

LIE GROUPS LOCALLY ISOMORPHIC TO GENERALIZED HEISENBERG GROUPS

HIROSHI TAMARU AND HISASHI YOSHIDA

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ABSTRACT. We classify connected Lie groups which are locally isomorphic to generalized Heisenberg groups. For a given generalized Heisenberg group N , there is a one-to-one correspondence between the set of isomorphism classes of connected Lie groups which are locally isomorphic to N and a union of certain quotients of noncompact Riemannian symmetric spaces.

1. INTRODUCTION

Generalized Heisenberg groups were introduced by Kaplan ([5]) and have been studied in many fields in mathematics. Especially, generalized Heisenberg groups, endowed with the left-invariant metric, provide beautiful examples in geometry. They provide many examples of symmetric-like Riemannian manifolds, that is, Riemannian manifolds which share some properties with Riemannian symmetric spaces (see [1] and the references). Compact quotients of some generalized Heisenberg groups provide isospectral but nonisometric manifolds (see Gordon [3] and the references). Generalized Heisenberg groups also provide remarkable examples of solvmanifolds by taking a certain solvable extension. The constructed solvmanifolds are now called Damek-Ricci spaces, since Damek-Ricci ([2]) showed that they are harmonic spaces (see also [1]).

The aim of this paper is to classify connected Lie groups which are locally isomorphic to generalized Heisenberg groups. They provide interesting examples of nilmanifolds, that is, Riemannian manifolds on which nilpotent Lie groups act isometrically and transitively. They might be a good prototype for the study of non-simply-connected and noncompact nilmanifolds. This is one of our motivations.

Other motivations come from the interesting structure of the moduli space, the set of isomorphism classes of connected Lie groups locally isomorphic to a generalized Heisenberg group. If one thinks about abelian Lie groups or classical Lie groups, the moduli space seems to be a finite set. But, in the case of a generalized Heisenberg group N , there is a continuous family of connected Lie groups which are locally isomorphic to the given N (except for the classical Heisenberg groups).

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Our main theorem states that the moduli space is bijective to

$$\prod_{r=0}^m (\mathrm{SL}_r(\mathbb{Z}) \backslash \mathrm{SL}_r(\mathbb{R}) / \mathrm{SO}_r(\mathbb{R})),$$

where m is the dimension of the center $Z(N)$. It seems surprising that the moduli spaces depend only on $\dim Z(N)$.

This paper is organized as follows. In Section 2, we study the structure of the automorphism group of a metric Lie algebra and of a two-step nilpotent metric Lie algebra. We recall the definition of a generalized Heisenberg group and describe the action of the automorphism group on the center in Section 3. In particular, this action is equivalent to the action of the conformal group $\mathrm{CO}_m(\mathbb{R})$. Our main theorem is proved in Section 4. For the proof, we have to classify lattices in \mathbb{R}^m up to $\mathrm{CO}_m(\mathbb{R})$ -conjugation. In the case of $\dim Z(N) = 2$, the component $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R})$ is the quotient of the real hyperbolic plane $\mathbb{R}H^2$ by the modular group $\mathrm{SL}_2(\mathbb{Z})$, which has been well studied. By using the structure theorem of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{R}H^2$, in Section 5 we give explicit descriptions of connected Lie groups which are locally isomorphic to N in this case.

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2. PRELIMINARIES ON AUTOMORPHISM GROUPS

For the classification of connected Lie groups locally isomorphic to a given simply-connected Lie group G , we need to know the action of the automorphism group on the center of G . In this section we study the structure of the automorphism group of a two-step nilpotent metric Lie algebra.

