

A MARKOV DILATION FOR SELF-ADJOINT SCHUR MULTIPLIERS

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ABSTRACT. We give a formula for Markov dilation in the sense of Anantharaman-Delaroche for real positive Schur multipliers on $\mathbb{B}(H)$.

The classical theory of semigroups has many applications and connections with ergodic theory, martingales and probability (see [12]). The recent developments of noncommutative integration in von Neumann provide analogues of these notions ([1], [5], [6]). For instance, classical Markov semigroups on a probability space are generalized to semigroups of unital completely positive maps preserving a given faithful state. It is natural to try to adapt techniques from the commutative theory to the noncommutative one. Dealing with C^* -algebras, Sauvageot [11] has given a construction of a Markov C^* -dilation for a semigroup in the spirit of Daniell-Kolmogorov. One of the main tools in the classical setting is Rota's dilation theorem [10], [12]. It states that any Markovian map (unital, positive, self-adjoint on L_2 and contractive on all L_p 's) has a nice dilation in terms of a reversed martingale, namely $Q^{2n} = \hat{E} \circ E_n$, where the E_n are conditional expectations from a decreasing filtration and \hat{E} is another conditional expectation. This is closely related to the construction of Markov chains. Anantharaman-Delaroche states a counterpart of it for von Neumann algebras in [1]. Unfortunately some extra technical condition is needed. She called it "factorization" or Markov dilation. It is unknown if this condition holds for any Markov operator. The aim of this paper is to discuss this factorization for some concrete and basic examples.

Let's start with Stinespring's dilation theorem for C^* -algebras. It asserts that given a unital completely positive map $u : A \rightarrow \mathbb{B}(H)$, one can find a Hilbert space K containing H , a representation π from A to $\mathbb{B}(K)$ so that u is just the composition of π and the natural conditional expectation onto $\mathbb{B}(H)$. More precisely, K is the Hilbert space $A \otimes_u H$ obtained by completion of $A \otimes H$ for the scalar product:

$$\forall a, a' \in A, h, h' \in H \quad \langle a \otimes h, a' \otimes h' \rangle_K = \langle h, u(a^* a') \cdot h' \rangle_H.$$

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The inclusion map $\pi : A \rightarrow \mathbb{B}(K)$ is given by $\pi(a)(a' \otimes h) = aa' \otimes h$. The embedding from H into K is $h \mapsto 1 \otimes h$. The formula for the projection P from K onto H is $P(a \otimes h) = 1 \otimes u(a).h$, and one has

$$u(a) = P\pi(a)|_H.$$

This result is used as the very basic step in Sauvageot's construction. Its main drawback is that when dealing with von Neumann algebras (with faithful state or trace) for applications in L_p spaces (see [1], [5], [6]), one would like to have a dilation that stays in this category. Therefore Claire Anantharaman-Delaroche introduces the notion of factorizable maps in [1], which we describe precisely now. We will use classical notation for von Neumann algebras as in [13], [1].

Let M and N be von Neumann algebras with normal faithful states ϕ and ψ .

Definition. A (ϕ, ψ) -Markov operator $u : (M, \phi) \rightarrow (N, \psi)$ is a normal unital completely positive map so that $\psi \circ u = \phi$ and that intertwines the modular groups of ϕ and ψ ($u \circ \sigma_t^\phi = \sigma_t^\psi \circ u$).

One says that u admits a factorization if there exist another von Neumann algebra \tilde{M} with a faithful state $\tilde{\phi}$ and normal representations $\pi : M \rightarrow \tilde{M}$, $\rho : N \rightarrow \tilde{M}$ that are $(\phi, \tilde{\phi})$ - and $(\psi, \tilde{\phi})$ -Markov maps, with

$$\psi(u(m)n) = \tilde{\phi}(\pi(m)\rho(n)).$$

We say that $(\tilde{M}, \tilde{\phi})$ is a Markov dilation for u . The conditions on modular groups imply that there is actually a $(\tilde{\phi}, \psi)$ -Markov conditional expectation $\mathbb{E} : \tilde{M} \rightarrow N$ and $u = \mathbb{E} \circ \pi$.

