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ESTIMATES OF GROMOV'S BOX DISTANCE

KEI FUNANO

(Communicated by Jon G. Wolfson)

This paper is dedicated to our advisors.

Abstract. In 1999, M. Gromov introduced the box distance function $\underline{\square}_\lambda$ on the space of all mm-spaces. In this paper, by using the method of T. H. Colding, we estimate $\underline{\square}_\lambda(\mathbb{S}^n, \mathbb{S}^m)$ and $\underline{\square}_\lambda(\mathbb{C}P^n, \mathbb{C}P^m)$, where $\mathbb{S}^n$ is the n-dimensional unit sphere in $\mathbb{R}^{n+1}$ and $\mathbb{C}P^n$ is the n-dimensional complex projective space equipped with the Fubini-Study metric. In particular, we give the complete answer to an exercise of Gromov's green book. We also estimate $\underline{\square}_\lambda\left(SO(n), SO(m)\right)$ from below, where $SO(n)$ is the special orthogonal group.

1. Introduction

In 1999, M. Gromov developed the theory of mm-spaces in [4, Chapter $3\frac{1}{2}_+$] by introducing two distance functions, called the *box distance function* $\underline{\square}_\lambda$ and the *observable distance function* $\underline{H}_\lambda\mathcal{L}\iota_1$, on the space $\mathcal{X}$ of all isomorphic classes of mm-spaces. Here, an *mm-space* is a triple (X, d_X, μ_X), where d_X is a complete separable metric on a set X and μ_X a finite Borel measure on (X, d_X). The notion of the distance function $\underline{\square}_\lambda$ is considered as a natural extension of the Gromov-Hausdorff distance function to the space $\mathcal{X}$. On the other hand, the notion of the distance function $\underline{H}_\lambda\mathcal{L}\iota_1$ is related to measure concentration. Roughly speaking, "measure concentration" amounts to saying that the push-forward measures $f_{n*}(\mu_n)$ on $\mathbb{R}$ concentrate to a point for any sequence of 1-Lipschitz functions $f_n : (X_n, d_n, \mu_n) \to \mathbb{R}$. For instance, the unit spheres in Euclidean spaces $\{\mathbb{S}^n\}_{n=1}^\infty$, the complex projective spaces $\{\mathbb{C}P^n\}_{n=1}^\infty$ equipped with the Fubini-Study metrics, and the special orthogonal groups $\{SO(n)\}_{n=1}^\infty$ have this property (we suppose that each space is equipped with its Riemannian volume measure normalized to have total volume 1). Gromov defined the distance $\underline{H}_\lambda\mathcal{L}\iota_1(X, Y)$ by using the Hausdorff distance between the space of 1-Lipschitz functions on X and that on Y, and showed that a sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces concentrates if and only if the sequence $\{X_n\}_{n=1}^\infty$ converges to a one-point space with respect to the distance function $\underline{H}_\lambda\mathcal{L}\iota_1$.

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The topology on $\mathcal{X}$ determined by $\underline{\square}_\lambda$ is strictly stronger than that of $\underline{H}_\lambda\mathcal{L}\iota_1$. In fact, the sequences $\{\mathbb{S}^n\}_{n=1}^\infty$, $\{\mathbb{C}P^n\}_{n=1}^\infty$, and $\{SO(n)\}_{n=1}^\infty$ are all divergent with respect to the distance $\underline{\square}_\lambda$ (see Proposition 3.1). This is related to the following exercise in Gromov's book:

Exercise (cf. [4, Section $3\frac{1}{2}$.18])**.** Estimate the distance $\underline{\square}_\lambda(\mathbb{S}^n, \mathbb{S}^m)$.

To solve the exercise, applying a method of [1, Lemma 5.10], we will estimate $\underline{\square}_\lambda(M, N)$ from below for compact Riemannian manifolds M and N with positive Ricci curvatures and the volume measures satisfying a homogeneity condition (see Lemma 3.4). As a result, we get the following proposition:

Proposition 1.1. *Assume that two sequences $\{n_k\}_{k=1}^\infty$, $\{m_k\}_{k=1}^\infty$ of natural numbers satisfy $n_k \leq C_1 k, m_k \leq C_2 k$ and $|n_k - m_k| \geq C_3 k$, $k = 1, 2, \cdots$, for some positive constants C_1, C_2, C_3. Then, we have*

$$\liminf_{k\to\infty} \underline{\square}_1(\mathbb{S}^{n_k}, \mathbb{S}^{m_k}), \liminf_{k\to\infty} \underline{\square}_1(\mathbb{C}P^{n_k}, \mathbb{C}P^{m_k}) \geq \min\left\{2^{-\frac{C_1}{C_3}}\pi^{-\frac{C_2}{C_3}}, 2^{-\frac{C_2}{C_3}}\pi^{-\frac{C_1}{C_3}}\right\}.$$

In particular, if in addition $|n_k - m_k| \geq C_4 k^\alpha$, $k = 1, 2, \cdots$, holds for some constant $C_4 > 0$ and a number $\alpha > 1$, then we have

$$\lim_{k\to\infty} \underline{\square}_1(\mathbb{S}^{n_k}, \mathbb{S}^{m_k}), \lim_{k\to\infty} \underline{\square}_1(\mathbb{C}P^{n_k}, \mathbb{C}P^{m_k}) = 1.$$

Note that $\mathrm{diam}(\mathcal{X}_1, \underline{\square}_1) = 1$, where $\mathcal{X}_1$ is the space of all mm-spaces with Borel probability measures.