We start with an arbitrary metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, a Lie algebra with an inner product. Let \mathfrak{z} be the center of \mathfrak{g} and \mathfrak{v} the orthogonal complement of \mathfrak{z} in \mathfrak{g} . Since the automorphism group $\mathrm{Aut}(\mathfrak{g})$ preserves \mathfrak{z} , we first study the structure of

$$N_{\mathrm{GL}(\mathfrak{g})}(\mathfrak{z}) := \{\sigma \in \mathrm{GL}(\mathfrak{g}); \sigma(\mathfrak{z}) \subset \mathfrak{z}\}.$$

Let us define an additive group

$$\mathrm{Hom}(\mathfrak{v}, \mathfrak{z}) := \{\beta : \mathfrak{v} \rightarrow \mathfrak{z} : \text{linear}\}.$$

Each element $\beta \in \mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$ acts on \mathfrak{g} by T_β in the natural way, that is,

$$T_\beta(z) := z, \quad T_\beta(v) := \beta(v) + v, \quad \text{for } z \in \mathfrak{z} \text{ and } v \in \mathfrak{v}.$$

Lemma 2.1. $N_{\mathrm{GL}(\mathfrak{g})}(\mathfrak{z}) = \mathrm{Hom}(\mathfrak{v}, \mathfrak{z}) \rtimes (\mathrm{GL}(\mathfrak{z}) \times \mathrm{GL}(\mathfrak{v}))$.

Proof. Let $(h, g) \in \mathrm{GL}(\mathfrak{z}) \times \mathrm{GL}(\mathfrak{v})$ and $\beta \in \mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$. It is easy to see that $(h, g)^{-1} \circ T_\beta \circ (h, g) = T_{h^{-1} \circ \beta \circ g}$. Thus $\mathrm{GL}(\mathfrak{z}) \times \mathrm{GL}(\mathfrak{v})$ normalizes $\mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$ and the inclusion “ \supset ” is clear. To show the converse, let $T \in N_{\mathrm{GL}(\mathfrak{g})}(\mathfrak{z})$. Since T preserves \mathfrak{z} , there exist $h \in \mathrm{GL}(\mathfrak{z})$, $g \in \mathrm{GL}(\mathfrak{v})$ and $\beta \in \mathrm{Hom}(\mathfrak{v}, \mathfrak{z})$ such that

$$T(z) = h(z), \quad T(v) = \beta(v) + g(v).$$

It concludes that $T = T_{\beta \circ g^{-1}} \circ (h, g) \in \mathrm{Hom}(\mathfrak{v}, \mathfrak{z}) \rtimes (\mathrm{GL}(\mathfrak{z}) \times \mathrm{GL}(\mathfrak{v}))$. \square

Let us now consider two-step nilpotent metric Lie algebras $(\mathfrak{n}, \langle, \rangle)$. In this case $\text{Aut}(\mathfrak{n})$ can be decomposed into two groups, according to Lemma 2.1. We need

$$\text{Aut}_{\mathfrak{v}}(\mathfrak{n}) := \text{Aut}(\mathfrak{n}) \cap (\text{GL}(\mathfrak{z}) \times \text{GL}(\mathfrak{v})).$$

Lemma 2.2. *For a two-step nilpotent metric Lie algebra $(\mathfrak{n}, \langle, \rangle)$, we have*

- (1) $\text{Hom}(\mathfrak{v}, \mathfrak{z}) \subset \text{Aut}(\mathfrak{n})$, and therefore $\text{Aut}(\mathfrak{n}) = \text{Hom}(\mathfrak{v}, \mathfrak{z}) \rtimes \text{Aut}_{\mathfrak{v}}(\mathfrak{n})$, and
- (2) $\text{Aut}_{\mathfrak{v}}(\mathfrak{n}) = \{(h, g) \in \text{GL}(\mathfrak{z}) \times \text{GL}(\mathfrak{v}); [gu, gv] = h[u, v] \ (\forall u, v \in \mathfrak{v})\}$.