A natural question is which maps are factorizable. The aim of this paper is to give a positive answer for real multipliers.

Viewing N acting as a subalgebra of $\mathbb{B}(L_2(N, \phi))$, one can notice that if u is factorizable, then $M \otimes_u L_2(N, \phi)$ consists exactly in the norm closure of $\pi(M).\rho(N)$ in $L_2(\tilde{M}, \tilde{\phi})$.

In the commutative setting, Stinespring's dilation is actually a Markov dilation. This follows from the fact that N acts on $M \otimes_u L_2(N, \phi)$ by right multiplications. The commutative von Neumann algebra generated by M and N in $\mathbb{B}(M \otimes_u L_2(N, \phi))$ is \tilde{M} and the state is $\tilde{\phi}(x) = \langle 1 \otimes 1, x.(1 \otimes 1) \rangle$. This is the classical Markov construction.

In the sequel, we are only interested in the case $M = N$ and $\phi = \psi$, and we keep the above notation.

We start with some remarks on the set of factorizable maps.

Any state-preserving homomorphism $\alpha : M \rightarrow M$ is factorizable; a dilation is obtained with $\tilde{M} = M$, $\tilde{\phi} = \phi$, $\pi = \alpha$ and $\rho = \text{Id}$.

Any Markovian conditional expectation $\mathbb{E} : M \rightarrow M$ onto a subalgebra N is factorizable. A dilation is given by the free product with amalgamation over N : $(\tilde{M}, \tilde{\phi}) = (M, \phi) *_N (M, \phi)$, π is the homomorphism onto the first copy of M in \tilde{M} and ρ onto the second one. We refer to [14], [9] for definitions. In particular, taking $N = \mathbb{C}.1$, ϕ is factorizable.

The dilation is nonunique in general. For instance, ϕ can also be dilated in $(M, \phi) \overline{\otimes}_{\min} (M, \phi)$ with the obvious inclusions.

The fixed point algebra N by a ϕ -Markov map plays a particular role in the dilation, as we must have $\pi(n) = \rho(n)$ for any $n \in N$.

If T is ϕ -Markovian, then it is known that the adjoint of T on $L^2(M, \phi)$ comes also from a ϕ -Markovian map denoted by T^* (see [1]), that is,

$$\phi(xT(y)) = \phi(T^*(x)y).$$

We say that T is self-adjoint if $T^* = T$.

Proposition. *The set of factorizable ϕ -Markovian operators on M is convex, stable by composition, by the involution $*$ and closed for the point weak- $*$ topology.*

Proof. About the involution, it suffices to exchange the roles of π and ρ as for analytic elements

$$\begin{aligned} \phi(T^*(y)x) &= \phi(T(\sigma_i^\phi(x))y) = \tilde{\phi}(\pi(\sigma_i^\phi(x))\rho(y)) \\ &= \tilde{\phi}(\sigma_i^{\tilde{\phi}}(\pi(x))\rho(y)) = \tilde{\phi}(\rho(y)\pi(x)). \end{aligned}$$

Let u_i be ϕ -Markovian on M with dilation $(\tilde{M}_i, \tilde{\phi}_i)$ and morphisms π_i and ρ_i . A dilation for $\lambda u_1 + (1 - \lambda)u_2$ is given by

$$(\tilde{M}_1 \oplus \tilde{M}_2, \lambda \tilde{\phi}_1 \oplus (1 - \lambda)\tilde{\phi}_2)$$

with morphisms $M \rightarrow \tilde{M}_1 \oplus \tilde{M}_2$, $\pi_1 \oplus \pi_2$ and $\rho_1 \oplus \rho_2$.

