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EQUIFOCALITY OF A SINGULAR RIEMANNIAN FOLIATION

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ABSTRACT. A singular foliation on a complete Riemannian manifold M is said to be Riemannian if each geodesic that is perpendicular to a leaf at one point remains perpendicular to every leaf it meets. We prove that the regular leaves are equifocal, i.e., the end point map of a normal foliated vector field has constant rank. This implies that we can reconstruct the singular foliation by taking all parallel submanifolds of a regular leaf with trivial holonomy. In addition, the end point map of a normal foliated vector field on a leaf with trivial holonomy is a covering map. These results generalize previous results of the authors on singular Riemannian foliations with sections.

1. INTRODUCTION

In this section, we will recall some definitions and state our main results as Theorem 1.5 and Corollary 1.6.

We start by recalling the definition of a singular Riemannian foliation (see the book of Molino [9]).

Definition 1.1 (s.r.f.)**.** A partition $\mathcal{F}$ of a complete Riemannian manifold M by connected immersed submanifolds (the *leaves*) is called a *singular Riemannian foliation* (s.r.f. for short) if it verifies condition (1) and (2):

(1) $\mathcal{F}$ is a *singular foliation*; i.e., the module $\mathcal{X}_F$ of smooth vector fields on M that are tangent at each point to the corresponding leaf acts transitively on each leaf. In other words, for each leaf L and each $v \in TL$ with footpoint p, there is $X \in \mathcal{X}_F$ with $X(p) = v$.

(2) $\mathcal{F}$ is *transnormal*; i.e., every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.

Let $\mathcal{F}$ be a singular Riemannian foliation on a complete Riemannian manifold M. A leaf L of $\mathcal{F}$ (and each point in L) is called *regular* if the dimension of L is maximal; otherwise L is called *singular*.

Typical examples of s.r.f. are the partition by orbits of an isometric action, by leaf closures of a Riemannian foliation, examples constructed by suspension of homomorphisms (see [2, 4]), and examples constructed by changes of metric and surgery (see [5]).

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A particular class of s.r.f. is the one which admits sections; i.e., for each regular point p the set $\Sigma := \exp(\nu_p L_p)$ is a complete immersed submanifold that meets each leaf orthogonally.

The concept of singular Riemannian foliations with sections (s.r.f.s. for short) was introduced in [2] and continued to be studied by the authors in [1, 3, 4, 11, 12, 5], by Lytchak and Thorbergsson in [8] and recently by Gorodski and the first author in [6]. In [7] Boualem dealt with a singular Riemannian foliation $\mathcal{F}$ on a complete manifold M such that the distribution of normal spaces of the regular leaves is integrable. It was proved in [4] that such an $\mathcal{F}$ must be an s.r.f.s.

An s.r.f.s. includes the partitions by orbits of a polar action and the well known class of isoparametric foliations on space forms, some of them with inhomogeneous leaves.

In [10], Terng and Thorbergsson introduced the concept of equifocal submanifolds with flat sections in compact symmetric spaces in order to generalize the definition of isoparametric submanifolds in a Euclidean space.

A connected immersed submanifold L of a complete Riemannian manifold M is called *equifocal* if it satisfies the following conditions:

(1) The normal bundle $\nu(L)$ is flat.
(2) For each parallel normal field ξ on a neighborhood $U \subset L$, the derivative of the map $\eta_\xi : U \to M$ defined by $\eta_\xi(x) := \exp_x(\xi)$ has constant rank.
(3) L has sections; i.e., for each $p \in L$, the set $\Sigma := \exp_p(\nu_p L_p)$, called a section, is a complete immersed totally geodesic submanifold.

There is almost an equivalence between the notions of an s.r.f.s. and equifocal submanifolds that is worked out in the authors' works [2] and [11].

On the one hand it was proved that a closed embedded equifocal submanifold induces an s.r.f.s. by taking all its parallel submanifolds ([11], [3]) if and only if there is exactly one section through every regular value of the normal exponential map of the equifocal submanifold. The global structure inherent to an s.r.f.s. was then used to generalize some results known for isoparametric submanifolds in Euclidean space.

On the other hand, it was proved in [2] that the leaves of an s.r.f.s. are equifocal (see [11] for an alternative proof). In a converse direction to the above the equifocality of an s.r.f.s. is also a very important tool in the theory of an s.r.f.s. For example, it allows us to have a Slice Theorem, singular holonomy, Weyl pseudogroups, a relation of an s.r.f.s. to transnormal maps and an extension of Weyl-invariant forms to basic forms.