We estimate $\underline{\square}_\lambda\big(SO(n), SO(m)\big)$ from below by the difference of their diameters (see Lemma 3.8). Consequently, we obtain the following proposition:

Proposition 1.2. *Assume that two sequences $\{n_k\}_{k=1}^\infty$, $\{m_k\}_{k=1}^\infty$ of natural numbers satisfy $n_k \leq C_1 k$, $m_k \leq C_2 k$ and $|n_k - m_k| \geq C_3\sqrt{k}$, $k = 1, 2, \cdots$, for some positive constants C_1, C_2, C_3. Then, we have*

$$\liminf_{k\to\infty} \underline{\square}_1\big(SO(n_k), SO(m_k)\big) \geq \min\left\{\frac{1}{2}, \frac{C_3}{\sqrt{C_1} + \sqrt{C_2}}\right\}.$$

In particular, if in addition $|n_k - m_k| \geq C_4 k^\alpha$, $k = 1, 2, \cdots$, holds for some constant $C_4 > 0$ and a number $\alpha > 1/2$, then we have

$$\liminf_{k\to\infty} \underline{\square}_1\big(SO(n_k), SO(m_k)\big) \geq \frac{1}{2}.$$

As it is related to the above Gromov's exercise, we also prove the following proposition. This proposition is also mentioned by Gromov in [4, Section $3\frac{1}{2}$.3, Exercise (e)].

Proposition 1.3. *We have*

$$\underline{\square}_1(\mathbb{S}^n, \mathbb{S}^{n-1}), \underline{\square}_1(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \to 0$$

as $n \to \infty$.

2. Preliminaries

2.1. Definition of Gromov's box distance function $\underline{\square}_\lambda$.

Definition 2.1. Let $\lambda \geq 0$ and (X, μ) be a measure space with $\mu(X) < +\infty$. For two maps $d_1, d_2 : X \times X \to \mathbb{R}$, we define a number $\square_\lambda(d_1, d_2)$ as the infimum of $\varepsilon > 0$ such that there exists a measurable subset $T_\varepsilon \subseteq X$ of measure at least $\mu(X) - \lambda\varepsilon$ satisfying $|\, d_1(x, y) - d_2(x, y)| \leq \varepsilon$ for any $x, y \in T_\varepsilon$.

It is easy to see that this is a distance function on the set of all functions on $X \times X$, and the two distance functions $\square_\lambda$ and $\square_{\lambda'}$ are equivalent to each other for any $\lambda, \lambda' > 0$.

Definition 2.2 (parameter)**.** Let X be an mm-space and $\mu(X) = m$. Then, there exists a Borel measurable map $\varphi : [0, m] \to X$ with $\varphi_*(\mathcal{L}) = \mu$, where $\mathcal{L}$ stands for the Lebesgue measure on $[0, m]$. We call φ a *parameter* of X.

Note that if the support of X is not one-point, then its parameter is not unique.

Definition 2.3 (Gromov's box distance function)**.** If two mm-spaces X, Y satisfy $\mu_X(X) = \mu_Y(Y) = m$, we define

$$\underline{\square}_\lambda(X, Y) := \inf \square_\lambda(\varphi_X^* \, d_X, \varphi_Y^* \, d_Y),$$

where the infimum is taken over all parameters $\varphi_X : [0, m] \to X,\ \varphi_Y : [0, m] \to Y$, and $\varphi_X^* \, d_X$ is defined by $\varphi_X^* \, d_X(s, t) := d_X(\varphi_X(s), \varphi_X(t))$ for $s, t \in [0, m]$. If $\mu_X(X) < \mu_Y(Y)$, putting $m := \mu_X(X), m' := \mu_Y(Y)$, we define

$$\underline{\square}_\lambda(X, Y) := \underline{\square}_\lambda \left(X, \frac{m}{m'} Y\right) + m' - m,$$

where $(m/m')Y := (Y, d_Y, (m/m')\mu_Y)$.

We recall that two mm-spaces are *isomorphic* to each other if there is a measure preserving isometry between the supports of their measures. $\underline{\square}_\lambda$ is a distance function on $\mathcal{X}$ for any $\lambda \geq 0$. See [2, Sections 1, 3] for a complete proof of that. Note that the distance functions $\underline{\square}_\lambda$ and $\underline{\square}_{\lambda'}$ are equivalent to each other for distinct $\lambda, \lambda' > 0$.

2.2. **Definition of observable distance functions $\underline{H}_\lambda \mathcal{L}\iota_1$.** For a measure space (X, μ) with $\mu(X) < +\infty$, we denote by $\mathcal{F}(X, \mathbb{R})$ the space of all functions on X. Given $\lambda \geq 0$ and $f, g \in \mathcal{F}(X, \mathbb{R})$, we put

$$\mathrm{me}_\lambda(f, g) := \inf\{\varepsilon > 0 \mid \mu\big(\{x \in X \mid |f(x) - g(x)| \geq \varepsilon\}\big) \leq \lambda\varepsilon\}.$$

Note that this me_λ is a distance function on $\mathcal{F}(X, \mathbb{R})$ for any $\lambda \geq 0$, and its topology on $\mathcal{F}(X, \mathbb{R})$ coincides with the topology of the convergence in measure for any $\lambda > 0$. Also, the distance functions me_λ for all $\lambda > 0$ are mutually equivalent.