A dilation for $u_2 \circ u_1$ comes from the free product construction. Consider $(\tilde{M}, \tilde{\phi})$ given by $(\tilde{M}_1, \tilde{\phi}_1) *_M (\tilde{M}_2, \tilde{\phi}_2)$, where the amalgamation is taken over the copy of M coming from ρ_1 in M_1 and π_2 in M_2 (note that there are indeed conditional expectations onto them). Let \mathbb{E} be the conditional expectation onto the amalgamated copy of M . From the definition of a dilation, we get that $\mathbb{E}(\pi_1(x)) = \rho_1(u_1(x))$ and $\mathbb{E}(\rho_2(y)) = \pi_2(u_2^*(y))$ for $x, y \in M$. A classical computation in free products gives

$$\begin{aligned} \tilde{\phi}(\pi_1(x)\rho_2(y)) &= \tilde{\phi}(\mathbb{E}(\pi_1(x))\mathbb{E}(\rho_2(y))) = \tilde{\phi}(\rho_1(u_1(x))\pi_2(u_2^*(y))) \\ &= \tilde{\phi}(\rho_1(u_1(x)u_2^*(y))) = \phi((u_2 \circ u_1)(x).y). \end{aligned}$$

The statement about the closure property is obtained by taking an ultraproduct (see [8]) and cutting with some projections to make representations normal and the state faithful. Technical details can be found in [4]. □

Among other permanence properties, it was observed in [5] that the free product of factorizable maps is still factorizable and the dilation is simply the free product of the dilations.

As a corollary of Proposition 2, it was pointed out to us by Claire Anantharaman-Delaroche that any tracial Markov map on \mathbb{M}_2 is factorizable as the extreme points of such maps are exactly the automorphisms.

We now come to Schur multipliers on $\mathbb{B}(\ell_2^I)$ with canonical orthonormal basis $(e_i)_{i \in I}$. We will assume that the state ϕ has a diagonal density $D = \sum \lambda_i e_i \otimes e_i$ for the canonical basis with respect to the trace. We have $\lambda_i > 0$ and $\sum \lambda_i = 1$. The modular group of ϕ is $\sigma_t^\phi(x) = D^{-it}x D^{it}$.

We represent elements in $\mathbb{B}(\ell_2^I)$ as matrices \mathbb{M}_I of size I . Given any matrix $T = (t_{i,j}) \in \mathbb{M}_I$, we say that T is a Schur multiplier if the following map is well defined:

$$M_T = \begin{cases} \mathbb{B}(\ell_2^I) & \rightarrow & \mathbb{B}(\ell_2^I) \\ (x_{i,j}) & \mapsto & (t_{i,j}x_{i,j}). \end{cases}$$

A characterization of bounded multipliers can be found in [7]. A multiplier is (completely) positive if and only if its symbol T is positive in the sense that for any

finite set $F \subset I$, $(t_{i,j})_{i,j \in F}$ is positive in M_F . M_T is unital if $t_{i,i} = 1$ for all $i \in I$. M_T is normal unital and completely positive iff there exist norm 1 vectors $x_i \in \ell_2^I$ so that $t_{i,j} = \langle x_i, x_j \rangle$.

From these observations, it is clear that any unital completely positive Schur multiplier is ϕ -Markovian.

In the opposite way, if a map u commutes with the modular group of ϕ and $(\log(\lambda_i))_{i \in I}$ is independent over \mathbb{Q} in \mathbb{R} , then u has to be a Schur multiplier.

The adjoint of M_T is M_{T^t} where $T^t = (t_{j,i})_{i,j}$. So any self-adjoint ϕ -Markovian Schur multiplier has to have real coefficients.