Proof. Since \mathfrak{n} is two-step nilpotent, one can immediately see that $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$. This easily yields that $\text{Hom}(\mathfrak{v}, \mathfrak{z}) \subset \text{Aut}(\mathfrak{n})$. Lemma 2.1 states that $\text{Aut}(\mathfrak{n}) \subset \text{Hom}(\mathfrak{v}, \mathfrak{z}) \rtimes (\text{GL}(\mathfrak{z}) \times \text{GL}(\mathfrak{v}))$. We can conclude (1) by the following group-theoretic property: let $K \subset H_1 \rtimes H_2$ and $H_1 \subset K$; then $K = H_1 \rtimes (H_2 \cap K)$ holds. The claim (2) is an easy consequence of the assumption that \mathfrak{n} is two-step nilpotent. \square

We note that Lemma 2.2 is obtained by Saal ([7, Proposition 2.3]) for generalized Heisenberg algebras.

3. AUTOMORPHISM GROUPS OF H -TYPE GROUPS

In this section, we recall the definition of a generalized Heisenberg algebra $(\mathfrak{n}, \langle, \rangle)$ and describe the action of the automorphism group $\text{Aut}(\mathfrak{n})$ on the center of \mathfrak{n} .

Definition 3.1. A two-step nilpotent metric Lie algebra $(\mathfrak{n}, \langle, \rangle)$ is called a *generalized Heisenberg algebra* or an *H-type algebra* if

$$J_z^2 = -\langle z, z \rangle \text{id}_{\mathfrak{v}} \quad (\forall z \in \mathfrak{z}),$$

where the operator $J : \mathfrak{z} \rightarrow \text{End}(\mathfrak{v})$ is defined by

$$\langle J_z v, u \rangle = \langle z, [v, u] \rangle \quad \text{for } z \in \mathfrak{z} \text{ and } u, v \in \mathfrak{v}.$$

The connected and simply-connected Lie group N with Lie algebra \mathfrak{n} , endowed with the induced left-invariant metric, is called a *generalized Heisenberg group* or an *H-type group*.

We refer the reader to [5] and [1] for H -type groups and algebras.

We would like to know the action of the automorphism group $\text{Aut}(N)$ on the center $Z(N)$ of an H -type group N . Since N is simply-connected and the exponential map $\exp : \mathfrak{n} \rightarrow N$ is a diffeomorphism, it is sufficient to investigate the action of the automorphism group $\text{Aut}(\mathfrak{n})$ on the center \mathfrak{z} of \mathfrak{n} . Note that the following proposition can be obtained as a corollary of the description of the automorphism group $\text{Aut}(\mathfrak{n})$ by Saal ([7]). But we will give a proof here, since we determine the action of $\text{Aut}(\mathfrak{n})$ on \mathfrak{z} directly and hence the arguments become slightly simpler. Denote the conformal group by

$$\text{CO}(\mathfrak{z}) := \{dg \in \text{GL}(\mathfrak{z}) \mid d \in \mathbb{R}^\times, g \in \text{O}(\mathfrak{z})\}.$$

Proposition 3.2. *For an H-type algebra $(\mathfrak{n}, \langle, \rangle)$, the action of $\text{Aut}(\mathfrak{n})$ on the center \mathfrak{z} is equivalent to the action of the conformal group $\text{CO}(\mathfrak{z})$ on \mathfrak{z} .*

Proof. First of all, Lemma 2.2 states that $\text{Aut}(\mathfrak{n}) = \text{Hom}(\mathfrak{v}, \mathfrak{z}) \rtimes \text{Aut}_{\mathfrak{v}}(\mathfrak{n})$. By definition, $\text{Hom}(\mathfrak{v}, \mathfrak{z})$ acts trivially on \mathfrak{z} ; therefore the actions of $\text{Aut}(\mathfrak{n})$ and $\text{Aut}_{\mathfrak{v}}(\mathfrak{n})$ coincide on \mathfrak{z} . Hence, the group we would like to know is nothing but

$$D := \{h \in \text{GL}(\mathfrak{z}); \exists g \in \text{GL}(\mathfrak{v}) : (h, g) \in \text{Aut}_{\mathfrak{v}}(\mathfrak{n})\}.$$

For the proof of $D = \text{CO}(\mathfrak{z})$, we need the following groups:

- Let $\text{Cliff}(\mathfrak{n})$ denote the subgroup of $\text{Aut}(\mathfrak{n})$ generated by $\{(-|z|^2\rho_z, J_z); z \in \mathfrak{z} - \{0\}\}$, where ρ_z is the reflection in \mathfrak{z} with respect to $(\mathbb{R}z)^\perp$.
- Let $\text{Pin}(\mathfrak{n})$ be the subgroup of $\text{Cliff}(\mathfrak{n})$ generated by $\{(-\rho_z, J_z); |z| = 1\}$.

