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LINEAR MAPS PRESERVING INVARIANTS

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ABSTRACT. Let $G \subset \mathrm{GL}(V)$ be a complex reductive group. Let G' denote $\{\varphi \in \mathrm{GL}(V) \mid p \circ \varphi = p \text{ for all } p \in \mathbb{C}[V]^G\}$. We show that, "in general", $G' = G$. In case G is the adjoint group of a simple Lie algebra $\mathfrak{g}$, we show that G' is an order 2 extension of G. We also calculate G' for all representations of SL_2.

1. Introduction

Our base field is $\mathbb{C}$, the field of complex numbers. Let $G \subset \mathrm{GL}(V)$ be a reductive group. Let $G' = \{\varphi \in \mathrm{GL}(V) \mid p \circ \varphi = p \text{ for all } p \in \mathbb{C}[V]^G\}$. Several authors have studied the problem of determining G'. If G is finite, then one easily sees that $G' = G$. Solomon [Sol05, Sol06] has classified many triples consisting of reductive groups $H \subset G$ and a G-module V such that $\mathbb{C}(V)^H = \mathbb{C}(V)^G$ (rational invariant functions). If G and H are semisimple, then this is the same thing as finding triples where we have equality of the polynomial invariants: $\mathbb{C}[V]^H = \mathbb{C}[V]^G$. We show that for "general" faithful G-modules V we have that $G = G'$. We also compute G' for all representations of SL_2.

First we study the case that G is the adjoint group of a simple Lie algebra $\mathfrak{g}$. Our interest in this case is due to the paper of Raïs [Rai07] where the question of determining G' is raised. The case that $\mathfrak{g} = \mathfrak{sl}_n$ was also settled by him [Rai72], where it is shown that G'/G is generated by the mapping $\mathfrak{sl}_n \ni X \mapsto X^t$ where X^t denotes the transpose of X. In §2 we show that, "in general", G'/G is generated by the element $-\psi$ where $\psi\colon \mathfrak{g} \to \mathfrak{g}$ is a certain automorphism of $\mathfrak{g}$ of order 2. In the case of $\mathfrak{sl}_n$, $\psi(X) = -X^t$, so that our result reproduces that of Raïs. The computation of G' for $\mathfrak{g}$ semisimple follows easily from the case that $\mathfrak{g}$ is simple. In §3 we prove our result that $G = G'$ for "general" G and "general" G-modules V. In §4 we consider representations of SL_2.

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2. The adjoint case

Proposition 2.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra, so we have $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ where the $\mathfrak{g}_i$ are simple ideals. Let $\varphi \in G'$. Then $\varphi(\mathfrak{g}_i) = \mathfrak{g}_i$ for all i, and $\varphi|_{\mathfrak{g}_i} = \pm\sigma_i$ where σ_i is an automorphism of $\mathfrak{g}_i$.*

Proof. By a theorem of Dixmier [Dix79] we know that the Lie algebra of G' is $\operatorname{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$. Thus φ acts on $\operatorname{ad}(\mathfrak{g}) \simeq \mathfrak{g}$ via an automorphism σ where $\varphi \circ \operatorname{ad} X \circ \varphi^{-1} = \operatorname{ad}\sigma(X)$ for $X \in \mathfrak{g}$. Since φ induces the identity on $\mathbb{C}[\mathfrak{g}]^G$, so does σ, and it follows that $\sigma = \prod_i \sigma_i$ where $\sigma_i \in \operatorname{Aut}(\mathfrak{g}_i)$, $i = 1, \dots, r$. By Schur's lemma, $\varphi \circ \sigma^{-1}$ restricted to $\mathfrak{g}_i$ is multiplication by some scalar $\lambda_i \in \mathbb{C}^*$, $i = 1, \dots, r$. Since $\operatorname{Aut}(\mathfrak{g}_i)$ and G' preserve the invariant of degree 2 corresponding to the Killing form on each $\mathfrak{g}_i$ we must have that $\lambda_i = \pm 1$, $i = 1, \dots, r$. □

From now on we assume that $\mathfrak{g}$ is simple. Let $\sigma \in \operatorname{Aut}(\mathfrak{g})$. Then we know that, up to multiplication by an element of $G = \operatorname{Aut}(\mathfrak{g})^0$, we can arrange that σ preserves a fixed Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Thus we may assume that φ preserves $\mathfrak{t}$. Let T denote the corresponding maximal torus of G.