Theorem. *Any positive self-adjoint ϕ -Markovian Schur multiplier is factorizable.*

To construct the dilation, we will need the fermion algebras. Let K be a real Hilbert space with complexification $K_{\mathbb{C}}$. We briefly recall their construction and the more general q -deformed algebras in the spirit of [2]. The q -Fock $(-1 \leq q < 1)$ space over K is

$$\mathcal{F}_q(K) = \mathbb{C}\Omega \oplus \bigoplus_k K_{\mathbb{C}}^{\otimes_q k},$$

where the scalar product on $K_{\mathbb{C}}^{\otimes_q n}$ is given by

$$\langle k_1 \otimes \dots \otimes k_n, h_1 \otimes \dots \otimes h_n \rangle_q = \sum_{\sigma \in S_n} q^{|\sigma|} \langle k_i, h_{\sigma_i} \rangle_{K_{\mathbb{C}}}$$

and where S_n is the symmetric group and $|\sigma|$ the number of inversions of the permutation σ . When $q = -1$, this is just the antisymmetric tensor product $K_{\mathbb{C}}^{\wedge n}$.

The creation operator for $e \in K$ is given by

$$l(e).(h_1 \otimes \dots \otimes h_n) = e \otimes h_1 \otimes \dots \otimes h_n.$$

They satisfy the q -relation

$$l(f)^*l(e) - ql(e)l(f)^* = \langle f, e \rangle_K Id.$$

The q -von Neumann algebra is

$$\Gamma_q(K) = \{\omega(e) = l(e) + l(e)^* ; e \in K\}''.$$

It is type II_1 with trace $\tau(x) = \langle \Omega, x.\Omega \rangle_{\mathcal{F}_q(K)}$.

We are mainly concerned with the fermion algebra when $q = -1$. If $e \in K$ has norm 1, then $\omega(e)$ is a symmetry, i.e. self-adjoint with $\omega(e)^2 = 1$. Moreover we have

$$\forall e, f \in K \quad \tau(\omega(e)\omega(f)) = \langle e, f \rangle_K.$$

Proof. We will use the notation $e_{i,j}$ for the canonical basis of $\mathbb{B}(\ell_2^I)$.

Let M_T be a self-adjoint Markovian Schur multiplier. As $t_{i,j} = \langle x_i, x_j \rangle$ is real, T defines a new scalar product on the real linear span of e_i 's by the formula

$$\langle \sum a_i e_i, \sum b_i e_i \rangle_T = \langle \sum a_i x_i, \sum b_i x_i \rangle_{\ell_2^I} = \sum_{i,j} a_i b_j t_{i,j}.$$

We call $\ell_{2,T}$ the real Hilbert space obtained after quotient and completion. We still denote by e_i the representative of e_i in $\ell_{2,T}$. We have

$$\langle e_i, e_j \rangle_T = t_{i,j} = t_{j,i}.$$

Let $\tilde{M} = \mathbb{B}(\ell_2^I) \overline{\otimes}_{\min} \Gamma_{-1}(\ell_{2,T})$ with normal faithful state $\tilde{\phi} = \phi \otimes \tau$. Let

$$d = \sum_i e_{i,i} \otimes \omega(e_i) \in \tilde{M}.$$

It is a unitary (symmetry) in the centralizer of $\tilde{\phi}$, as $e_{i,i}$ are in the centralizer of ϕ .

For $x \in \tilde{M}$, define

$$\mathcal{U}(x) = dx d.$$

This is a $\tilde{\phi}$ -Markovian map and a representation as $d^2 = 1$.

Let $\pi : M \rightarrow \tilde{M}$ be the obvious inclusion $\text{Id} \otimes 1$. Define $\rho : M \rightarrow \tilde{M}$ as $\rho = \mathcal{U} \circ \pi$. It is clear that π and ρ are $(\phi, \tilde{\phi})$ -Markovian and for $x = (x_{i,j})$ and $y = (y_{i,j})$ finite matrices:

$$\begin{aligned} \tilde{\phi}(\pi(x)\rho(y)) &= \phi \otimes \tau \left((x \otimes 1)d(y \otimes 1)d \right) \\ &= \phi \otimes \tau \left((x_{i,j} \cdot 1) \cdot (y_{i,j} \omega(e_i) \omega(e_j)) \right) \\ &= \sum_{i,j} \lambda_i x_{i,j} y_{j,i} \tau(\omega(e_i) \omega(e_j)) \\ &= \sum_{i,j} \lambda_i x_{i,j} y_{j,i} t_{i,j} \\ &= \phi(T(x)y). \end{aligned} \quad \square$$

When $T = \text{Id}$, M_T is a conditional expectation, and this dilation is very different from the one obtained by free product.