While the existence of sections has interesting structural implications, it naturally restricts the number of cases that are covered. This can best be seen in the case of homogenous s.r.f. when comparing an arbitrary isometric action with a polar action. The latter is best exemplified by the action of a compact Lie group on itself by conjugation. In this paper we want to drop the condition on the existence of sections and prove that regular leaves of an s.r.f. are also equifocal. In order to make this statement precise, we will drop the first and third conditions in the definition of an equifocal submanifold, and we will also need to change the concept of parallel normal fields to foliated vector fields. Note that the restriction $\mathcal{F}_r$ of $\mathcal{F}$ to the regular stratum of M is a regular foliation. We recall that a vector field ξ in the normal bundle of the foliation over an open subset U in the regular stratum is called *foliated* if for each vector field $Y \in \mathcal{X}_F$ the Lie bracket $[\xi, Y]$ also belongs to $\mathcal{X}_F$. If we consider a local submersion π which describes the plaques of $\mathcal{F}$ in a neighborhood of a point of L, then a normal foliated vector field is a normal projectable/basic vector field with respect to π.

Remark 1.2. A Bott or basic connection ∇ of a foliation $\mathcal{F}$ is a connection of TM with $\nabla_X Y = [X, Y]^{\nu\mathcal{F}}$ whenever $X \in \mathcal{X}_F$ and Y is a vector field of the normal bundle $\nu\mathcal{F}$ of the foliation. Here the superscript $\nu\mathcal{F}$ denotes projection onto $\nu\mathcal{F}$. A foliated vector field clearly is parallel with respect to the Bott connection. This connection can be restricted to the normal bundle of a leaf.

Definition 1.3. Let L be a regular leaf of an s.r.f. A normal vector field along L is said to be *foliated* if it is Bott-parallel or, in other words, if it is locally the restriction of a foliated vector field of $\mathcal{F}_r$ to a neighborhood $U \subset L$.

Remark 1.4. Note that if the s.r.f. admits sections, then a normal foliated vector field is a parallel normal field along each regular leaf L with respect to the induced Levi-Civita connection on νL and vice versa. In other words, in the case of sections the induced Levi-Civita connection is a Bott connection.

We are finally ready to state our result precisely.

Theorem 1.5. *Let $\mathcal{F}$ be an s.r.f. on a complete Riemannian manifold M. Then for each regular point p there exists a neigborhood U of p in L_p such that*

1. *For each normal foliated vector field ξ along U the derivative of the map $\eta_\xi : U \to M$, defined as $\eta_\xi(x) := \exp_x(\xi)$, has constant rank.*
2. *$W := \eta_\xi(U)$ is an open set of $L_{\eta_\xi(p)}$.*

Corollary 1.6. *Let L_p be a regular leaf with trivial holonomy and let Ξ denote the set of all normal foliated vector fields along L_p.*

1. *Let $\xi \in \Xi$. Then $\eta_\xi : L_p \to L_q$ is a covering map if $q = \eta_\xi(p)$ is a regular point.*
2. *$\mathcal{F} = \{\eta_\xi(L_p)\}_{\xi \in \Xi}$; i.e., we can reconstruct the singular foliation by taking all parallel submanifolds of the regular leaf L_p.*

This paper is organized as follows. In Section 2 we present the propositions needed to prove the theorem. In particular we prove two propositions which contain some improvements of Molino's results on the local analysis of an s.r.f. More precisely, we review a local decomposition result and a product theorem due to Molino (see Proposition 2.2 and Proposition 2.3). In Section 3 we prove Theorem 1.5, and in Section 4 we prove Corollary 1.6.

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2. Properties of an s.r.f.

In this section we will present the propositions needed to prove Theorem 1.5. Throughout this section we assume that $\mathcal{F}$ is an s.r.f. on a complete Riemannian manifold M.

We start by recalling the so-called *Homothetic Transformation Lemma* of Molino (see Lemma 6.2 of [9]).

By conjugating the homothetic transformations of the normal bundle of a plaque P via the normal exponential map, one defines for small strictly positive real numbers λ, a homothetic transformation h_λ with proportionality constant λ with respect to the plaque P.

Proposition 2.1 ([9]). *The homothetic transformation h_λ sends plaque to plaque and therefore respects the singular foliation $\mathcal{F}$ in the tubular neigborhood* $\mathrm{Tub}(P)$ *where it is defined.*

The next two propositions contain some improvements on Molino's results (compare with Theorem 6.1 and Proposition 6.5 of [9]).

