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CHEEGER'S CONSTANT IN BALLS AND ISOPERIMETRIC INEQUALITY ON RIEMANNIAN MANIFOLDS

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ABSTRACT. We prove isoperimetric inequality on a Riemannian manifold, assuming that the Cheeger constant for balls satisfies a certain estimation.

1. Introduction and the main result

The isoperimetric property of a ball, which maximizes the volume for a given surface area, was well known to the ancient Greeks. Nowadays, isoperimetric inequalities have also become a part of Analysis, due to seminal contributions of Lord Rayleigh [12], Faber [4], Krahn [7], Pólya and Szegö [11] et al. For a recent account of isoperimetric inequalities in relation to Analysis, see Chavel [2].

Let M be a Riemannian manifold of dimension n and let $\Omega \subset M$ be an open set with smooth boundary. Denote by $V(\Omega)$ the n-dimensional Riemannian volume of Ω, and by $A(\partial\Omega)$ the $(n-1)$-dimensional Riemannian area of $\partial\Omega$. Let $\rho(p, q)$ be a distance on M, and let $\rho_p(q)$ be the C^2 distance function from the point $p \in M$, such that $|\nabla\rho_p| \leq 1$ for all $p \in M$.

Denote by $B_r(p)$ the open ball centered at $p \in M$ of radius $r > 0$. The Cheeger constant $\overline{h}(B)$ of a ball $B = B_r(p)$ is defined by

$$\overline{h}(B) := \inf_{\Omega \subset B} \frac{A(\partial\Omega)}{V(\Omega)}, \tag{1.1}$$

where inf is taken over all open sets $\Omega \subset B$ with smooth boundaries and compact closure. We say that a function $h(r)$ on an interval $(0, r_{max})$ is a *Cheeger function* if it is positive, decreasing, and for any $p \in M$ and $r \in (0, r_{max})$,

$$\overline{h}(B_r(p)) \geq h(r). \tag{1.2}$$

Here r_{max} is either a positive number or $+\infty$. Our main result is as follows.

Theorem 1.1. *Let M be a Riemannian manifold. Assume that M admits a Cheeger function,*

$$h(r) = \frac{n}{r} - h_0(r), \tag{1.3}$$

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where $h_0(r) \geq 0$ is a continuous function on $(0, r_{max})$, such that

$$J := \int_0^{r_{max}} h_0(r)\, dr < \infty. \tag{1.4}$$

Then, for any open set $\Omega \subset M$ with smooth boundary and compact closure, such that $V(\Omega) \leq v_{max}$, the following inequality takes place:

$$A(\partial\,\Omega) \geq CV(\Omega)^{1-\frac{1}{n}}, \tag{1.5}$$

where $C = 4^{-n}\left(\frac{3}{7}\omega_n\right)^{\frac{1}{n}} \exp\left(-\frac{J}{n}\right)$ and v_{max} is given by

$$v_{max} = \frac{3}{7}\omega_n r_{max}^n \exp\left(-J\right), \tag{1.6}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Remark 1.2. The case when M admits a Cheeger function $h(r) = \frac{n}{r}$ was considered by Chung, Grigor'yan and Yau [3]. They proved the isoperimetric inequality (1.5) using the method of Michael and Simon [9]; our proof is based on the same approach.

2. The proof

Proof. The proof is divided into three steps.

First step. Fix a open set $\Omega \subset M$ with smooth boundary and compact closure, a point $p \in M$ and define two functions

$$\begin{cases} m(r) &= V(\Omega \cap B_r), \\ s(r) &= A(\partial\,\Omega \cap B_r). \end{cases} \tag{2.1}$$

where $B_r = B_r(p)$.