Combining the previous example with the permanence properties gives a wide class of factorizable maps. If I is finite and ϕ is the trace, we can take compositions of multipliers in different bases (and with conditional expectations, representations) and convex combinations of them. We do not know whether we can achieve all tracial Markovian maps for M_n . It follows from Grothendieck’s theorem that any completely bounded Schur multiplier is a multiple (less than the Grothendieck constant) of a convex combination of rank one multipliers (see [7]). In terms of Markov maps, any Markov Schur multiplier can be obtained as a linear combination of representations. Unfortunately it cannot be a convex one (or the Grothendieck constant would have to be exactly 1).

If one looks carefully, the von Neumann algebra generated by $\pi(\mathbb{B}(\ell_2^I))$ and $\rho(\mathbb{B}(\ell_2^I))$ is exactly $\mathbb{B}(\ell_2^I) \overline{\otimes} \Gamma_{-1}^e(\ell_{2,T})$, where $\Gamma_{-1}^e(\ell_{2,T})$ is the subalgebra of $\Gamma_{-1}(\ell_{2,T})$ generated by even elements of the form $\omega(e_i)\omega(e_j)$.

Now that we have a dilation for Schur multipliers, thanks to the construction in [1, chapter 6], we have a noncommutative Markov chain. In our concrete setting, one can avoid this abstract construction (and free products). We follow the notation of [1] and classical notation for infinite tensor products (we drop the completion symbol). Let

$$M = \mathbb{B}(\ell_2^I) \otimes \Gamma_{-1}^e(\ell_{2,T})^{\otimes \infty} \subset \Gamma_{-1}(\ell_{2,T})^{\otimes \infty}$$

be equipped with the tensor product state. Actually, because of the commutation relations,

$$\Gamma_{-1}^e(\ell_{2,T})^{\otimes \infty} \subset \Gamma_{-1}(\ell_{2,T} \otimes \ell_2).$$

Let $J_0 : \mathbb{B}(\ell_2^I) \rightarrow M$ be the natural inclusion given by

$$J_0(x) = x \otimes 1 \otimes 1 \otimes \dots$$

The letter S stands for the shift on $\Gamma_{-1}^e(\ell_{2,T})^{\otimes\infty}$:

$$S(x_1 \otimes \dots \otimes x_n \otimes 1 \otimes \dots) = 1 \otimes x_1 \otimes \dots \otimes x_n \otimes 1 \otimes \dots$$

We will also need the symmetry

$$d_1 = \sum_i e_{i,i} \otimes \omega(e_i) \otimes 1 \otimes \dots \in \mathbb{B}(\ell_2^I) \overline{\otimes} \Gamma_{-1}(\ell_{2,T})^\infty.$$

We have an injective morphism $\beta : M \rightarrow M$ given by

$$\beta(x) = d_1 S(x) d_1.$$

If $J_q = \beta^q \circ J_0$, the q^{th} copy of $\mathbb{B}(\ell_2^I)$ is

$$J_q(e_{i,j}) = e_{i,j} \otimes \underbrace{\omega(e_i)\omega(e_j) \otimes \dots \otimes \omega(e_i)\omega(e_j)}_{q \text{ times}} \otimes 1 \otimes \dots$$

The algebra \mathcal{B}_n generated by the first n copies of $\mathbb{B}(\ell_2^I)$ is exactly

$$\mathcal{B}_n = \mathbb{B}(\ell_2^I) \otimes \Gamma_{-1}^e(\ell_{2,T})^{\otimes n} \otimes \mathbb{C}^{\otimes\infty},$$

and $\mathcal{B}_{[n]}$, generated by $J_q(\mathbb{B}(\ell_2^I))$ with $q \geq n$, is

$$\mathcal{B}_{[n]} = J_n(\mathbb{B}(\ell_2^I)) \cdot (\mathbb{C}^{\otimes n} \otimes \Gamma_{-1}^e(\ell_{2,T})^{\otimes\infty}).$$

