the fibers of the induced maps on asymptotic cones are ultrametric. In view of Corollary 4.3 and Theorem 3.3 one can improve his result by stating that asymptotic cones of $BS_{p,q}$ are of Assouad-Nagata dimension 1.

Question 4.4. Is there a finitely generated group whose asymptotic cones have bounded Assouad-Nagata dimension but the group itself has infinite asymptotic Assouad-Nagata dimension?

Question 4.5. Is there (X, ρ) of positive asymptotic Assouad-Nagata dimension whose asymptotic cones are of Assouad-Nagata dimension at most 0?

5. Assouad-Nagata dimension of the discrete Heisenberg group

We do not know of a finitely presented group G of finite asymptotic dimension whose asymptotic Assouad-Nagata dimension is larger but finite. Piotr Nowak [20] identified finitely generated groups G_n of asymptotic dimension $n \geq 1$ and $\mathrm{asdim}_{AN}(G_n) = \infty$. A.N. Dranishnikov communicated to us that there was a suggestion by John Roe that perhaps the Heisenberg group (see [23] for its discussion) may provide an example where the dimensions differ. However, that is not the case.[1]

Proposition 5.1. *The asymptotic dimension and Assouad-Nagata dimension of the discrete Heisenberg group $H_3(\mathbf{Z})$ both equal 3.*

Proof. Recall (see [15], p. 51) that the continuous Heisenberg group $H_3(\mathbf{R})$ is the only non-Abelian nilpotent three-dimensional group. It contains, as a cocompact lattice, the discrete Heisenberg group $H_3(\mathbf{Z})$ generated by a, b, and c with the relations $c = [a, b]$, $[a, c] = 1$, and $[b, c] = 1$. Since $G = H_3(\mathbf{Z})$ is of Hirsch length 3, its asymptotic dimension is 3 (see Theorem 3.5 in [9]). To observe $\mathrm{asdim}_{AN}(G) \leq 3$ consider the exact sequence $1 \to \mathbf{Z} \to G \to G/\mathbf{Z} \to 1$, where $\mathbf{Z}$ is generated by c. As $G/\mathbf{Z} = \mathbf{Z} \oplus \mathbf{Z}$, it suffices to show $(\mathbf{Z}, d)$ is quasi-symmetrically equivalent to $\mathbf{Z}$ equipped with the regular metric, where d is a metric on $\mathbf{Z}$ induced from a word metric on G. Indeed, in view of Theorem 2.1 one gets $\mathrm{asdim}_{AN}(\mathbf{Z}, d) = 1$, and applying Theorem 3.4 one derives $3 = \mathrm{asdim}(G) \leq \mathrm{asdim}_{AN}(G) \leq 3$.

As suggested in [15] on p. 52 the metric d is quasi-isometric to the square root of the regular metric on $\mathbf{Z}$ (the equality $[a^n, b^n] = c^{n^2}$ is cited). Since that statement is not totally obvious, let us provide a detailed exposition of the following:

Claim 1. Let $K > 16$ be sufficiently large so that $4(n+1) + K \cdot \sqrt{n} \leq K \cdot n$ for all $n \geq 2$ (equivalently, $K \geq \frac{4(n+1)}{n - \sqrt{n}}$ for all $n \geq 2$). For every integer k there is a word w in a and b (including inverses) such that $w = c^k$ in G and $l(w) \leq K\sqrt{|k|}$, where $l(w)$ is the length of w.