Set $\Omega_r = \Omega \cap B_r$ and observe that

$$\partial\Omega_r = (\Omega \cap \partial B_r) \cup (\partial\Omega \cap B_r)\,. \tag{2.2}$$

Since $\Omega_r \subset B_r$, we have by definition of a Cheeger function,

$$h(r)\, V(\Omega_r) \leq A(\partial\,\Omega_r). \tag{2.3}$$

Clearly,

$$\begin{aligned} A(\partial\Omega_r) &\leq A(\Omega \cap \partial B_r) + A(\partial\Omega \cap B_r) \\ &= m'(r) + s(r), \end{aligned} \tag{2.4}$$

where $m'(r)$ is the derivative with respect to r. From (2.3) and (2.4) we obtain

$$h(r)\, m(r) \leq s(r) + m'(r) \tag{2.5}$$

or

$$-\left(m'(r) - h(r)\, m(r)\right) \leq s(r). \tag{2.6}$$

Fix some $a \in (0, r_{max})$ and set

$$H(r) := \int_a^r h(t) dt. \tag{2.7}$$

Note that the function $H(r)$ is a differentiable and increasing function defined on $[0, r_{max}]$ onto $[H_{min}, H_{max}]$, where

$$-\infty \leq H(0) =: H_{min} < H(a) = 0 < H(r_{max}) =: H_{max} \leq \infty.$$

We prove this as follows: we already know that $h(r)$ is a continuous function, so that $H'(r) = h(r) \neq 0$ for all $r \in (0, r_{max})$, and therefore by the Inverse

Function Theorem there exists an inverse function denoted by $H^{-1}(s)$ where $s \in [H_{min}, H_{max}]$. Moreover $H^{-1}(s)$ is a differentiable function in (H_{min}, H_{max}), with derivative given by

$$\left(H^{-1}\right)'(s) = \frac{1}{H'(H^{-1}(s))} = \frac{1}{h(H^{-1}(s))} > 0\,;$$

therefore $H^{-1}(s)$ is also an increasing function.

Multiplying by the factor $F(r) := \exp(-H(r))$ gives the differential inequality

$$-\frac{d}{dr}\left(F(r)\,m(r)\right) \leq F(r)\,s(r). \tag{2.8}$$

Second step. In the sequel we will use the following lemma, which was proved in [3].

Lemma 2.1. *Let $f(r)$ be an absolutely continuous positive function on $(0, \infty)$. Suppose that for some $r_0 \in (0, \infty)$ and all $r \in (0, r_0)$ the inequality is satisfied:*

$$-\frac{d}{dr}f(r) < \frac{1}{r_0}f(4r). \tag{2.9}$$

Then

$$\sup_{(0,\,r_0)} f < \frac{7}{3}\sup_{(r_0,\,4\,r_0)} f. \tag{2.10}$$

Assuming that $V(\Omega) < v_{max}$, we obtain

$$\ln\left(\frac{7}{3}V(\Omega)\,\omega^{-1}\right) < H_{max} := H(r_{max}), \tag{2.11}$$

where $\omega := a^n\omega_n\exp\left(-\int_0^a h_0(r)dr\right)$. Indeed we have

$$H_{max} := \int_a^{r_{max}} h(r)dr = n\ln\left(\frac{r_{max}}{a}\right) - \int_a^{r_{max}} h_0(r)dr, \tag{2.12}$$

whence

$$\omega\exp\left(H_{max}\right) = \omega_n r_{max}^n\exp\left(-J\right) = \frac{7}{3}v_{max}\,. \tag{2.13}$$

Therefore $V(\Omega) < v_{max}$ implies

$$V(\Omega) < \frac{3}{7}\omega\exp\left(H_{max}\right); \tag{2.14}$$

hence (2.11) follows. Therefore $\frac{7}{3}V(\Omega)\,\omega^{-1} \in [H_{min}, H_{max}]$, and we can set

$$r_0 := H^{-1}\left(\ln\left(\frac{7}{3}V(\Omega)\,\omega^{-1}\right)\right). \tag{2.15}$$

We will prove that there exists $r \in (0, r_0)$ such that

$$s(r) \geq \frac{1}{r_0}m(4r)\frac{F(4r)}{F(r)}. \tag{2.16}$$

Assume on the contrary that (2.16) is false. That means for all $r \in (0\ r_0)$,

$$F(r)\,s(r) < \frac{1}{r_0}F(4\,r)\,m(4\,r), \tag{2.17}$$

which implies by (2.8)

$$-\frac{d}{dr}\left(F(r)m(r)\right) < \frac{1}{r_0}F(4r)\,m(4r). \tag{2.18}$$

Applying Lemma 2.1 to $f(r) = F(r)m(r)$ we conclude that

$$\sup_{(0,\,r_0)} F(r)\,m(r) < \frac{7}{3}\sup_{(r_0,\,4\,r_0)} F(r)\,m(r). \tag{2.19}$$