We recall that the *Hausdorff distance* between two closed subsets A and B in a metric space X is defined by

$$d_H(A, B) := \inf\{\varepsilon > 0 \mid A \subseteq B_\varepsilon, B \subseteq A_\varepsilon\},$$

where A_ε is a closed ε-neighborhood of A.

Let (X, μ) be a measure space with $\mu(X) < +\infty$. For a semi-distance d on X, we indicate by $\mathcal{L}ip_1(d)$ the space of all 1-Lipschitz functions on X with respect to d. Note that $\mathcal{L}ip_1(d)$ is a closed subset in $(\mathcal{F}(X, \mathbb{R}), \mathrm{me}_\lambda)$ for any $\lambda \geq 0$.

Definition 2.4. For $\lambda \geq 0$ and two semi-distance functions d, d' on X, we define

$$H_\lambda \mathcal{L}\iota_1(d, d') := d_H\left(\mathcal{L}ip_1(d), \mathcal{L}ip_1(d')\right),$$

where d_H stands for the Hausdorff distance function in $(\mathcal{F}(X, \mathbb{R}), \mathrm{me}_\lambda)$.

This $H_\lambda \mathcal{L}\iota_1$ is actually a distance function on the space of all semi-distance functions on X for all $\lambda \geq 0$, and the two distance functions $H_\lambda \mathcal{L}\iota_1$ and $H_{\lambda'} \mathcal{L}\iota_1$ are equivalent to each other for any $\lambda, \lambda' > 0$.

Lemma 2.5. *For any two semi-distance functions d, d' on X, we have*

$$H_\lambda \mathcal{L}\iota_1(d, d') \leq \square_\lambda(d, d').$$

Proof. For any $\varepsilon > 0$ with $\square_\lambda(X, Y) < \varepsilon$, there exists a measurable subset $T_\varepsilon \subseteq X$ such that $\mu(X \setminus T_\varepsilon) \leq \lambda\varepsilon$ and $|\, d(x, y) - d'(x, y)| \leq \varepsilon$ for any $x, y \in T_\varepsilon$. Given arbitrary $f \in \mathcal{L}ip_1(d)$, we define $\widetilde{f} \in \mathcal{F}(X, \mathbb{R})$ by $\widetilde{f}(x) := \inf\{f(y) + d'(x, y) \mid y \in T_\varepsilon\}$. We easily see that $\widetilde{f} \in \mathcal{L}ip_1(d')$ and $\widetilde{f}(x) \leq f(x)$ for any $x \in T_\varepsilon$. Taking any $x \in T_\varepsilon$, we have

$$\begin{aligned} |f(x) - \widetilde{f}(x)| &= f(x) - \widetilde{f}(x) \\ &= \sup\{f(x) - f(y) - d'(x, y) \mid y \in T_\varepsilon\} \\ &\leq \sup\{d(x, y) - d'(x, y) \mid y \in T_\varepsilon\} \\ &\leq \varepsilon. \end{aligned}$$

Therefore, we get $\mathrm{me}_\lambda(f, \widetilde{f}) \leq \varepsilon$, which implies $\mathcal{L}ip_1(d) \subseteq \big(\mathcal{L}ip_1(d')\big)_\varepsilon$. Similarly, we also have $\mathcal{L}ip_1(d') \subseteq \big(\mathcal{L}ip_1(d)\big)_\varepsilon$, which yields $H_\lambda \mathcal{L}\iota_1(d, d') \leq \varepsilon$. This completes the proof. $\square$

Definition 2.6 (Observable distance function)**.** If two mm-spaces X, Y satisfy $\mu_X(X) = \mu_Y(Y) = m$, we define

$$\underline{H}_\lambda \mathcal{L}\iota_1(X, Y) := \inf H_\lambda \mathcal{L}\iota_1(\varphi_X^* \, d_X, \varphi_Y^* \, d_Y),$$

where the infimum is taken over all parameters $\varphi_X : [0, m] \to X,\ \varphi_Y : [0.m] \to Y$. If $\mu_X(X) < \mu_Y(Y)$, putting $m := \mu_X(X), m' := \mu_Y(Y)$, we define

$$\underline{H}_\lambda \mathcal{L}\iota_1(X, Y) := \underline{H}_\lambda \mathcal{L}\iota_1 \left(X, \frac{m}{m'} Y\right) + m' - m.$$

Hence, $\underline{H}_\lambda \mathcal{L}\iota_1$ is a distance function on $\mathcal{X}$ for any $\lambda \geq 0$. See [2, Section 3] for a complete proof. Note that the distance functions $\underline{H}_\lambda \mathcal{L}\iota_1$ and $\underline{H}_{\lambda'} \mathcal{L}\iota_1$ are equivalent to each other for any $\lambda, \lambda' > 0$.

For a Borel measure ν on $\mathbb{R}$ with $m := \nu(\mathbb{R}) < +\infty$ and $\kappa > 0$, we put

$$\mathrm{diam}(\nu, m - \kappa) := \inf\{\mathrm{diam}\, Y \mid Y \subseteq \mathbb{R} \text{ is a Borel subset such that } \nu_Y(Y) \geq m - \kappa\},$$

and call it the *partial diameter* on ν.