All maps preserve the involved modular groups, and if $\mathbb{E}_{[n]}$ and $\mathbb{E}_{[n]}$ are the conditional expectations onto \mathcal{B}_n and $\mathcal{B}_{[n]}$, one can check the Markov properties

$$\begin{aligned} \mathbb{E}_{[n]} \circ J_q &= J_n \circ T^{q-n}, & q \geq n, \\ \mathbb{E}_{[n+q]} \circ \beta^q &= \beta^q \circ \mathbb{E}_{[n]}, \\ \mathbb{E}_{[n]} \circ J_0 &= J_n \circ T^n. \end{aligned}$$

In particular Rota’s dilation is

$$J_0 \circ T^{2n} = \mathbb{E}_{[0]} \circ \mathbb{E}_{[n]} \circ J_0.$$

The above construction can also be carried out for Fourier multipliers on discrete groups. Let G be a discrete countable group and $L(G) \subset \mathbb{B}(\ell_2 G)$ its left von Neumann algebra. It is the bicommutant of the left translations by $g \in G$, denoted as usual by $\lambda(g)$. It is a type II_1 algebra with trace given by

$$\tau(\lambda(g)) = \langle \delta_e, \lambda(g)\delta_e \rangle = \delta_{g,e}.$$

A function $t : G \rightarrow \mathbb{C}$ defines a Fourier multiplier M_t if the following map is well defined on $L(G)$:

$$M_t(\lambda(g)) = t_g \lambda(g).$$

Actually a Fourier multiplier is completely bounded if and only if the Schur multiplier $(t_{h^{-1}g})_{h,g}$ is bounded on $\mathbb{B}(\ell_2 G)$ (see [7]). It is unital completely positive iff $t_e = 1$ (e is the unit of G) and t is positive definite; in this case it is τ -Markovian. M_t is self-adjoint if $t_g = t_{g^{-1}} \in \mathbb{R}$.

Corollary. *Any self-adjoint unital completely positive Fourier multiplier on a discrete group is factorizable.*

Proof. The dilation can be obtained directly from the previous one but can be reinterpreted in terms of a cross product as follows.

Let $\ell_{2,T}$ be the Hilbert space obtained by quotient and completion of the real span of $\lambda(g)$ for the scalar product

$$\langle \lambda(g), \lambda(h) \rangle_T = t_{g^{-1}h}.$$

We let h be the class of $\lambda(h)$ in $\ell_{2,T}$. It is clear that G acts unitarily on $\ell_{2,T}$ by left multiplications. So G also acts by automorphisms on $\Gamma_{-1}(\ell_{2,T})$ by, for $g, h \in G$,

$$\alpha(g).\omega(h) = \omega(gh).$$

Let \tilde{M} be the crossed product $\Gamma_{-1}(\ell_{2,T}) \rtimes_{\alpha} G$ (see [13] for definitions). This is again a type II_1 algebra, $\tilde{\phi}$ is the canonical trace and for $g \in G, x \in \ell_{2,T}$,

$$\lambda(g)\omega(x)\lambda(g^{-1}) = \alpha(g).\omega(x) = \omega(g.x).$$

Then π is just the natural copy of $L(G)$ in \tilde{M} and ρ is given by $\rho(x) = \omega(e)\pi(x)\omega(e)$ for $x \in L(G)$. Note that $\omega(e)$ is a symmetry. We have

$$\begin{aligned} \tilde{\phi}(\pi(\lambda(g))\rho(\lambda(h))) &= \tilde{\phi}(\lambda(g)\omega(e)\lambda(h)\omega(e)) \\ &= \tilde{\phi}(\lambda(g)\lambda(h)\omega(h^{-1})\omega(e)) \\ &= \delta_{gh,e}t_h = \delta_{gh,e}t_g \\ &= \tau(M_t(\lambda(g))\lambda(h)). \end{aligned} \quad \square$$