Proposition 2.2. *Let g be the original metric on M and $q \in M$. Then there exists a tubular neighborhood* $\mathrm{Tub}(P_q)$ *and a new metric $\tilde{g}$ on* $\mathrm{Tub}(P_q)$ *with the following properties:*

(a) *For each $x \in \mathrm{Tub}(P_q)$ the normal space of the leaf L_x is tangent to the slice $S_{\tilde{q}}$ which contains x, where $\tilde{q} \in P_q$.*
(b) *Let $\pi : \mathrm{Tub}(P_q) \to P_q$ be the orthogonal projection. Then the restriction $\pi|_{P_x}$ is a Riemannian submersion.*
(c) *$\mathcal{F} \cap \mathrm{Tub}(P_q)$ is an s.r.f.*
(d) *$\mathcal{F} \cap S_{\tilde{q}}$ is an s.r.f. for each $\tilde{q} \in P_q$.*
(e) *The associated transverse metric is not changed; i.e., the distance between the plaques with respect to g is the same distance between the plaques with respect to $\tilde{g}$.*
(f) *If a curve γ is a geodesic orthogonal to P_q with respect to the original metric g, then γ is a geodesic orthogonal to P_q with respect to the new metric $\tilde{g}$.*

Proof. Let $X_1, \dots, X_r \in \mathcal{X}_F$ (i.e., vector fields that are always tangent to the leaves) so that $\{X_i(q)\}_{i=1,\dots,r}$ is a linear basis of $T_q P_q$. Let $\varphi^1_{t_1}, \dots, \varphi^r_{t_r}$ denote the associated one parameter groups and define $\varphi(t_1, \dots, t_r, y) := \varphi^1_{t_1} \circ \cdots \circ \varphi^r_{t_r}$ where $y \in S_q$ and $(t_1, \dots, t_r)$ belongs to a neighborhood U of $0 \in \mathbb{R}^r$. Then, reducing U and $\mathrm{Tub}(P_q)$ if necessary, one can guarantee the existence of a regular foliation $\mathcal{F}^2$ with plaques $P^2_y = \varphi(U, y)$. We note that the plaques $P^2_z \subset P_z$ and each plaque P^2 cuts each slice at exactly one point. Using the fact that $\pi|_{P^2_y} : P^2_y \to P_q$ is a diffeomorphism, we can define a metric on each plaque P^2_y as $\tilde{g}^2 := (\pi|_{P^2_y})^* g$.

Now we want to define a metric $\tilde{g}^1$ on each slice $S \in \{S_{\tilde{q}}\}_{\tilde{q} \in P_q}$. Set $D_p := \nu_p L^2_p$ and define $\Pi : T_p M \to D_p$ as the orthogonal projection with respect to g. The fact that each plaque P^2 cuts each slice at one point implies that $\Pi|_{T_p S} : T_p S \to D_p$ is an isomorphism. Finally we define $\tilde{g}^1 := (\Pi|_{T_p S})^* g$ and $\tilde{g} := \tilde{g}^1 + \tilde{g}^2$, meaning that $\mathcal{F}^2$ and the slices meet orthogonally. Items (a) and (b) follow directly from the definition of $\tilde{g}$.

To prove item (c) it suffices to prove that the plaques of $\mathcal{F}$ are locally equidistant to each other. Let $x \in S_{\tilde{q}}$, P_x be a plaque of $\mathcal{F}$. We know that the plaques of $\mathcal{F}$ are contained in the leaves of the foliation by distance-cylinders $\{C\}$ with axis P_x with respect to g. We will prove that each C is also a distance-cylinder with axis P_x with respect to the new metric $\tilde{g}$. These facts and the arbitrary choice of x will imply that the plaques of $\mathcal{F}$ are locally equidistant to each other.

First we recall that a smooth function $f : M \to \mathbb{R}$ is called a *transnormal function* with respect to the metric g if there exists a $C^2(f(M))$ function b such that $g(\mathrm{grad}\ f, \mathrm{grad}\ f) = b \circ f$. Let $f : \mathrm{Tub}(P_x) \to \mathbb{R}$ be a smooth transnormal function with respect to the metric g so that each regular level set $f^{-1}(c)$ is a cylinder C with axis P_x, e.g. $f(y) = d(y, P_x)^2$. Let $\widetilde{\mathrm{grad}}\ f$ denote the gradient of f with respect to the metric $\tilde{g}$. It follows from the construction of $\tilde{g}$ that

$$\widetilde{\mathrm{grad}}\ f = \mathrm{grad}\ f + l, \tag{2.1}$$

where l is a vector tangent to a plaque of $\mathcal{F}^2$ and in particular to a plaque of $\mathcal{F}$.

Indeed, let $v \in D_p$ and $w := (\Pi|_{T_pS})^{-1}(v)$. Then

$$\begin{aligned} g(\text{grad } f, v) &= df(v) \\ &= df(w) \\ &= \tilde{g}(\widetilde{\text{grad}}\ f, w) \\ &= \tilde{g}^1(\widetilde{\text{grad}}\ f, w) \\ &= g(\Pi\widetilde{\text{grad}}\ f, \Pi w) \\ &= g(\Pi\widetilde{\text{grad}}\ f, v). \end{aligned}$$

We conclude from the arbitrary choice of $v \in D_p$ that grad $f = \Pi\widetilde{\text{grad}}\ f$ and hence $\widetilde{\text{grad}}\ f = \text{grad } f + l$.