Note that $(-|z|^2\rho_z, J_z) \in \text{GL}(\mathfrak{z}) \times \text{GL}(\mathfrak{v})$ defines an automorphism of \mathfrak{n} . Let $l := \dim \mathfrak{v}$, $m := \dim \mathfrak{z}$, and $\{z_1, \dots, z_m\}$ be an orthonormal basis of \mathfrak{z} .

Claim 1: the action of $\text{Cliff}(\mathfrak{n})$ on \mathfrak{z} coincides with that of $\text{CO}(\mathfrak{z})$. In case m is even, the action of $\text{Pin}(\mathfrak{n})$ on \mathfrak{z} coincides with that of $\text{O}(\mathfrak{z})$, since $\det(-\rho_z) = -1$. Therefore $\text{Cliff}(\mathfrak{n})$ acts on \mathfrak{z} as $\text{CO}(\mathfrak{z})$. In case m is odd, the action of $\text{Pin}(\mathfrak{n})$ on \mathfrak{z} coincides with that of $\text{SO}(\mathfrak{z})$, since $\det(-\rho_z) = 1$. Thus $\text{Cliff}(\mathfrak{n})$ acts on \mathfrak{z} as $\mathbb{R}^\times \cdot \text{SO}(\mathfrak{z})$, but m is odd, which implies $\mathbb{R}^\times \cdot \text{SO}(\mathfrak{z}) = \mathbb{R}^\times \cdot \text{O}(\mathfrak{z}) = \text{CO}(\mathfrak{z})$.

Claim 2: $D \supset \text{CO}(\mathfrak{z})$. This is a direct consequence of Claim 1.

Claim 3: $D \subset \text{CO}(\mathfrak{z})$. Let us take $h \in D$. By definition, there exists $g \in \text{GL}(\mathfrak{v})$ such that $(h, g) \in \text{Aut}_{\mathfrak{v}}(\mathfrak{n})$. Elementary linear algebra yields that

$$\exists k_1, k_2 \in \text{O}(\mathfrak{z}) \text{ such that } k_1 h k_2 = \text{diag}(d_1, \dots, d_m) \text{ and } d_i > 0.$$

Put $h_0 := \text{diag}(d_1, \dots, d_m)$, which is the diagonal matrix (note that we consider matrix representation with respect to the basis $\{z_1, \dots, z_m\}$). Since Claim 2 implies that $\text{Cliff}(\mathfrak{n})$ has a subgroup which acts on \mathfrak{z} as $\text{O}(\mathfrak{z})$, there exists $g_1, g_2 \in \text{GL}(\mathfrak{v})$ such that $(k_1, g_1), (k_2, g_2) \in \text{Cliff}(\mathfrak{n}) \subset \text{Aut}_{\mathfrak{v}}(\mathfrak{n})$. Let $g_0 := g_1 g g_2$. One has

$$(h_0, g_0) = (k_1, g_1)(h, g)(k_2, g_2) \in \text{Aut}_{\mathfrak{v}}(\mathfrak{n}).$$

The condition for (h_0, g_0) to be an automorphism is

$$[g_0 u, g_0 v] = h_0[u, v] \quad (\forall u, v \in \mathfrak{v}).$$

By taking the inner product with z_i ($i = 1, \dots, m$), we have

$$\langle z_i, [g_0 u, g_0 v] \rangle = \langle z_i, h_0[u, v] \rangle \quad (\forall u, v \in \mathfrak{v}).$$