Definition 2.7 (Observable diameter)**.** Let (X, d, μ) be an mm-space and let $m := \mu(X)$. For any $\kappa > 0$ we define the *observable diameter* of X by

$$\begin{aligned} &\mathrm{diam}(X \xrightarrow{\mathrm{Lip}_1} \mathbb{R}, m - \kappa) \\ &\qquad := \sup\{\mathrm{diam}(f_*(\mu), m - \kappa) \mid f : X \to \mathbb{R} \text{ is a 1-Lipschitz function}\}. \end{aligned}$$

The idea of the observable diameter comes from quantum and statistical mechanics; that is, we think of μ as a state on a configuration space X and f is interpreted as an observable. We define a sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces as a *Lévy family* if $\mathrm{diam}(X_n \xrightarrow{\mathrm{Lip}_1} \mathbb{R}, m_n - \kappa) \to 0$ as $n \to \infty$ for any $\kappa > 0$, where m_n is the total measure of the mm-space X_n. This is equivalent to the fact that for any $\varepsilon > 0$ and any sequence $\{f_n : X_n \to \mathbb{R}\}_{n=1}^\infty$ of 1-Lipschitz functions, we have

$$(\diamondsuit) \qquad \mu_n(\{x \in X_n \mid |f_n(x) - m_{f_n}| \geq \varepsilon\}) \to 0 \text{ as } n \to \infty,$$

where m_{f_n} is some constant determined by f_n.

For a compact connected Riemannian manifold M, we denote by μ_M the Riemannian volume measure of M normalized as $\mu_M(M) = 1$ and by d_M the Riemannian distance of M. We shall consider M as an mm-space (M, d_M, μ_M).

Example 2.8. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of compact connected Riemannian manifolds and assume that $Ric_{M_n} \geq \kappa_n \to +\infty$ as $n \to \infty$. Then, by virtue of Lévy-Gromov's isoperimetric inequality, the sequence $\{M_n\}_{n=1}^{\infty}$ is a Lévy family (cf. [5, Section 1, Remark 2]). For example, $\{\mathbb{S}^n\}_{n=1}^{\infty}$ and $\{\mathbb{C}P^n\}_{n=1}^{\infty}$ are Lévy families. Recall that the Fubini-Study metric on $\mathbb{C}P^n$ is the unique Riemannian metric on $\mathbb{C}P^n$ such that the canonical projection $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$ is a Riemannian submersion. Since $Ric_{SO(n)} \geq (n-1)/4$, the sequence $\{SO(n)\}_{n=1}^{\infty}$ is a Lévy family. Since the distance function induced from the Hilbert-Schmidt norm on $SO(n)$ is not greater than that of the Riemannian distance function, $\{SO(n)\}_{n=1}^{\infty}$ is also a Lévy family with respect to the Hilbert-Schmidt norms.

Example 2.9 (Hamming cube)**.** Let μ_n be the normalized counting measure on $\{0,1\}^n$ and d_n be the *Hamming distance function* on $\{0,1\}^n$; that is,

$$d_n\left((x_i)_{i=1}^n, (y_i)_{i=1}^n\right) := \frac{1}{n}\,\mathrm{Card}\left(\{i \in \{1, \cdots, n\} \mid x_i \neq y_i\}\right).$$

The mm-space $\{0,1\}^n$ is called the *Hamming cube*. The sequence $\left\{\{0,1\}^n\right\}_{n=1}^{\infty}$ is a Lévy family (cf. [4, Section $3\frac{1}{2}$.42]).

Gromov showed the following proposition by considering a constant m_{f_n} in ($\diamondsuit$) as a Lipschitz function from a one-point space $\{*_n\}$ with total measure $\mu_n(X_n)$.

Proposition 2.10 (cf. [4, Section $3\frac{1}{2}$.45])**.** *A sequence $\{X_n\}_{n=1}^{\infty}$ of mm-spaces is a Lévy family if and only if $\underline{H}_\lambda\mathcal{L}\iota_1(X_n, \{*_n\}) \to 0$ as $n \to \infty$ for any $\lambda > 0$.*

3. Estimates of Gromov's box distance function

Let X be a metric space. Denote by $B_X(x, r)$ the closed ball in X centered at $x \in X$ with radius $r > 0$. A Borel measure μ on X is said to be *uniformly distributed* if

$$0 < \mu\big(B_X(x, r)\big) = \mu\big(B_X(y, r)\big) < +\infty$$

for any $r > 0$ and $x, y \in X$.

From Lemma 2.5, we see that the topology on $\mathcal{X}$ determined by $\underline{\square}_\lambda$ is not weaker than that of $\underline{H}_\lambda\mathcal{L}\iota_1$ for any $\lambda \geq 0$. For a Borel measure μ on a metric space, we denote by $\mathrm{Supp}\,\mu$ its support.

Proposition 3.1. *Let $\{(X_n, d_n, \mu_n)\}_{n=1}^{\infty}$ be a Lévy family such that μ_n is a uniformly distributed probability measure satisfying $X_n = \mathrm{Supp}\,\mu_n$ and $\inf_{n\in\mathbb{N}} \mathrm{diam}\, X_n > 0$. Then, the sequence $\{X_n\}_{n=1}^{\infty}$ does not converge with respect to the distance function $\underline{\square}_\lambda$ for any $\lambda \geq 0$.*

Proof. Suppose that $\{X_n\}_{n=1}^{\infty}$ converges and let X be its limit. Since $\{X_n\}_{n=1}^{\infty}$ is a Lévy family, by using Proposition 2.10, X must be a one-point space. Fix $\varepsilon > 0$ with $\varepsilon < \min\{3, \inf_{n\in\mathbb{N}} \mathrm{diam}\, X_n\}/3$. For any sufficiently large $n \in \mathbb{N}$, there exist a parameter $\varphi_n : [0,1] \to X_n$ of X_n and a Borel subset $T_n \subseteq [0,1]$ such that $\mathcal{L}(T_n) > 1 - \varepsilon/2$ and $d_n\left(\varphi_n(s), \varphi_n(t)\right) < \varepsilon/2$ for any $s, t \in T_n$. Fix a point $t_n \in T_n$.