By definition of F and H, we have

$$F(r) = \left(\frac{a}{r}\right)^n \exp\left(-\int_r^a h_0(r)d\,r\right). \tag{2.20}$$

Noting that $m(r) \sim \omega_n r^n$ as $r \to 0$, we obtain from (2.20)

$$\lim_{r\to 0} F(r)m(r) = \omega_n a^n \exp\left(-\int_0^a h_0(r)d\,r\right) = \omega. \tag{2.21}$$

Hence, by (2.19) and (2.21),

$$\omega < \frac{7}{3}\sup_{(0,\,r_0)} F(r)m(r). \tag{2.22}$$

On the other hand, $F(r)$ is decreasing and $m(r) \le V(\Omega)$, which implies

$$\sup_{(r_0,\,4r_0)} F(r)m(r) \le F(r_0)V(\Omega)\,. \tag{2.23}$$

Combining the above two lines, we obtain

$$\omega < \frac{7}{3}F(r_0)V(\Omega)\,. \tag{2.24}$$

However, by the choice of r_0,

$$F(r_0) = \exp\left(-H(r_0)\right) = \frac{3}{7}V(\Omega)^{-1}\,\omega, \tag{2.25}$$

whence $\frac{7}{3}V(\Omega)\,F(r_0) = \omega$, contradicting (2.24).

Third step. In the previous step we proved that there exists $r \in (0, r_0)$ such that

$$A(\partial\Omega \cap B_r) \ge \frac{1}{r_0}\left(\frac{F(4r)}{F(r)}\right)V(\Omega \cap B_{4r}). \tag{2.26}$$

Since $h(r) \le \frac{n}{r}$ it follows that

$$\frac{F(4r)}{F(r)} = \exp\left(-\int_r^{4r} h(r)d\,r\right) \ge 4^{-n}. \tag{2.27}$$

Then by (2.26) and (2.27) we obtain

$$A\left(\partial\Omega \cap B_r\right) \ge \frac{1}{4^n r_0}V\left(\Omega \cap B_{4r}\right). \tag{2.28}$$

Next we will use the following lemma, proved in [5] (see also [3] and [8]).

Lemma 2.2. *Let (M, d) be a metric space with countable base. Suppose that any point x from a set $\Omega \subset M$ is assigned a metric ball $B_{r_x}(x)$ of radius $r_x \in (0, r_0)$. Then there exists an (at most countable) set $S \subset \Omega$ such that all balls $B_{r_x}(x)$, $x \in S$, are disjoint, whereas the union of the balls $B_{4r_x}(x)$, $x \in S$, covers all the set Ω.*

Let $S = \{x_i\}$ be the countable set given by Lemma 2.2, and set $r_i = r_{x_i}$. We have by (2.26):

$$\begin{aligned} A(\partial\Omega) &\geq \sum_i A(\partial\Omega \cap B_{r_i}) \\ &\geq \sum_i \frac{1}{r_0}\frac{1}{4^n} V(\Omega \cap B_{4r_i}) \\ &\geq \frac{1}{r_0}\frac{1}{4^n} V(\Omega)\,. \end{aligned} \tag{2.29}$$

To conclude the proof of the theorem, we need to prove that

$$r_0 \leq K V(\Omega)^{\frac{1}{n}}, \tag{2.30}$$

where $K = \left(\frac{7}{3}\right)^{\frac{1}{n}} \omega_n^{-\frac{1}{n}} \exp\left(\frac{J}{n}\right)$.