This yields that ${}^t g_0 J_{z_i} g_0 = d_i J_{z_i}$. By taking the determinant of each side, one has $(\det g_0)^2 = d_i^l$. This means that d_i is independent of i (recall that $d_i > 0$). Therefore $h_0 = d \cdot \text{id}_{\mathfrak{z}}$, where $d := d_1 = d_2 = \dots = d_m$. We conclude that $h = d k_1^{-1} k_2^{-1} \in \text{CO}(\mathfrak{z})$, which completes the proof. \square

4. MAIN THEOREM

In this section, we classify connected Lie groups which are locally isomorphic to H -type groups N . We refer the reader to [4], [8] for the theory of Lie groups and homogeneous spaces.

Let G be a connected and simply-connected Lie group, and denote it by

$$LI(G) := \{G'; G' \text{ is a connected Lie group and locally isomorphic to } G\}/\text{isom}.$$

An arbitrary connected Lie group can be represented by G/Γ , where G is a connected and simply-connected Lie group and Γ is a discrete subgroup contained in the center $Z(G)$ of G . Furthermore, G/Γ_1 and G/Γ_2 are isomorphic to each other if and only if Γ_1 and Γ_2 are conjugate by $\text{Aut}(G)$. Therefore, one has

$$LI(G) \cong \{\Gamma \subset Z(G); \Gamma \text{ is a discrete subgroup}\}/\text{Aut}(G).$$

We now study H -type groups N . Denote the Lie algebra of N by $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where \mathfrak{z} is the Lie algebra of $Z(N)$. Via a Lie group isomorphism $\exp : \mathfrak{z} \rightarrow Z(N)$, discrete

subgroups in $Z(N)$ correspond to lattices in $\mathfrak{z} \cong \mathbb{R}^m$, where $m = \dim \mathfrak{z}$. Note that $\text{Aut}(N) \cong \text{Aut}(\mathfrak{n})$, since N is simply-connected. Therefore we have that

$$LI(N) \cong \{\Lambda \subset \mathfrak{z}; \Lambda \text{ is a lattice}\} / \text{Aut}(\mathfrak{n}).$$

Hence, by Proposition 3.2,

$$LI(N) \cong \{\Lambda \subset \mathfrak{z}; \Lambda \text{ is a lattice}\} / \text{CO}(\mathfrak{z}).$$

Theorem 4.1. *Let N be an H -type group with m -dimensional center. The moduli space $LI(N)$ of isomorphism classes of connected Lie groups which are locally isomorphic to N satisfies*

$$LI(N) \cong \prod_{r=0}^m (\text{SL}_r(\mathbb{Z}) \backslash \text{SL}_r(\mathbb{R}) / \text{SO}_r(\mathbb{R})),$$

where we define $\text{SL}_0(\mathbb{R}) \backslash \text{SL}_0(\mathbb{Z}) / \text{SO}_0(\mathbb{R}) = \{0\}$.

Proof. We fix an orthonormal basis $\{z_1, \dots, z_m\}$ of \mathfrak{z} . Then one can identify $\mathfrak{z} = \mathbb{R}^m$, and thus

$$LI(N) = \{\Lambda \subset \mathbb{R}^m; \Lambda \text{ is a lattice}\} / \text{CO}_m(\mathbb{R}).$$

Here we define

$$LI_r(N) := \{\Lambda \subset \mathbb{R}^m; \Lambda \text{ is a lattice of rank } r\} / \text{CO}_m(\mathbb{R}).$$