There exists a point $x_n \in X_n$ such that $d_n\big(\varphi_n(t_n), x_n\big) \geq \operatorname{diam} X_n/3 > \varepsilon$, and hence $B_{X_n}(\varphi_n(t_n), \varepsilon/2) \cap B_{X_n}(x_n, \varepsilon/2) = \emptyset$. Therefore, we get

$$\begin{aligned} 1 &\geq \mu_n\big(B_{X_n}(\varphi_n(t_n), \varepsilon/2) \cup B_{X_n}(x_n, \varepsilon/2)\big) \\ &= 2\mu_n\big(B_{X_n}(\varphi_n(t_n), \varepsilon/2)\big) \\ &= 2\mathcal{L}\big(\varphi_n^{-1}\big(B_{X_n}(\varphi_n(t_n), \varepsilon/2)\big)\big) \geq 2\mathcal{L}(T_n) \geq 2 - \varepsilon > 1, \end{aligned}$$

which gives a contradiction. This completes the proof. □

From Proposition 3.1, we see that many Lévy families such as $\{\mathbb{S}^n\}_{n=1}^{\infty}$, $\{\mathbb{C}P^n\}_{n=1}^{\infty}$, $\{SO(n)\}_{n=1}^{\infty}$, and $\big\{\{0,1\}^n\big\}_{n=1}^{\infty}$ have no convergent subsequences with respect to the distance function $\underline{\square}_\lambda$. Therefore, the distance function $\underline{\square}_\lambda$ determines the topology on $\mathcal{X}$ strictly stronger than that of the distance function $\underline{H}_\lambda \mathcal{L}\iota_1$ for any $\lambda > 0$. However, since the proof of Proposition 3.1 is by contradiction, we do not estimate $\underline{\square}_\lambda(X_n, X_m)$ from below for $n, m \in \mathbb{N}$.

The proof of the following lemma is an analogue of the proof of [1, Lemma 5.10].

Lemma 3.2. *Let $(X, d_X, \mu_X), (Y, d_Y, \mu_Y)$ be mm-spaces and assume that μ_X, μ_Y are uniformly distributed Borel probability measures. Denote by $v_X(r)$ (respectively, $v_Y(r)$) the measure of a closed ball of X (respectively, Y) with radius $r > 0$ and assume that $v_X(a + c) \leq (1 - c)v_Y(a/2)$ for some $a, c > 0$ with $c < 1$. Then, we have $\underline{\square}_1(X, Y) \geq c$.*

Proof. Let us prove the lemma by contradiction. Suppose that $\underline{\square}_1(X, Y) < c$. Then, there exist a compact subset $T \subseteq [0, 1]$ and two parameters $\varphi_X : [0, 1] \to X$, $\varphi_Y : [0, 1] \to Y$ such that

(1) $\mathcal{L}(T) > 1 - c$,
(2) $\varphi_X|_T : T \to X$, $\varphi_Y|_T : T \to Y$ are continuous,
(3) $|\, d_X\big(\varphi_X(s), \varphi_X(t)\big) - d_Y\big(\varphi_Y(s), \varphi_Y(t)\big)| < c$ for any $s, t \in T$.

By (1) and (2), $\varphi_Y(T)$ is compact. Put

$$\begin{aligned} l := \max\{k \in \mathbb{N} \mid \text{there exist points } p_i \in \varphi_Y(T),\ i = 1, \cdots, k, \text{ such that} \\ B_Y(p_i, a/2) \cap B_Y(p_j, a/2) = \emptyset \text{ for any } i, j \text{ with } i \neq j\}. \end{aligned}$$

Then, there exist points $p_i \in \varphi_Y(T)$, $i = 1, \cdots, l$, such that $B_Y(p_i, a/2) \cap B_Y(p_j, a/2) = \emptyset$ for any i, j with $i \neq j$. Hence, we get

$$1 \geq \mu_Y(\bigcup_{i=1}^{l} B_Y(p_i, a/2)) = \sum_{i=1}^{l} \mu_Y(B_Y(p_i, a/2)) = l \cdot v_Y(a/2).$$

It also follows from the definition of l that $\varphi_Y(T) \subseteq \bigcup_{i=1}^{l} B_Y(p_i, a)$. For any $i = 1, \cdots, l$, we fix $t_i \in T$ with $p_i = \varphi_Y(t_i)$.

Claim 3.3.

$$\varphi_X(T) \subseteq \bigcup_{i=1}^{l} B_X\big(\varphi_X(t_i), a + c\big).$$

Proof. Take an arbitrary $q = \varphi_X(s) \in \varphi_X(T)$, $s \in T$. Since $\varphi_Y(s) \in \varphi_Y(T) \subseteq \bigcup_{i=1}^{l} B_Y(p_i, a)$, there exists i with $1 \le i \le l$ such that $d_Y\left(\varphi_Y(s), p_i\right) \le a$. Therefore, by using (2), we obtain

$$d_X\left(\varphi_X(s), \varphi_X(t_i)\right) < d_Y\left(\varphi_Y(s), p_i\right) + c \le a + c.$$