Equation (2.1) implies that f is a also a transnormal function with respect to the metric $\tilde{g}$, i.e.,

$$\tilde{g}(\widetilde{\text{grad}}\ f, \widetilde{\text{grad}}\ f) = b \circ f. \tag{2.2}$$

Indeed,

$$\begin{aligned} \tilde{g}(\widetilde{\text{grad}}\ f, \widetilde{\text{grad}}\ f) &= df(\widetilde{\text{grad}}\ f) \\ &= df(\text{grad } f) \\ &= g(\text{grad } f, \text{grad } f) \\ &= b \circ f. \end{aligned}$$

Using a local version of Q.-M. Wang's theorem [13], we conclude that each regular level set of f (i.e., C) is a distance cylinder around P_x with respect to the metric $\tilde{g}$.

Now we want to prove item (d). Set $P_x^s = P_x \cap S_{\tilde{q}}$ and $C^s := C \cap S_{\tilde{q}}$. It suffices to note that the singular foliation $\{C^s\}$ is a foliation by cylinders with axis P_x^s with respect to the new metric $\tilde{g}$. This follows from the fact that $\nu_x P_x \subset T_x S_{\tilde{q}}$ and that each geodesic orthogonal to P_x at x is contained in $S_{\tilde{q}}$ (see item (a)).

In particular we conclude that the distance between C and P_x and the distance between C^s and P_x^s with respect to the metric $\tilde{g}$ are the same.

To prove item (e) we have to prove that the distance between the cylinder C and the plaque P_x is the same for both metrics. Let f be the transnormal function (with respect to g) defined above. According to Q.-M. Wang [13] for $k = f(P_x)$ and a regular value c we have $d(P_x, f^{-1}(c)) = \int_c^k \frac{ds}{\sqrt{b(s)}}$. Since f is also a transnormal function with respect to $\tilde{g}$ (see equation (2.2)), we conclude that $d(P_x, C) = \tilde{d}(P_x, C)$, for $C = f^{-1}(c)$.

Finally we prove item (f). We consider the transnormal function f above with $x = q$. In this case, equation (2.1) and the fact that grad $f \in D_p \cap T_pS$ imply that grad $f = \widetilde{\text{grad}}\ f$. On the other hand, the integral curves of the gradient of a transnormal function are geodesic segments up to reparametrization (see e.g. [13]). Therefore the radial geodesics of P_q coincide in both metrics. This finishes the proof. $\square$

Proposition 2.3. *Let $\tilde{g}$ be the metric defined in Proposition* 2.2. *Then there exists a new metric g_0 on* $\text{Tub}(P_q)$ *so that:*

(a) *We may consider the tangent space* $T_{\tilde q}S_{\tilde q}$ *with the metric* $\tilde g$ *and* $S_{\tilde q}$ *with the metric* g_0. *Then* $\exp_{\tilde q} : T_{\tilde q}S_{\tilde q} \to S_{\tilde q}$ *is an isometry.*
(b) *For this new metric* g_0 *we have that* $\mathcal{F} \cap S_{\tilde q}$ *and* $\mathcal{F}$ *restricted to* $\mathrm{Tub}(P_q)$ *are also an s.r.f.*
(c) *For each* $x \in \mathrm{Tub}(P_q)$ *the normal space of the leaf* L_x *is tangent to the slice* $S_{\tilde q}$ *which contains* x, *where* $\tilde q \in P_q$.

Remark 2.4. Clearly a curve γ which is a geodesic orthogonal to P_q with respect to the original metric remains a geodesic orthogonal to P_q with respect to the new metric g_0.

Proof. Let Π_1 be the orthogonal projection to the slices, and recall that $\tilde g^1 = \tilde g \circ \Pi_1$ and $\tilde g^2 = \tilde g \circ d\,\pi$. Let h_λ denote the homothetic transformation with respect to P_q. Define $g_\lambda = \frac{1}{\lambda^2} h_\lambda^* \tilde g^1 + \tilde g^2$. Note that the metric g_λ tends uniformly to a metric g_0 for $\lambda \to 0$. This metric g_0 restricted to $S_{\tilde q}$ is the induced metric on $\nu P_{\tilde q}$, where $\tilde q \in P_q$.

This implies that L_λ tends uniformly to L_0, where L_λ is the length function. It follows then that

$$\lim_{\lambda \to 0} d_\lambda(x, P) = d_0(x, P) \tag{2.3}$$

where P is a plaque.