Proof of Claim 1. We will use equality $[a^u, b^v] = c^{u \cdot v}$, which we leave to the reader to prove. It suffices to consider k positive and the proof is by induction on k. If $k = n^2$ for some natural n, then we put $w = [a^n, b^n]$ whose length is $4 \cdot n$. For $k = 2$ one takes $w = [a, b]^2$. Similarly, $k = 3$ is handled. If there is natural $n \geq 2$ such that $n^2 < k \leq n^2 + n$, then we pick a word w' for $k' = k - n^2$ and we put $w = [a^n, b^n] \cdot w'$. Notice $l(w) \leq 4n + l(w') \leq 4n + K\sqrt{k - n^2} \leq Kn \leq K\sqrt{k}$. Otherwise there is $n \geq 2$ such that $(n+1)^2 - n \leq k < (n+1)^2$, in which case we put $k' = k - (n+1)^2$ and $w = [a^{n+1}, b^{n+1}] \cdot w'$, where $w' = c^{k'}$ and $l(w') \leq K\sqrt{n}$. $\qquad\square$

Claim 2. If w is a word in a and b (including inverses) such that $w = c^k$ in G for some k, then $2\sqrt{|k|} \leq l(w)$.

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[1]When the paper was written we learned from Dranishnikov that he and P. Nowak proved this fact independently.

Proof of Claim 2. By contradiction. Find w of minimal length such that $w = c^k$ and $l(w) < 2\sqrt{|k|}$. Among such words (minimizing $l(w)$) choose a word maximizing $|k|$ (that maximum cannot be infinity as there are only finitely many words of a given length). Notice $l(w) > 4$. Since c is in the center of G, equality $x \cdot y = c^k$ implies $y \cdot x = c^k$. Therefore (via a cyclic permutation) we may assume $w = a^{m(1)} \cdot b^{n(1)} \cdot \ldots \cdot a^{m(s)} \cdot b^{n(s)}$, where all exponents are non-zero and $l(w) = \sum(|m(i)| + |n(i)|)$. Notice $\sum m(i) = \sum n(i) = 0$. Also, we may assume $k > 0$.

The first observation is $m(i) \cdot n(i) > 0$ for all i. Indeed, if $m(i) \cdot n(i) < 0$ for some i we may assume $i = s$ without loss of generality and observe $a^{m(s)} b^{n(s)} = b^{n(s)} a^{m(s)} c^{m(s)n(s)}$, so one can find either a shorter word w' or larger $k' = k - m(s)n(s)$ satisfying $w' = c^{k'}$, a contradiction.

In the same way, the sign of $m(i)$ cannot be the same as that of $m(i+1)$ for any i. Indeed $b^{n(i)} a^{m(i+1)} = c^{-n(i)m(i+1)} a^{m(i+1)} b^{n(i)}$ and as above we get a contradiction.

If one looks for the smallest absolute value of the exponents $m(i)$ and $n(i)$, then (up to cyclic permutation) one can express w as either $x(a^u \cdot b^u \cdot a^{-u})y$ or $x(b^u \cdot a^{-u} \cdot b^{-u})y$ with $l(w) = l(x) + l(y) + 3|u|$. In the first case one has $a^u \cdot b^u \cdot a^{-u} = c^{u^2} b^u$, so $x \cdot b^u \cdot y = c^{k-u^2}$. By induction $l(w) - 2|u| \geq 2\sqrt{|k - u^2|}$, so $l(w) \geq 2|u| + 2\sqrt{|k - u^2|} \geq 2\sqrt{k}$, a contradiction. The second case is similar. $\square$

Now $d(c^m, c^n) = d(c^{m-n}, 1) \geq 2\sqrt{|m-n|}$ by Claim 2, and $d(c^m, c^n) \leq K\sqrt{|m-n|}$ by Claim 1. That means $(\mathbf{Z}, d)$ is bi-Lipschitz equivalent to $(\mathbf{Z}, \sqrt{|m-n|})$. Since $(\mathbf{Z}, \sqrt{|m-n|})$ is quasi-symmetrically equivalent to $(\mathbf{Z}, |m-n|)$, we are done. $\square$

It is not clear if using methods of [21] one can generalize Proposition 5.1 and prove $\mathrm{asdim}(G) = \mathrm{asdim}_{AN}(G)$ for any virtually nilpotent group G.

Question 5.2. Is there a finitely generated group of Assouad-Nagata dimension 1 that is not finitely presented?

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