This completes the proof of the claim. □

Applying Claim 3.3, we get

$$1 \le \sum_{i=1}^{l} \frac{\mu_X\big(B_X\big(\varphi_X(t_i), a+c\big)\big)}{\mu_X\big(\varphi_X(T)\big)} = l \cdot \frac{v_X(a+c)}{\mu_X\big(\varphi_X(T)\big)} \le \frac{v_X(a+c)}{v_Y(a/2) \cdot \mu_X\big(\varphi_X(T)\big)}.$$

Since $\mu_X\big(\varphi_X(T)\big) \ge \mathcal{L}\big(\varphi_X^{-1}\big(\varphi_X(T)\big)\big) \ge \mathcal{L}(T) > 1 - c$, we obtain

$$1 \le \frac{v_X(a+c)}{v_Y(a/2) \cdot \mu_X\big(\varphi_X(T)\big)} < \frac{v_X(a+c)}{v_Y(a/2) \cdot (1-c)} \le 1,$$

which is a contradiction. This completes the proof of Lemma 3.2. □

For a compact Riemannian manifold M, we denote by $vol(M)$ the total Riemannian volume of M. We indicate by Γ the Gamma function.

Lemma 3.4. *Let M (respectively, N) be an m (respectively, n)-dimensional compact Riemannian manifold having a uniformly distributed Riemannian volume measure. Assume that $Ric_M \ge (m-1)\kappa_1 > 0$ and $Ric_N > 0$, and put $a_N := vol(N)/\, vol(\mathbb{S}^n)$. If a positive number c with $c < 1$ satisfies*

$$c^{n-m} \le (1-c)\frac{n a_N (\kappa_1)^{m/2} \Gamma\big(\frac{m+1}{2}\big)\Gamma\big(\frac{n}{2}\big)}{m 2^{n+1} \pi^{m-1} \Gamma\big(\frac{m}{2}\big)\Gamma\big(\frac{n+1}{2}\big)} \text{ and } c\sqrt{\kappa_1} \le \pi,$$

then we have $\underline{\square}_1(M, N) \ge c$.

Proof. For $r > 0$, we put $v_M(r) := \mu_M\big(B_M(x,r)\big)$ for $x \in M$ and $v_N(r) := \mu_N\big(B_N(y,r)\big)$ for $y \in N$. From the Bishop-Gromov volume comparison theorem, we get

$$v_M(c/2) = \frac{vol\left(B_M(x, c/2)\right)}{vol(M)} \ge v_{\mathbb{S}^m}\big((c\sqrt{\kappa_1})/2\big) = \frac{vol(\mathbb{S}^{m-1})}{vol(\mathbb{S}^m)} \int_0^{(c\sqrt{\kappa_1})/2} \sin^{m-1}\theta d\theta.$$

From $c\sqrt{\kappa_1} \le \pi$, we have $\sin\theta \ge (\pi\theta)/2$ for any $\theta \in [0, (c\sqrt{\kappa_1})/2]$. Hence, we obtain

$$v_M(c/2) \ge \frac{2^{m-1}\, vol(\mathbb{S}^{m-1})}{\pi^{m-1}\, vol(\mathbb{S}^m)} \int_0^{(c\sqrt{\kappa_1})/2} \theta^{m-1} d\theta = \frac{c^m (\kappa_1)^{m/2}\, vol(\mathbb{S}^{m-1})}{2m\pi^{m-1}\, vol(\mathbb{S}^m)}.$$

Let κ_2 be a positive number such that $Ric_N \ge (n-1)\kappa_2$. We also obtain from the Bishop inequality that

$$v_N(2c) \le \frac{v_{\mathbb{S}^n}(2c\sqrt{\kappa_2})}{a_N(\kappa_2)^{n/2}} = \frac{vol(\mathbb{S}^{n-1})}{a_N(\kappa_2)^{n/2}\, vol(\mathbb{S}^n)} \int_0^{2c\sqrt{\kappa_2}} \sin^{n-1}\theta d\theta < \frac{(2c)^n\, vol(\mathbb{S}^{n-1})}{n a_N\, vol(\mathbb{S}^n)}.$$

Recall that $vol(\mathbb{S}^n) = 2\pi^{(n+1)/2}/\Gamma\big((n+1)/2\big)$. Therefore, combining the above calculations with Lemma 3.2, we complete the proof. □

Remark 3.5. In the proof of Lemma 3.4, we use only the Bishop inequality and the Bishop-Gromov volume comparison theorem. Therefore, Lemma 3.4 also holds for general mm-spaces satisfying the Bishop inequality and the Bishop-Gromov volume comparison theorem.

Proof of Proposition 1.1. Without loss of generality, it may be assumed that $n_k \geq m_k$.

First, we consider the case of $\{\mathbb{S}^n\}_{n=1}^{\infty}$. From the assumption, we have $c^{n_k-m_k} \leq c^{C_3k}$ for any $0 < c < 1$. Substituting $n := n_k$ and $m := m_k$, we estimate the right-hand side of the inequality of Lemma 3.4 by

$$(1-c)\frac{n_k\Gamma\left(\frac{m_k+1}{2}\right)\Gamma\left(\frac{n_k}{2}\right)}{m_k2^{n_k}\pi^{m_k-1}\Gamma\left(\frac{m_k}{2}\right)\Gamma\left(\frac{n_k+1}{2}\right)} \geq (1-c)2^{-C_1k}\pi^{-C_2k+1}\frac{n_k\Gamma\left(\frac{n_k}{2}\right)}{m_k\Gamma\left(\frac{n_k+1}{2}\right)}.$$

Therefore, if

$$c \leq \left\{(1-c)\frac{n_k\Gamma\left(\frac{n_k}{2}\right)}{m_k\Gamma\left(\frac{n_k+1}{2}\right)}\right\}^{\frac{1}{C_3k}}2^{-\frac{C_1}{C_3}}\pi^{-\frac{C_2}{C_3}+\frac{1}{C_3k}},$$

then we obtain from Lemma 3.4 that $\underline{\square}_1(\mathbb{S}^{n_k},\mathbb{S}^{m_k}) \geq c$. Since

$$\left\{(1-c)\frac{n_k\Gamma\left(\frac{n_k}{2}\right)}{m_k\Gamma\left(\frac{n_k+1}{2}\right)}\right\}^{\frac{1}{C_3k}} \to 1 \text{ as } k \to \infty,$$

we have completed the proof for $\{\mathbb{S}^n\}_{n=1}^{\infty}$.