Now we claim that $\mathcal{F}$ is an s.r.f. with respect to g_λ. Indeed, since $h_\lambda^* \tilde g^2 = \tilde g^2$ and the homothetic transformation h_λ sends plaque to plaque (see Proposition 2.1), it suffices to prove that $\mathcal{F}$ is an s.r.f. with respect to $\frac{1}{\lambda^2}\tilde g^1 + \tilde g^2$. Let $f : \mathrm{Tub}(P_x) \to \mathbb{R}$ be a smooth transnormal function with respect to the metric $\tilde g$ so that each regular level set $f^{-1}(c)$ is a cylinder with axis P_x. Note that f is also a transnormal function with respect to the metric $\frac{1}{\lambda^2}\tilde g^1 + \tilde g^2$, because $\widetilde{\mathrm{grad}}\, f$ is tangent to the slice. Using a local version of Q.-M. Wang's theorem [13], we conclude that each regular level set of f is a tube over P_x with respect to the metric $\frac{1}{\lambda^2}\tilde g^1 + \tilde g^2$. Therefore the plaques are equidistant to P_x, and hence we conclude that $\mathcal{F}$ is an s.r.f. with respect to $\frac{1}{\lambda^2}\tilde g^1 + \tilde g^2$.

Finally let x and y be points which belong to the same plaque. Using equation (2.3) and the fact that $\mathcal{F}$ is an s.r.f. with respect to g_λ, we conclude that

$$\begin{aligned} 0 &= \lim_{\lambda \to 0} (\, d_\lambda(x, P) - d_\lambda(y, P)\,) \\ &= d_0(x, P) - d_0(y, P). \end{aligned}$$

The above equation implies that the plaques are locally equidistant and hence that the singular foliation $\mathcal{F}$ is Riemannian.

Now we want to prove item (c). Let P_x be a plaque with $x \in S$. Note that for each metric g_λ the normal space H_λ of P_x at x (with respect to the metric g_λ) is tangent to S. This fact will imply that the normal space of P_x at x with respect to g_0 is also tangent to S. Indeed, we can find a sequence of normal spaces $H_{1/n}$ such that $H_{1/n}$ converge to a subspace H_0 tangent to S at x. Then we can find a subsequence of frames $\{e_i^n\}$ which converge to a frame $\{e_i\}$ such that $\{e_i^n\}$ and $\{e_i\}$ are bases of $H_{1/n}$ and H_0 respectively. Since

$$\frac{1}{\lambda^2} h_\lambda^* \tilde g^1 (d(\exp_q)_V Y, d(\exp_q)_V Z) = \tilde g^1 (d(\exp_q)_{\lambda V} Y, d(\exp_q)_{\lambda V} Z),$$

we conclude that

$$g_0(e_i, l) = \lim_{n \to \infty} g_{1/n}(e_i^n, l) = 0$$

where l is tangent to the plaque. The last equation implies that H_0 is the normal space of P_x at x with respect to g_0. □

Proposition 2.5. *Let S_q be a slice at q and $\varphi : S_q \to S_q$ be the geodesic symmetry at q, i.e., $\varphi = \exp_q \circ(-\mathrm{id}) \circ \exp_q^{-1}$. Then the map φ is $\mathcal{F} \cap S_q$ foliated; i.e., the foliation $\mathcal{F} \cap S_q$ is invariant by the involution φ.*

Proof. It follows from Proposition 2.3 and Remark 2.4 that we can lift $\mathcal{F}$ via the exponential map in a neighborhood of q to an s.r.f. of T_qS_q. Therefore we assume that $\mathcal{F}$ is an s.r.f. of $\mathbb{R}^n$ with a Euclidean metric which has $\{0\}$ as a leaf.

Lemma 2.6. *The induced singular foliation on the unit sphere $\mathcal{F}' := \mathcal{F}|S^{n-1}$ is an s.r.f.*

Proof. First note that every leaf of $\mathcal{F}$ that has a point in S^{n-1} lies in S^{n-1}. Clearly $\mathcal{F}'$ is a singular foliation. We now want to show transnormality. Let $v \in S^{n-1}$ and $\xi \in \nu_v L_v \cap T_v S^{n-1}$ be a unit vector. We denote by γ_ξ the geodesic in S^{n-1} with initial vector ξ. We want to show that $\xi(t) := \dot{\gamma}_\xi(t) \in \nu_w L_w$, where $w = \gamma_\xi(t)$. First we assume $t \in (0, \pi)$. Then the two unit radial vectors of S^{n-1} in v and w span a 2-plane of $\mathbb{R}^n$ containing the origin. As this plane contains the straight line from v to w, it lies in $\nu_w L_w$ by transnormality of $\mathcal{F}$. The intersection of this 2-plane with S^{n-1} is exactly the geodesic γ_ξ. Therefore $\xi(t) \in \nu_w L_w$. This shows that $\gamma_\xi|[0, t)$ and consequently $\gamma_\xi|[0, \pi)$ is transnormal. To prove transnormality of $\gamma_\xi|[0, 2\pi)$ we repeat the argument with w, respectively $\xi(t)$, as our new v, respectively ξ. Since the geodesic γ_ξ is closed of period 2π only a third step is needed to show its transnormality. □