On the other hand, the natural inclusion map $\text{SL}_r(\mathbb{R}) \rightarrow \text{GL}_r(\mathbb{R})$ induces

$$\text{SL}_r(\mathbb{Z}) \backslash \text{SL}_r(\mathbb{R}) / \text{SO}_r(\mathbb{R}) \cong \text{GL}_r(\mathbb{Z}) \backslash \text{GL}_r(\mathbb{R}) / \text{CO}_r(\mathbb{R}),$$

since $\text{CO}_r(\mathbb{R}) \cap \text{SL}_r(\mathbb{R}) = \text{SO}_r(\mathbb{R})$ and $\text{GL}_r(\mathbb{Z}) \cap \text{SL}_r(\mathbb{R}) = \text{SL}_r(\mathbb{Z})$. We will show the theorem by constructing a bijective map from $\text{GL}_r(\mathbb{Z}) \backslash \text{GL}_r(\mathbb{R}) / \text{CO}_r(\mathbb{R})$ to $LI_r(N)$. First of all, we define

$$\varphi : \text{GL}_r(\mathbb{R}) \rightarrow \{\Lambda \subset \mathbb{R}^m; \text{rank } \Lambda = r\} : \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} \mapsto \mathbb{Z}\tilde{u}_1 + \dots + \mathbb{Z}\tilde{u}_r,$$

where $u_i \in M_{1,r}(\mathbb{R})$ and $\tilde{u}_i := (u_i, 0, \dots, 0) \in M_{1,m}(\mathbb{R})$. Denote by $\tilde{\varphi}$ the induced map

$$\tilde{\varphi} : \text{GL}_r(\mathbb{Z}) \backslash \text{GL}_r(\mathbb{R}) / \text{CO}_r(\mathbb{R}) \rightarrow LI_r(N) : u \mapsto [\varphi(u)],$$

where $[\varphi(u)]$ is the $\text{CO}_m(\mathbb{R})$ -conjugate class of the lattice $\varphi(u)$.

Claim 1: $\tilde{\varphi}$ is well-defined. Let $u \in \text{GL}_r(\mathbb{R})$, $C \in \text{GL}_r(\mathbb{Z})$ and $g \in \text{CO}_r(\mathbb{R})$. One has $[\varphi(Cug)] = [\varphi(Cu)]$, since $\text{CO}_r(\mathbb{R})$ is a subgroup of $\text{CO}_m(\mathbb{R})$ naturally. It can also be easily checked that $\varphi(u) = \varphi(v)$ (that is, they are the same lattice) if and only if $u = Cv$ for some $C \in \text{GL}_r(\mathbb{Z})$. Hence one has $\varphi(Cu) = \varphi(u)$. We conclude that $[\varphi(Cug)] = [\varphi(u)]$.

Claim 2: $\tilde{\varphi}$ is surjective. Let $\Lambda = \mathbb{Z}u'_1 + \dots + \mathbb{Z}u'_r$ be a lattice so that $[\Lambda] \in LI_r(N)$. Then there exists $g \in \text{CO}_m(\mathbb{R})$ such that $u'_i g = (u_i, 0, \dots, 0)$ (for every $i = 1, \dots, r$). Since $\text{rank } \Lambda = r$, one has

$$u := \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} \in \text{GL}_r(\mathbb{R})$$

and $\varphi(u)g^{-1} = \Lambda$. We conclude that $[\varphi(u)] = [\Lambda]$.

Claim 3: $\tilde{\varphi}$ is injective. Let $u, v \in \text{GL}_r(\mathbb{R})$, and assume that $[\varphi(u)] = [\varphi(v)]$. Then there exists $g \in \text{CO}_m(\mathbb{R})$ such that $\varphi(u) = \varphi(v)g$. Since all the bases of these lattices are of the form $(u_i, 0, \dots, 0)$, one can see that there exists $g' \in \text{CO}_r(\mathbb{R})$ such that $\varphi(u) = \varphi(v)g' = \varphi(vg')$. They are the same lattice, which implies that $u = Cv g'$ for some $C \in \text{GL}_r(\mathbb{Z})$. We conclude that u and v are in the same equivalent class. \square

5. THE CASE OF TWO DIMENSIONAL CENTER

In this section we explicitly describe the classification of Lie groups which are locally isomorphic to H -type groups with two dimensional center.