Corollary 2.2. *We may modify φ by an element of G so that φ is the identity on $\mathfrak{t}$.*

Proof. By Chevalley's theorem, restriction to $\mathfrak{t}$ gives an isomorphism of $\mathbb{C}[\mathfrak{g}]^G$ with $\mathbb{C}[\mathfrak{t}]^W$, where W is the Weyl group of $\mathfrak{g}$. Thus the restriction of φ to $\mathfrak{t}$ coincides with an element of W, where every element of W is the restriction of an element of G stabilizing $\mathfrak{t}$. Thus we may assume that φ is the identity on $\mathfrak{t}$. □

Let φ be the set of roots and φ^+ a choice of positive roots. Let Π denote the set of simple roots. Since $\varphi = \pm\sigma$ is the identity on $\mathfrak{t}$, $\sigma(x) = c_\sigma x$ for all $x \in \mathfrak{t}$, where $c_\sigma = \pm 1$. Hence either σ sends each $\mathfrak{g}_\alpha$ to itself or it sends each $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$, $\alpha \in \varphi$. Choose nonzero elements $x_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Pi$, and choose elements $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $(x_\alpha, y_\alpha, [x_\alpha, y_\alpha])$ is an $\mathfrak{sl}_2$-triple. Let ψ denote the unique order 2 automorphism of $\mathfrak{g}$ such that $\psi(x) = -x$, $x \in \mathfrak{t}$ and $\psi(x_\alpha) = -y_\alpha$, $\alpha \in \Pi$ (see [Hum72, 14.3]).

Proposition 2.3. (1) *If $c_\sigma = 1$, then σ is inner.*
(2) *If $c_\sigma = -1$, then σ differs from ψ by an element of $\operatorname{Ad}(T)$.*

Proof. If $c_\sigma = 1$, then $\sigma(x_\alpha) = c_\alpha x_\alpha$, $c_\alpha \in \mathbb{C}$, $\alpha \in \Pi$. There is a $t \in T$ such that $\operatorname{Ad}(t)(x_\alpha) = c_\alpha x_\alpha$, $\alpha \in \Pi$. It follows that $\sigma = \operatorname{Ad}(t) \in G$. If $c_\sigma = -1$, we can modify σ by an element of T so that it becomes ψ. □

Proposition 2.4. *Let $\mathfrak{g}$ be simple. Then the following are equivalent.*

(1) *Every representation of $\mathfrak{g}$ is self-dual.*
(2) *The automorphism ψ is inner.*
(3) *The generators of $\mathbb{C}[\mathfrak{g}]^G$ have even degree.*
(4) *$\mathfrak{g}$ is of the following type:*
  (a) $\mathsf{B_n}$, *$n \geq 1$,*
  (b) $\mathsf{C_n}$, *$n \geq 3$,*
  (c) $\mathsf{D_{2n}}$, *$n \geq 2$,*
  (d) $\mathsf{E_7}$, $\mathsf{E_8}$, $\mathsf{F_4}$ *or* $\mathsf{G_2}$.

Proof. The equivalence of (1), (3) and (4) is well-known. Now given a highest weight vector λ of $\mathfrak{g}$, the highest weight vector of the corresponding dual representation

$V(\lambda)^*$ is $-\rho(\lambda)$, where ρ is the unique element of the Weyl group W which sends φ^+ to φ^- ([Hum72, §21, Exercise 6]). Suppose that we have (2). Then, since ψ is inner and it normalizes $\mathfrak{t}$, it gives an element of W, namely ρ, so that $V(\lambda)^* \simeq V(\lambda)$ for all λ and (1) holds. Conversely, if (1) holds, then $-\rho$ is the identity on the set of weights; hence $\rho(\alpha) = -\alpha$ for all $\alpha \in \varphi$. It follows that $\rho \circ \psi$ is an automorphism of $\mathfrak{g}$ which is the identity on $\mathfrak{t}$ and sends $\mathfrak{g}_\alpha$ to $\mathfrak{g}_\alpha$ for all α. Then $\rho \circ \psi \in \mathrm{Ad}(T)$ so that ψ is inner. □

Theorem 2.5. *The group G'/G has order 2, generated by $-\psi$.*

Proof. If $\varphi = \sigma \in \mathrm{Aut}(\mathfrak{g})$, then Proposition 2.3 shows that $\varphi = \sigma \in G$. If $\varphi = -\sigma$, then by Proposition 2.3 we may assume that $\varphi = -\psi$. Now $-\psi$ induces an automorphism of $\mathbb{C}[\mathfrak{g}]^G$ and $-\psi$ is the identity on $\mathfrak{t}$. Hence Chevalley's theorem shows that $-\psi \in G'$, and we know that $-\psi$ generates G'/G. Moreover, $-\psi$ is not in $\mathrm{Aut}(\mathfrak{g})$, so that $-\psi \notin G$. □

Corollary 2.6. *Suppose that ψ is inner. Then G'/G is generated by multiplication by -1.*

We leave it to the reader to formulate versions of Theorem 2.5 and Corollary 2.6 for the semisimple case.