Now let $v \in S^{n-1}$ and let L_v be a leaf through v. Here we denote by $\nu'_v L_v$ the normal space of L_v in S^{n-1}. For any unit vector $\xi \in \nu'_v L_v$ the geodesic γ_ξ in S^{n-1} meets the leaf L_{-v} in the antipodal point $-v$ orthogonally, i.e., $-\xi = \dot{\gamma}_\xi(\pi) \in \nu'_{-v} L_{-v}$. So as vector spaces in $\mathbb{R}^n$ we have $\nu'_v L_v \subset \nu'_{-v} L_{-v}$, and by symmetry we have equality for every $v \in S^{n-1}$. In other words $-\mathrm{id}$ respects the normal bundle and therefore also the tangent bundle of $\mathcal{F}$. From this we conclude that $-\mathrm{id}$ respects $\mathcal{F}$ on S^{n-1}. □

Corollary 2.7. *Let γ be a geodesic orthogonal to a regular leaf of an s.r.f. Then the singular points are isolated on γ.*

Proof. The codimensions are invariant when restricting the foliation to the slice in the first singular point on γ. Now we apply the previous lemma. □

3. Proof of the theorem

In this section we will apply the above propositions to prove the theorem. We start by proving a local version of Theorem 1.5.

Proposition 3.1. *Let $\mathrm{Tub}(P_q)$ be a tubular neighborhood of a plaque P_q, $x_0 \in \mathrm{Tub}(P_q)$, a regular point and $\xi \in \nu P_{x_0}$ such that $\exp_{x_0}(\xi) = q$. Then we can find a neighborhood U of x_0 in P_{x_0} with the following properties:*

1. *We can extend ξ to a foliated normal vector field ξ on U.*
2. *The geodesic segment that is orthogonal to P_q and contains a point $x \in U$ is $\gamma_x(t) := \exp_x((t+1)\,\xi)$ where $t \in [-1, 1]$.*
3. *$\eta_{(t+1)\,\xi}(U)$ is an open subset of $L_{\gamma_{x_0}(t)}$.*

(4) $\eta_{(t+1)\,\xi} : U \to \eta_{(t+1)\,\xi}(U)$ *is a diffeomorphism for* $t \neq 0$.
(5) $\dim \operatorname{rank} D\eta_{\xi}$ *is constant on* U.

Proof. The proof of item (1) is straightforward. The proof of item (2) follows from the Homothetic Transformation Lemma by Molino (Proposition 2.1).

Lemma 3.2. *Let* $\alpha(s)$ *be a curve in* U. *Define* $f(s,t) = \exp_{\alpha(s)}((t+1)\xi)$ *and* $J(t) = \frac{\partial f}{\partial s}(0,t)$. *Then to prove items* (3), (4) *and* (5) *it suffices to prove that the Jacobi field* J *is always tangent to the leaves.*

Proof. The fact that J is always tangent to the leaves, Corollary 2.7, and Proposition 2.1 imply that $\eta_{(t+1)\,\xi}(U) \subset L_{\gamma_{x_0}(t)}$, for all t. Proposition 2.1 implies that, for $t < 0$, $\eta_{(t+1)\,\xi}(U)$ is an open subset of $L_{\gamma_{x_0}(t)}$ and that $\eta_{(t+1)\,\xi} : U \to \eta_{(t+1)\,\xi}(U)$ is a diffeomorphism. We conclude from the fact that $\pi : \operatorname{Tub}(P_q) \to P_q$ is a submersion that $\dim \operatorname{rank} D\eta_{\xi}$ is constant on U. Finally, consider $t > 0$. The fact that $\pi : \operatorname{Tub}(P_q) \to P_q$ is a submersion implies that $\ker D\eta_{(t+1)\,\xi}$ is tangent to the slice. On the other hand, $\eta_{(t+1)\,\xi}$ restricted to the slice is a diffeomorphism. Therefore $\eta_{(t+1)\,\xi}|_U$ is a local diffeomorphism. It is not difficult to see that it is bijective and hence is a diffeomorphism. □

In what follows we will prove that the Jacobi field J defined above is always tangent to the leaves.

Let g_0 be the metric defined in Proposition 2.3. Then Remark 2.4 and item (2) imply that the Jacobi field J defined in Lemma 3.2 was not changed when the metric was modified.