It is known that an H -type group N with two dimensional center is the complex Heisenberg group, that is,

$$N = \left\{ \begin{bmatrix} 1 & \alpha_1 & \cdots & \alpha_k & \gamma \\ & 1 & & & \beta_k \\ & & \ddots & & \vdots \\ & & & 1 & \beta_1 \\ & & & & 1 \end{bmatrix} ; \alpha_i, \beta_i, \gamma \in \mathbb{C} \right\}.$$

We denote the element in the center \mathfrak{z} of the Lie algebra \mathfrak{n} , for $\gamma \in \mathbb{C}$, by

$$u(\gamma) := \begin{bmatrix} 0 & \cdots & 0 & \gamma \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix}.$$

Obviously $\{u(1), u(i)\}$ is an orthonormal basis of \mathfrak{z} . Define

$$D_1 := \{\rho \in \mathbb{C}; \text{Im } \rho > 0, |\rho| \leq 1, |\rho - 1| \geq 1, |\rho + 1| \geq 1\}.$$

Theorem 5.1. *Let N be a complex Heisenberg group. A connected (real) Lie group which is locally isomorphic to N is isomorphic to either*

- (1) N ,
- (2) $N/\mathbb{Z} \exp u(1)$, or
- (3) $N/\mathbb{Z} \exp u(1) \times \mathbb{Z} \exp u(\rho)$ with $\rho \in D_1$.

For $\rho, \rho' \in D_1$, two Lie groups $N/\mathbb{Z}u(1) \times \mathbb{Z} \exp u(\rho)$ and $N/\mathbb{Z}u(1) \times \mathbb{Z} \exp u(\rho')$ are isomorphic if and only if

- (i) $\{\rho, \rho'\} = \{ie^{i\theta}, ie^{-i\theta}\}$ for $0 \leq \theta \leq \pi/6$, or
- (ii) $\{\rho, \rho'\} = \{e^{i\theta} - 1, -e^{-i\theta} + 1\}$ for $0 < \theta \leq \pi/3$.

Proof. Theorem 4.1 states that the moduli space of connected Lie groups which are locally isomorphic to N is bijective to

$$\prod_{r=0}^2 (\text{SL}_r(\mathbb{Z}) \backslash \text{SL}_r(\mathbb{R}) / \text{SO}_r(\mathbb{R})).$$

Note that $\text{SL}_0(\mathbb{Z}) \backslash \text{SL}_0(\mathbb{R}) / \text{SO}_0(\mathbb{R}) = \{\text{pt}\} = \{N\}$, which corresponds to the simply-connected one. Furthermore, $\text{SL}_1(\mathbb{Z}) \backslash \text{SL}_1(\mathbb{R}) / \text{SO}_1(\mathbb{R}) = \{\text{pt}\}$, which means that $N/\mathbb{Z}u(\rho)$ are isomorphic to each other for all $\rho \in \mathbb{C}$. Therefore, in this case, it is isomorphic to $N/\mathbb{Z}u(1)$.

Now we have to study $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R})$. It is known that $\mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R}) = \mathbb{R}^+ \cdot \mathrm{SL}_2(\mathbb{R}) / \mathrm{CO}_2^+(\mathbb{R})$ is bijective to the real hyperbolic plane (or the upper half complex plane)

$$\mathbb{R}H^2 := \{z \in \mathbb{C} ; \mathrm{Im} z > 0\}.$$

Note that the bijective correspondence is induced from the following linear fractional transformation:

$$p : \mathbb{R}^+ \cdot \mathrm{SL}_2(\mathbb{R}) \rightarrow \mathbb{R}H^2 : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (ai + b)/(ci + d).$$

Furthermore, note that the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}^+ \cdot \mathrm{SL}_2(\mathbb{R}) / \mathrm{CO}_2^+(\mathbb{R})$ induced from the left action on $\mathbb{R}^+ \cdot \mathrm{SL}_2(\mathbb{R})$ is equivalent to the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}H^2$ via the linear fractional transformation.