3. The general case

We have a finite dimensional vector space $V$and G is a reductive subgroup of $\mathrm{GL}(V)$. Let $G' := \{\varphi \in \mathrm{GL}(V) \mid p \circ \varphi = p \text{ for all } p \in \mathbb{C}[V]^G\}$. We show that, "in general", we have $G' = G$.

Let U denote the subset of V consisting of closed G-orbits with trivial stabilizer. It follows from Luna's slice theorem [Lun73] that U is open in V.

Theorem 3.1. *Suppose that $V \setminus U$ is of codimension 2 in V. Then $G' = G$.*

Proof. Let $\varphi \in G'$ and let $x \in U$. Then $\varphi(x) = \psi(x) \cdot x$ where $\psi \colon U \to G$ is a well-defined morphism. Since G is affine, we may consider ψ as a mapping from $U \to G \subset \mathbb{C}^n$ for some n where G is Zariski closed in $\mathbb{C}^n$. Our condition on the codimension of $V \setminus U$ guarantees that each component of ψ is a regular function on V; hence ψ extends to a morphism defined on all of V, with image in G. Now let $x \in U$. Then

$$\varphi(x) = \lim_{t\to 0} \varphi(tx)/t = \lim_{t\to 0} \psi(tx)tx/t = \psi(0)(x).$$

Thus φ is just the action of $\psi(0) \in G$, so $G' = G$. □

4. Representations of SL_2

As an illustration, we consider representations of $G = \mathrm{SL}_2$ or $G = \mathrm{SO}_3$. We only consider representations with no nonzero fixed subspace. We let R_j denote the irreducible representation of dimension $j+1$, $j \geq 0$, and kR_j denotes the direct sum of k copies of R_j, $k \geq 1$. When we have a representation only containing copies of R_j for j even, then we are considering representations of SO_3. From [Sch95, 11.9] we know that all representations of G satisfy the hypotheses of Theorem 3.1 except for the following cases, where we compute G'.

(1) For R_1 we have $G' = \mathrm{GL}_2$, for $2R_1$ we have $G' = \mathrm{O}_4$ and for $3R_1$ we have $G' = G$.

(2) For R_2 we have $G' = \mathrm{O}_3$ and for $2R_2$ we have $G' = \mathrm{O}_3$. (Here $G = \mathrm{SO}_3$.)
(3) For $R_2 \oplus R_1$ we have $G' = \{g' \in \mathrm{GL}_2 \mid \det(g') = \pm 1\}$.
(4) For R_3 the group G' is the same as in case (3).
(5) For R_4 we have $G' = G = \mathrm{SO}_3$.

Most of the calculations are easy. We mention some details for some of the non-obvious cases.

Suppose that our representation is R_4, which has generating invariants of degrees 2 and 3. The Lie algebra $\mathfrak{g}'$ acts irreducibly on R_4; hence it is the sum of a center and a semisimple Lie algebra ([Jac62, Ch. II, Theorem 11]). Clearly we cannot have a nontrivial center, so that $\mathfrak{g}'$ is semisimple. Now a case-by-case check of the possibilities forces $\mathfrak{g}' = \mathfrak{g}$. Suppose that $g' \in G' \setminus G$. Then conjugation by g' gives an inner automorphism of G; hence we can correct g' by an element of G so that g' commutes with G. Thus g' acts on R_4 as a scalar. But to preserve the invariants the scalar must be 1. Thus we have $G' = G$. Similar considerations give that $\mathfrak{g}' = \mathfrak{g}$ in case (4), so that G'/G is generated by scalar multiplication by i (since the generating invariant of R_3 has degree 4), which shows that G' is as claimed.

In case (3), one sees that $\mathfrak{g}' = \mathfrak{g}$, so that generators of G'/G act as scalars on R_2 and R_1. Now generators of the invariants have degrees $(2,0)$ and $(1,2)$ so that G'/G is generated by an element which is multiplication by -1 on R_2 and multiplication by i on R_1. Hence G' is as claimed.

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