Indeed we have by definition of r_0 (equation (2.15)),

$$\ln\left(\frac{7}{3}V(\Omega)\,\omega^{-1}\right) = H(r_0) = n \ln\frac{r_0}{a} - \int_a^{r_0} h_0(r)\,dr. \tag{2.31}$$

Therefore,

$$r_0 \leq a\left(\frac{7}{3}V(\Omega)\,\omega^{-1}\right)^{\frac{1}{n}} \exp\left(\frac{1}{n}\int_a^{r_0} h_0(r)\,dr\right). \tag{2.32}$$

Substituting ω, we obtain

$$r_0 \leq \left(\frac{7}{3}\right)^{\frac{1}{n}} \omega_n^{-\frac{1}{n}} \exp\left(\frac{J}{n}\right) V(\Omega)^{\frac{1}{n}}, \tag{2.33}$$

which completes the proof. □

3. Examples

Example 3.1 (Hadamard-Cartan manifolds)**.** A manifold M is called a Hadamard-Cartan (H-C) manifold, if and only if M is a geodesic complete manifold, simply connected, noncompact and with nonpositive sectional curvature.

Any H-C manifold (including $\mathbb{R}^n$ and $\mathbb{H}^n_\kappa$) admits as a Cheeger function $h(r) = \frac{n}{r}$. We prove this as follows: let M be an H-C manifold. Grigory'an [6] and García León [5] report that if M is an H-C manifold and ρ_p denotes a C^2 distance function from the point $p \in M$ such that $|\nabla\rho_p| \leq 1$, then its Laplacian satisfies

$$\triangle\rho_p \geq \frac{n-1}{\rho_p}. \tag{3.1}$$

Let r_p be the geodesic distance from the point $p \in M$. It is well known that $|\nabla r_p| = 1$; hence by (3.1),

$$\triangle r_p^2 = 2\left(r_p \triangle r_p + |\nabla r_p|^2\right) \geq 2\,n. \tag{3.2}$$

Let $r \in (0, r_{max})$ and $\Omega \subset B_r(p)$ be an open set with smooth boundary and compact closure. Hence by (3.2) we obtain

$$\int_\Omega \triangle r_p^2 dV \geq 2\,n\,V(\Omega)\,. \tag{3.3}$$

On the other hand, by the Green's formula,

$$2\,r\,A\,(\partial\Omega)\,\geq\,\int_{\Omega}\triangle r_p^2 dV; \tag{3.4}$$

thus from equations (3.3) and (3.4) we obtain

$$\overline{h}\,(B_r(p))\,\geq\,\frac{n}{r}. \tag{3.5}$$

Now let $r_{max} = \infty$ and $h_0(r) = 0, \forall\, r > 0$. Applying Theorem 1.1, we conclude that if M is an H-C manifold, then the inequality (1.5) is satisfied for all open sets Ω with smooth boundary and compact closure.

Example 3.2. If the curvature of M is allowed to be positive, one should expect a Cheeger function to be smaller than $\frac{n}{r}$. Indeed, $\mathbb{S}^2$ has a Cheeger function $h(r) = \cot\left(\frac{r}{2}\right),\ 0 < r < \pi$, which is smaller than $\frac{2}{r}$, but the function $h_0 = \frac{2}{r} - \cot\left(\frac{r}{2}\right)$ is bounded and hence satisfies our condition (1.4). A similar statement is true for $\mathbb{S}^n$.

Example 3.3 (Surfaces with bounded mean curvature)**.** Let M be a surface in $\mathbb{R}^n$, let $p,\, q \in M$ and let $d(p,\, q)$ be the extrinsic distance. We know that if ρ_p denotes the extrinsic distance function, then $\rho_p \leq r_p$, where r_p denotes the geodesic distance function from the point $p \in M$. Moreover it implies that $|\nabla \rho_p| \leq 1,\ \forall\, p \in M$.