Now consider a geodesic segment γ orthogonal to the leaves of $\mathcal{F}$ so that $\gamma(0) = q$ and $\gamma(1)$ is a regular point contained in S_q. It follows from Corollary 2.7 that $\gamma(t)$ is always regular for $-1 \leq t < 0$ and $0 < t \leq 1$.

We define σ as the submanifold contained in S_q which is the image by $\exp_q$ of a subspace and so that σ is orthogonal to L_x at x.

By Proposition 2.5, Proposition 2.3 and Proposition 2.1 we have that the plaques $P_{\gamma(t)} \cap S_q$ are orthogonal to σ for $-1 \leq t \leq 1$. Then it follows from Proposition 2.3 that the plaques $P_{\gamma(t)}$ are orthogonal to σ for $-1 \leq t \leq 1$.

Consider a geodesic segment β so that $\beta(0) = \gamma(t)$ and β is orthogonal to $P_{\gamma(t)}$. Then Proposition 2.3 implies that β is contained in S_q. Since S_q is identified with $T_q S_q$ we can consider β as a straight line. Since $P_{\gamma(t)} \cap S_q$ is orthogonal to σ and σ is identified with a subspace, we conclude that β is contained in σ.

Therefore $\exp_{\gamma(t)}(\nu(P)_{\gamma(t)} \cap B_{\epsilon}(0))$ is an open set of σ. A standard argument from Riemannian geometry implies that the second form is null at $\gamma(t)$; i.e., σ is geodesic at $\gamma(t)$. In particular the curvature tensor R of σ is the same as the ambient space at $\gamma(t)$. This fact and the fact that $R(\gamma', \cdot)\gamma'$ is self-adjoint imply that $T_{\gamma(t)}\sigma$ as well $(T_{\gamma(t)}\sigma)^{\perp}$ are families of a parallel subspace along γ which are invariant by $R(\gamma', \cdot)\gamma'$.

Finally consider the L_x-Jacobi field J defined in Lemma 3.2. This Jacobi field has initial conditions at $(T_{\gamma(1)}\sigma)^{\perp}$ and satisfies the Jacobi equation. So $J(t) \in (T_{\gamma(t)}\sigma)^{\perp}$ for $-1 \leq t \leq 1$.

As remarked above plaques $P_{\gamma(t)}$ are orthogonal to σ for $-1 \leq t \leq 1$. Since $P_{\gamma(t)}$ are regular plaques for $t \neq 0$ (see Corollary 2.7), we conclude that $J(t)$ is always tangent to $P_{\gamma(t)}$. □

We are finally ready to prove Theorem 1.5.

Let L be a leaf of $\mathcal{F}$ and ξ be a normal foliated vector field along a neighborhood U of L. Let $p \in U$. Since singular points are isolated along $\gamma_p(t) = \exp_p(t\,\xi)|_{[-\epsilon,1+\epsilon]}$, there exists a partition $0 = t_0 < \cdots < t_n = 1$ such that $\gamma(t_i)$ are the only possible singular points.

Let $P_{\gamma_p(r_i)}$ be regular plaques that belong to $\mathrm{Tub}(P_{\gamma_p(t_{i-1})}) \cap \mathrm{Tub}(P_{\gamma_p(t_i)})$, where $t_{i-1} < r_i < t_i$. Applying Proposition 3.1 we can find an open set $U_0 \subset P_p$, of the plaque P_p, an open set U_{n+1} of $P_{\gamma_p(1)}$, open sets $U_i \subset P_{\gamma_p(r_i)}$ of the plaques $P_{\gamma_p(r_i)}$ (for $1 \leq i \leq n$) and normal foliated vector fields ξ_i along U_i (for $0 \leq i \leq n$) with the following properties:

1) For each U_i, the normal foliated vector field ξ_i is tangent to the geodesics $\gamma_x(t)$, where $x \in U_0$.
2) $\eta_{\xi_i} : U_i \to U_{i+1}$ is surjective and for $i < n$ a diffeomorphism.
3) $\eta_\xi|_{U_0} = \eta_{\xi_n} \circ \eta_{\xi_{n-1}} \circ \cdots \circ \eta_{\xi_0}|_{U_0}$.

Because $\dim \operatorname{rank} d\eta_{\xi_i}$ is constant on U_i, it follows that $\dim d\eta_\xi$ is constant on U_0. Since this holds for each $p \in U$, $\dim d\eta_\xi$ is constant on U. It also follows that $\eta_\xi(U)$ is an open set of $L_{\eta_\xi(U)}$.

4. Proof of Corollary 1.6

Let L_p be a regular leaf with trivial holonomy and ξ a normal foliated vector field along L_p. It follows from Theorem 1.5 that $\eta_\xi(L_p)$ is an open set of L_q, where $q = \eta_\xi(p)$. In this section we will prove that $\eta_\xi(L_p)$ is also a closed set in L_q and hence that $\eta_\xi(L_p) = L_q$. In addition, when q is a regular point, we will also prove that $\eta_\xi : L_p \to L_q$ is a covering map.