Next, we consider $\{\mathbb{C}P^n\}_{n=1}^{\infty}$. It is well-known that $vol(\mathbb{C}P^n) = \pi^n/n!$ and the sectional curvature of $\mathbb{C}P^n$ is bounded from below by 1 (cf. [3, Section 3.D.2, 3.H.3]). Hence, we get

$$a_{\mathbb{C}P^n} = \frac{\Gamma\left(n+\frac{1}{2}\right)}{2\sqrt{\pi}n!}.$$

For any $0 < c < 1$, we have $c^{2n_k-2m_k} \leq c^{2C_3k}$. Substituting $n := 2n_k$ and $m := 2m_k$, we calculate the right-hand side of the inequality of Lemma 3.4 by

$$(1-c)\frac{\Gamma\left(m_k+\frac{1}{2}\right)}{2\sqrt{\pi}m_k2^{2n_k+1}\pi^{2m_k-1}\Gamma(m_k)} \geq (1-c)\frac{1}{2\sqrt{\pi}C_2k}\cdot 2^{-2C_1k-1}\pi^{-2C_2k+1}.$$

So, if

$$c \leq \left\{(1-c)\frac{1}{2\sqrt{\pi}C_2k}\right\}^{\frac{1}{2C_3k}}2^{-\frac{C_1}{C_3}-\frac{1}{2C_3k}}\pi^{-\frac{C_2}{C_3}+\frac{1}{2C_3k}},$$

then we get by using Lemma 3.4 that $\underline{\square}_1(\mathbb{C}P^{n_k},\mathbb{C}P^{m_k}) \geq c$. Since

$$\left\{(1-c)\frac{1}{2\sqrt{\pi}C_2k}\right\}^{\frac{1}{2C_3k}} \to 1 \text{ as } k \to \infty,$$

we complete the proof of the proposition. □

Lemma 3.6 (J. Christensen, cf. [6, Section 3.3])**.** *Let X be a metric space and μ,ν be uniformly distributed Borel measures on X. Then, there exists a positive number $c > 0$ such that $\mu = c\nu$.*

Proof of Proposition 1.3. We identify $\mathbb{S}^{n-1}$ with $\{(x_1,\cdots,x_n,0)\in\mathbb{S}^n \mid (x_1,\cdots,x_n) \in \mathbb{S}^{n-1}\}$. Given an arbitrary $\varepsilon > 0$, since the sequence $\{\mathbb{S}^n\}_{n=1}^{\infty}$ is a Lévy family, we have $r_n := \mu_n\left((\mathbb{S}^{n-1})_\varepsilon\right) \to 1$ as $n \to \infty$. Hence, there is $m \in \mathbb{N}$ such that $1-r_n < \varepsilon$ for any $n \geq m$. Suppose that $n \geq m$. Taking two parameters $\Phi_1 : [0,r_n] \to (\mathbb{S}^{n-1})_\varepsilon$

and $\Phi_2 : (r_n, 1] \to \mathbb{S}^n \setminus (\mathbb{S}^{n-1})_\varepsilon$, we define a Borel measurable map $\Phi : [0,1] \to \mathbb{S}^n$ by

$$\Phi(t) := \begin{cases} \Phi_1(t), & t \in [0, r_n], \\ \Phi_2(t), & t \in (r_n, 1]. \end{cases}$$

The map Φ is a parameter of $\mathbb{S}^n$. Let $\psi : \mathbb{S}^n \setminus \{(0, \cdots, 0, 1), (0, \cdots, 0, -1)\} \to \mathbb{S}^{n-1}$ be the projection; that is, $\psi(x)$ is the unique element of $\mathbb{S}^{n-1}$ satisfying $d_n\big(x, \psi(x)\big) = d_n(x, \mathbb{S}^{n-1})$. Put $\varphi_1 := \psi \circ \Phi_1 : [0, r_n] \to \mathbb{S}^{n-1}$.

Claim 3.7. $\varphi_{1*}(\mathcal{L}) = r_n \mu_{n-1}$.