As before, one has nice formulas for the associated Markov chain. Briefly speaking, $M = \Gamma_{-1}(\ell_{2,T} \otimes \ell_2) \rtimes_{\alpha} G$, where α is the diagonal action of $G : \alpha(g).\omega(h \otimes v) = \omega(gh \otimes v)$ for $g, h \in G$ and $v \in \ell_2$. The inclusion J_0 of $L(G)$ is the natural one. The morphism β is given by the shift on the canonical basis (e_i) of ℓ_2 and a conjugation

$$\beta(\lambda(g)\omega(h \otimes e_i)) = \omega(e \otimes e_0)\lambda(g)\omega(h \otimes e_{i+1})\omega(e \otimes e_0).$$

Then one defines $\mathcal{B}_{[n]}$ and $\mathcal{B}_{\{n\}}$ as above to get Rota's construction.

There is another family of maps close to multipliers which can be seen to have a dilation quite easily. It consists of the maps arising from the second quantization on q -deformed algebras (see [2]). For any real Hilbert space K , let $\omega_K : \Gamma_q(K) \rightarrow \mathcal{F}_q(K)$ be $\omega_K(x) = x.\Omega_K$. This is an injective mapping with dense range; we denote by ω_K^{-1} its inverse (not defined everywhere).

If K and L are real Hilbert spaces, any contraction $T : K \rightarrow L$ gives rise to a tracial Markov map $\Gamma(T) : \Gamma_q(K) \rightarrow \Gamma_q(L)$ satisfying

$$T(\omega_K^{-1}(k_1 \otimes \dots \otimes k_n)) = \omega_L^{-1}(T(k_1) \otimes \dots \otimes T(k_n)).$$

If T is isometric (unitary), then $\Gamma(T)$ is an injective representation (automorphism). So if $K \subset L$, we can see $\Gamma_q(K)$ as a subalgebra of $\Gamma_q(L)$. In this situation, if P is the orthogonal projection of L onto K , $\Gamma(P)$ is then the trace-preserving conditional expectation from $\Gamma_q(L)$ onto $\Gamma_q(K)$.

Any contraction $T : K \rightarrow K$ can be dilated to a unitary say U (to a symmetry if T is self-adjoint) on a bigger Hilbert space L . From $T = PU|_K$, one sees that a dilation of $\Gamma(T)$ is given by $\Gamma_q(L)$ with its trace, with π the natural injection and $\rho = \Gamma(U^*) \circ \pi$.

To get Rota's dilation is also easy in this case, assuming that T is self-adjoint. Indeed, let U be a strong dilation of T on L , that is, $PU|_K^k = T^k$ (see [7], Theorem 1.1). Let $K_n = \text{span}\{U^l(K) ; l \geq n\}$ and P_n be the corresponding projection.

Then one has

$$PP_nP = T^{2n}.$$

Indeed if $k \in K$ and $x \in K$ for $l \geq 0$,

$$\begin{aligned} \langle k, U^{n+l}(x) \rangle &= \langle k, PU^{n+l}P(x) \rangle = \langle T^{n+l}(k), x \rangle \\ &= \langle T^n(k), U^l(x) \rangle = \langle U^n(T^n(k)), U^{n+l}(x) \rangle, \end{aligned}$$

so $P_nP(k) = U^n(T^n(x))$.

Going to the second quantization, with $\mathbb{E}_n = \Gamma(P_n)$ being the conditional expectation from $\Gamma_q(L)$ onto $\Gamma_q(K_n)$, and $\hat{\mathbb{E}} = \Gamma(P)$ being the conditional expectation onto $\Gamma_q(K)$ (adjoint of the inclusion J), one has

$$\hat{\mathbb{E}} \circ \mathbb{E}_n \circ J = \Gamma(PP_nP) \circ J = J \circ \Gamma(T)^{2n}.$$

The same dilation works also for the q -deformed versions of the Araki-Woods factors of Hiai ([3]).

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