It is well known (see [6], [9], for example) that the representative of the orbit under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}H^2$ can be taken as

$$D_2 := \{z \in \mathbb{R}H^2; |z| \geq 1, -1/2 \leq \mathrm{Re} z \leq 1/2\}.$$

Note that $z, z' \in D_2$ are in the same $\mathrm{SL}_2(\mathbb{Z})$ -orbit if and only if

- (i)' $\{z, z'\} = \{ie^{i\theta}, ie^{-i\theta}\}$ for $0 \leq \theta \leq \pi/6$, or
- (ii)' $\{z, z'\} = \{1/2 + iy, -1/2 + iy\}$ for $y \geq \sqrt{3}/2$.

We translate these results into matrices via the linear fractional transformation p . Direct calculations show that

$$p \left(\begin{bmatrix} 1 & 0 \\ r \cos \theta & r \sin \theta \end{bmatrix} \right) = (1/r)e^{i\theta}.$$

Therefore the $\mathrm{SL}_2(\mathbb{Z})$ -orbit in $\mathbb{R}^+ \cdot \mathrm{SL}_2(\mathbb{R}) / \mathrm{CO}_2^+(\mathbb{R})$ has to meet $p(D_3)$, where $D_3 \subset \mathbb{R}^+ \cdot \mathrm{SL}_2(\mathbb{R})$ is defined by

$$D_3 := \left\{ \begin{bmatrix} 1 & 0 \\ r \cos \theta & r \sin \theta \end{bmatrix} ; 0 < \theta < \pi, 0 < r \leq 1, -1/2 \leq (1/r) \cos \theta \leq 1/2 \right\}.$$

The above matrix represents the lattice $\mathbb{Z}u(1) + \mathbb{Z}u(re^{i\theta})$ in \mathfrak{z} and hence represents the discrete subgroup $\mathbb{Z} \exp u(1) \times \mathbb{Z} \exp u(re^{i\theta})$ in $Z(N)$. The domain D_1 can be obtained by the expression of D_3 , and the equivalence conditions (i) and (ii) come from the conditions (i)' and (ii)', respectively. \square

REFERENCES

1. J. Berndt, F. Tricerri, L. Vanhecke, *Generalized Heisenberg groups and Damek-Ricci harmonic spaces*. Lecture Notes in Mathematics, **1598**. Springer-Verlag, Berlin, 1995. MR1340192 (97a:53068)
2. E. Damek, F. Ricci, *Harmonic analysis on solvable extensions of H -type groups*. J. Geom. Anal. **2** (1992), no. 3, 213–248. MR1164603 (93d:43006)
3. C. Gordon, *Isospectral closed Riemannian manifolds which are not locally isometric*. J. Differential Geom. **37** (1993), no. 3, 639–649. MR1217163 (94b:58098)
4. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*. Graduate Studies in Mathematics, **34**. Amer. Math. Soc., Providence, RI, 2001. MR1834454 (2002b:53081)
5. A. Kaplan, *Riemannian nilmanifolds attached to Clifford modules*. Geom. Dedicata **11** (1981), no. 2, 127–136. MR621376 (82h:22008)
6. D. Mumford, *Tata lectures on theta. I*. With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman. Progress in Mathematics, **28**. Birkhäuser Boston, Inc., Boston, MA, 1983. MR688651 (85h:14026)
7. L. Saal, *The automorphism group of a Lie algebra of Heisenberg type*. Rend. Sem. Mat. Univ. Politec. Torino **54** (1996), no. 2, 101–113. MR1490017 (99c:17033)

8. M. Sugiura, *Theory of Lie Groups*. (in Japanese), Kyoritsu Shuppan Co., Ltd., Tokyo, 2000.
9. H. Umemura, *Theory of Elliptic Functions: Analysis of Elliptic Curves* (in Japanese), University of Tokyo Press, 2000.

DEPARTMENT OF MATHEMATICS, HIROSHIMA UNIVERSITY, HIGASHI-HIROSHIMA 739-8526, JAPAN
E-mail address: `tamaru@math.sci.hiroshima-u.ac.jp`

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, CHIKUSA-KU, NAGOYA 464-8602,
JAPAN