Proof. Take any Borel subset $A \subseteq \mathbb{S}^{n-1}$. For any $g \in SO(n-1)$, we have

$$\varphi_{1*}(\mathcal{L})(gA) = r_n \mu_n\big(\psi^{-1}(gA)\big) = r_n \mu_n\big(\psi^{-1}(A)\big) = \varphi_{1*}(\mathcal{L})(A).$$

Hence, $\varphi_{1*}(\mathcal{L})$ is an $SO(n-1)$-invariant Borel measure. From Lemma 3.6, we complete the proof of the claim. □

Taking a parameter $\phi : (0,1] \to \mathbb{S}^{n-1}$ of $\mathbb{S}^{n-1}$, we define a Borel measurable map $\varphi_2 : (r_n, 1] \to \mathbb{S}^{n-1}$ by $\varphi_2(t) := \phi\big((t - r_n)/(1 - r_n)\big)$. Then, we have $\varphi_{2*}(\mathcal{L}) = (1 - r_n)\mathcal{L}$. Therefore, defining a Borel measurable map $\varphi : [0,1] \to \mathbb{S}^{n-1}$ by

$$\varphi(t) := \begin{cases} \varphi_1(t), & t \in [0, r_n], \\ \varphi_2(t), & t \in (r_n, 1], \end{cases}$$

we see that the map φ is a parameter of $\mathbb{S}^{n-1}$. Since

$$\begin{aligned} &|\, d_n\big(\Phi(s), \Phi(t)\big) - d_{n-1}\big(\varphi(s), \varphi(t)\big)| \\ &= |\, d_n\big(\Phi_1(s), \Phi_1(t)\big) - d_{n-1}\big(\varphi_1(s), \varphi_1(t)\big)| \le 2\varepsilon \end{aligned}$$

for any $s, t \in [0, r_n]$, we get

$$\underline{\square}_1(\mathbb{S}^n, \mathbb{S}^{n-1}) \le \square_1(\Phi^*\, d_n, \varphi^*\, d_{n-1}) \le \max\{2\varepsilon, 1 - r_n\} = 2\varepsilon.$$

Consequently, we obtain $\underline{\square}_1(\mathbb{S}^n, \mathbb{S}^{n-1}) \to 0$ as $n \to \infty$. A similar argument shows that $\underline{\square}_1(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \to 0$ as $n \to \infty$. This completes the proof of Proposition 1.3. □

Lemma 3.8. *For any $n, m \in \mathbb{N}$, we have*

$$\underline{\square}_1\big(SO(n), SO(m)\big) \ge c(n,m) := \min\Big\{\frac{1}{2}, |\operatorname{diam} SO(n) - \operatorname{diam} SO(m)|\Big\}.$$

Proof. Suppose that $n > m$ and $\underline{\square}_1\big(SO(n), SO(m)\big) < c(n,m)$. There exist a compact subset $T \subseteq [0,1]$ and two parameters $\varphi_n : [0,1] \to SO(n)$, $\varphi_m : [0,1] \to SO(m)$ such that

(1) $\mathcal{L}(T) > 1 - c(n,m) \ge 1/2$,
(2) $\varphi_n|_T : T \to SO(n)$, $\varphi_m|_T : T \to SO(m)$ are continuous,
(3) $|\, d_n\big(\varphi_n(s), \varphi_n(t)\big) - d_m\big(\varphi_m(s), \varphi_m(t)\big)| < c(n,m)$ for any $s, t \in T$.

Claim 3.9. There exist $s_0, t_0 \in T$ such that $d_n\big(\varphi_n(s_0), \varphi_n(t_0)\big) = \operatorname{diam} SO(n)$.

Proof. Take $A_0, B_0 \in SO(n)$ such that $\operatorname{diam} SO(n) = d_n(A_0, B_0)$ and define a map $\psi : SO(n) \to SO(n)$ by $\psi(A) := AA_0^{-1}B$. Then, $\psi_*(\mu_n) = \mu_n$ and $d_n\big(A, \psi(A)\big) =$

$\operatorname{diam} SO(n)$ for any $A \in SO(n)$. Suppose that $d_n\big(\varphi_n(s), \varphi_n(t)\big) < \operatorname{diam} SO(n)$ for any $s, t \in T$. Then, we get $\psi\big(\varphi_n(T)\big) \cap \varphi_n(T) = \emptyset$, which leads to

$$\begin{aligned}\mu_n\big(\psi\big(\varphi_n(T)\big) \cap \varphi_n(T)\big) &= \mu_n\big(\psi\big(\varphi_n(T)\big)\big) + \mu_n\big(\varphi_n(T)\big)\\ &= \mu_n\big(\psi^{-1}\big(\psi\big(\varphi_n(T)\big)\big)\big) + \mu_n\big(\varphi_n(T)\big)\\ &\geq 2\mu_n\big(\varphi_n(T)\big) > 1.\end{aligned}$$

This is a contradiction, and thus we complete the proof of the claim. □

By Claim 3.9, we obtain

$$\operatorname{diam} SO(n) - \operatorname{diam} SO(m) \leq |\, d_n\left(\varphi_n(s_0), \varphi_n(t_0)\right) - d_m\left(\varphi_m(s_0), \varphi_m(t_0)\right)| < c(n, m),$$

which is a contradiction. This completes the proof of Lemma 3.8. □

Proof of Proposition 1.2. An easy calculation shows that $2\sqrt{n-1} \leq \operatorname{diam} SO(n) \leq 2\sqrt{n}$. Therefore, supposing $n_k \geq m_k$, we have

$$\begin{aligned}\operatorname{diam} SO(n_k) - \operatorname{diam} SO(m_k) &\geq 2\sqrt{n_k - 1} - 2\sqrt{m_k}\\ &= 2\frac{n_k - m_k - 1}{\sqrt{n_k - 1} + \sqrt{m_k}} \geq 2\frac{C_3 - 1/\sqrt{k}}{\sqrt{C_1 - 1/k} + \sqrt{C_2}}.\end{aligned}$$

Thus, applying Lemma 3.8, we complete the proof. □

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Mathematical Institute, Tohoku University, Sendai 980-8578, Japan
E-mail address: `sa4m23@math.tohoku.ac.jp`