At first suppose that L_q is a regular leaf.

For a point $z_0 \in L_q$ assume that there exists a point $z_1 \in \eta_\xi(L_p)$ which also belongs to the plaque P_{z_0}. Let x_α be a point in L_p such that $\eta_\xi(x_\alpha) = z_1$. Let $\hat{\xi}_\alpha$ be the vector in $T_{z_1}M$ tangent to the geodesic $\exp_{x_\alpha}(t\,\xi)$ so that $\exp_{z_1}(\hat{\xi}_\alpha) = x_\alpha$. We can extend $\hat{\xi}_\alpha$ along the plaque P_{z_0} to a normal foliated vector field. Theorem 1.5 implies that $\eta_{\hat{\xi}_\alpha} : P_{z_0} \to L_p$. Let A be the set of points $z \in P_{z_0}$ such that $\hat{\xi}_\alpha(z)$ is tangent to the geodesic $\exp_x(t\,\xi)$ and $\exp_z(\hat{\xi}_\alpha) = x$ for $x \in L_p$. The fact that $\eta_\xi : L_p \to L_q$ is a local diffeomorphism (see Theorem 1.5) implies that A is an open set of P_{z_0}.

Let $z_2 \in \partial A$. By Theorem 1.5 there exist a neighborhood U of z_2 in P_q and a neighborhood W of $\eta_{\hat{\xi}_\alpha}(z_2)$ in L_q such that $\eta_{\hat{\xi}_\alpha} : U \to W$ is a diffeomorphism. Let $\xi_- := \frac{\partial f}{\partial s}(1,z)$ be a vector field along W, where $f(s,z) := \exp_z(s\hat{\xi}_\alpha)$. It follows from Theorem 1.5 that ξ_- is a normal foliated vector field. Since $z_2 \in \partial A$, we can conclude that $\xi_- = -\xi$ and hence $z_2 \in A$. This implies that A is a closed set of P_{z_0}. Therefore $A = P_{z_0}$. This means that $z_0 \in \eta_\xi(L_p)$ and hence that $\eta_\xi(L_p)$ is a closed set in L_q.

Now we want to prove that $\eta_\xi : L_p \to L_q$ is a covering map for a regular point q. For a plaque P_z consider all points $x_\alpha \in L$ so that $\eta_\xi(x_\alpha) = z$. For each x_α let $\hat{\xi}_\alpha$ be the vector in T_zM tangent to the geodesic $\exp_{x_\alpha}(t\,\xi)$ so that $\exp_z(\hat{\xi}_\alpha) = x_\alpha$. Extend $\hat{\xi}_\alpha$ to a normal foliated vector field along P_z and set $W_\alpha = \eta_{\hat{\xi}_\alpha}(P_z)$. As we have seen above, $\frac{\partial f}{\partial s}(1,\tilde{z}) = -\xi(f(1,\tilde{z}))$ where $f(s,\tilde{z}) := \exp_{\tilde{z}}(s\hat{\xi}_\alpha)$ for $\tilde{z}$ in a neighborhood of z. This fact and the fact that L_p has trivial holonomy imply that

$\eta_\xi : W_\alpha \to P_z$ is a diffeomorphism with inverse $\eta_{\xi_\alpha} : P_z \to W_\alpha$. Finally, note that $\eta_\xi^{-1}(P_z) = \bigcup_\alpha W_\alpha$. We conclude that $\eta_\xi : L_p \to L_q$ is a covering map.

At last, suppose that L_q is a singular leaf.

For a point $z_0 \in L_q$ assume that there exists a point $z_1 \in \eta_\xi(L_p)$ which also belongs to the plaque P_{z_0}. There exists $x_1 \in L_p$ such that $z_1 = \eta_\xi(x_1) \in P_{z_0}$. We can find an $s < 1$ such that $y_1 = \eta_{s\,\xi}(x_1)$ is a regular point. Since y_1 is a regular point, the plaque P_{y_1} is an open set of $\eta_{s\,\xi}(L_p)$. There exists a parallel normal field $\hat{\xi}$ along P_{y_1} such that $\eta_{\hat{\xi}} \circ \eta_{s\,\xi} = \eta_\xi$. It follows that $\eta_{\hat{\xi}}(P_{y_1}) \subset P_{z_0}$. On the other hand, since the foliation is singular, the plaque P_{y_1} intersects the slice S_{z_0}. These two facts imply that $z_0 \in \eta_{\hat{\xi}}(P_{y_1})$. Therefore $z_0 \in \eta_\xi(L_p)$, and hence $\eta_\xi(L_p)$ is a closed set in L_q.

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