If we denote by H the mean curvature vector and we assume that

$$\sup|H|\,\leq\,K, \tag{3.6}$$

where the sup runs on all points of M, $0 < K < \infty$ is a constant and $r_{max} > 0$ is an arbitrary and fixed number. Osserman [10] and Grigor'yan [6] proved that: If ρ_p is the extrinsic distance, shifting $x_1, \ldots, x_N$ coordinates in $\mathbb{R}^N$ and taking p as the origin, then

$$\begin{aligned}\triangle\rho_p^2 &= 2\left(\sum_{i=1}^{N} x_i H_i\,+\,\sum_{i=1}^{N}\left|\nabla x_i^2\right|\right)\\ &= 2\left(\sum_{i=1}^{N} x_i H_i\,+\,n\right),\end{aligned} \tag{3.7}$$

where H_i means coordinates of the mean curvature vector, and by computation we obtain that it is true that $\sum_{i=1}^{N}\left|\nabla x_i^2\right| = n$. By Schwarz inequality it is obtained that

$$\sum_{i=1}^{N} x_i H_i\,\leq\,\rho_p\,|H|\,, \tag{3.8}$$

whence

$$\begin{aligned}\triangle\rho_p^2 &\geq 2\left(n\,-\,\sum_{i=1}^{N} x_i\,H_i\right)\\ &\geq 2\,(n\,-\,\rho_p\,|H|)\,.\end{aligned} \tag{3.9}$$

We already know that $|H| \leq K$, where $K > 0$ is constant. By (3.9), we obtain

$$\triangle\,\rho_p^2\,\geq\,2\,(n\,-\,K\,\rho_p)\,. \tag{3.10}$$

Let us define $\varphi(r)$ by

$$\varphi(r) = n - K r, \quad 0 < r < r_{max},$$

where we take

$$r_{max} = \frac{n}{K} > 0.$$

We will construct a continuous function $h_0 : (0, r_{max}) \longrightarrow \mathbb{R}$ such that

- $h_0(r) \geq 0, \forall r \in (0, r_{max})$, and
- $\int_0^{r_{max}} h_0(r)\, dr \leq J$, for all $p \in M$, where $0 < J < \infty$ is a constant.

Note that the function $\varphi(r)$ is in fact a decreasing and continuous function. Let $r \in (0, r_{max})$, $p \in M$ and let $\Omega \subset B_r(P)$ be an open set with smooth boundary and compact closure. Hence integrating (3.10) on Ω, and by the fact that $\varphi(r)$ is a decreasing function,

$$\begin{aligned} \int_\Omega \triangle \rho_p^2 dV &\geq 2 \int_\Omega \varphi(\rho_p) dV \\ &\geq 2\varphi(r) V(\Omega). \end{aligned} \tag{3.11}$$

On the other hand, by the Green's formula,

$$2\, r\, A(\partial\Omega) \geq \int_\Omega \triangle \rho_p^2 dV. \tag{3.12}$$

Hence by (3.11) and (3.12), for all open sets with smooth boundary and compact closure, it is satisfied that

$$\frac{A(\partial\Omega)}{V(\Omega)} \geq \frac{\varphi(r)}{r}. \tag{3.13}$$

We conclude by definition of Cheeger's constant on the ball that

$$\overline{h}(B_r(p)) \geq \frac{\varphi(r)}{r} = \frac{n}{r} - K. \tag{3.14}$$

Let $h_0 : (0, r_{max}) \longrightarrow \mathbb{R}$ be the constant function $h_0(r) = K \geq 0$. Thus estimating its integral on $(0, r_{max})$,

$$\int_0^{r_{max}} h_0(r) dr = K\, r_{max} = n =: J < \infty. \tag{3.15}$$

Now by Theorem 1.1, if $V(\Omega) < v_{max}$, then

$$A(\partial\Omega) \geq C\, V(\Omega)^{1 - \frac{1}{n}}, \tag{3.16}$$

where

$$\begin{aligned} C &= 4^{-n} \left(\tfrac{3}{7}\right)^{\frac{1}{n}} e^{-1}, \\ v_{max} &= \tfrac{3}{7} \left(\tfrac{n}{e}\right)^n K^{-n}. \end{aligned} \tag{3.17}$$

Remark 3.4. On minimal manifolds we already know that $|H| = 0$, and in this case we obtain that $r_{max} = v_{max} = \infty$. Therefore if M is a minimal manifold, then M satisfies the isoperimetric property.

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