ON SEMILOCAL RINGS

HONGBO ZHANG

(Communicated by Birge Huisgen-Zimmermann)

ABSTRACT. In this paper, semilocal rings are characterized in different ways; in particular, it is proved that a ring R is semilocal if and only if every descending chain of principal right ideals of R, $a_0 R \supseteq a_1 R \supseteq a_2 R \supseteq \cdots \supseteq a_n R \supseteq \cdots$ with $a_{i+1} = a_i - a_i b_i a_i$ eventually terminates. Then modules with semilocal endomorphism rings are characterized by chain conditions.

1. INTRODUCTION

All rings in this paper are associative with identity; modules are unital right R-modules.

A finite set $A_1, \cdots, A_n$ of proper submodules of M is said to be *coindependent* if for each i, $1 \le i \le n$, $A_i + \bigcap_{j \ne i} A_j = M$, and a family of submodules of M is said to be coindependent if each of its finite subfamilies is coindependent. The module M is said to have *finite hollow dimension* (or finite dual Goldie dimension) if every coindependent family of submodules of M is finite. It can be shown that, in this case, there is a maximal coindependent family of submodules of M. If this set is finite, then its cardinality (denoted by $h.dim(M)$ or $codim(M)$) is uniquely determined and is called the *hollow dimension* of M (or dual Goldie dimension of M). A module M with hollow dimension 1 is said to be *hollow*, and a cyclic hollow module is said to be *local*.

A ring R with Jacobson radical $J(R)$ is said to be semilocal if $R/J(R)$ is a semisimple ring. Semilocal rings are characterized as those rings with finite hollow dimension (see [2], Proposition 2.43). For a semilocal ring R,

$$h.dim(R) = \text{composition length of the right } R\text{-module } R/J(R).$$

It is well known that semilocal rings have stable range one, and so any modules M with semilocal endomorphism rings can cancel from the direct sum; i.e. $M \oplus B \cong M \oplus C$ implies $B \cong C$. Facchini, Herbera, Levy and Vámos [3] proved that if M has a semilocal endomorphism ring, then M has the "n-th root property"; i.e., $M^n \cong N^n$ implies $M \cong N$.

In 1993, Camps and Dicks [1] obtained a number of new characterizations of semilocal rings. Facchini stated two of their many characterizations in [2], Theorem 4.2 as follows.

Theorem 1 (Camps and Dicks). *The following conditions are equivalent for a ring R.*

   (a)   *R is semilocal.*
   (b)   *There exists an integer $n \geq 0$ and a function $d : R \to \{0, 1, \cdots, n\}$ such that*
      (i)   *for every $a, b \in R$, $d(a - aba) = d(a) + d(1 - ab)$;*
      (ii)   *if $a \in R$ and $d(a) = 0$, then $a \in U(R)$.*
   (c)   *There exists a partial order $\leq$ on the set R such that*
      (iii)   *$(R, \leq)$ is an Artinian partial order set;*
      (iv)   *if $a, b \in R$ and $1 - ab \notin U(R)$, then $a - aba < a$.*

In this paper, inspired by Camps and Dicks's Theorem 1, a new notion "hollow length" is introduced (Definition 2). Then some properties of hollow length are presented in Proposition 6, Corollary 7 and Proposition 8. We give different characterizations of semilocal rings in Theorem 9 and Corollary 10. Modules with a semilocal endomorphism ring are characterized in Theorem 11. Theorem 3 (1), (2) of [4] and Theorem 5 of [1] are extended in Corollaries 12 and 13.

Refer to [2], [5] and [6] for details concerning hollow dimension and semilocal rings.

Throughout the paper, $J(R)$ will denote the Jacobson radical of a ring R. $\overline{R}$ and $\overline{a}$ denote respectively $R/J(R)$ and $a + J(R)$. Denote by $dim(M)$ the *Goldie dimension* of M and by $U(R)$ the group of units in the ring R. $r.U(R)$ (resp. $l.U(R)$) denotes the set of right (resp. left) invertible elements of R.

2. Main results

Definition 2. Let R be a ring, $a \in R$. A **right hollow chain** of a is a strictly descending chain

$$a_0 R \supsetneq a_1 R \supsetneq a_2 R \supsetneq a_3 R \supsetneq \cdots ,$$

with $a_0 = a$ and for all $n \geq 0$, $a_{n+1} = a_n - a_n b_n a_n$ for some $b_n \in R$.

Set $r = \sup\{n \in \mathbb{Z} \mid a_0 R \supsetneq a_1 R \supsetneq a_2 R \supsetneq \cdots \supsetneq a_{n-1} R \supsetneq a_n R$ is a hollow chain of $a_0 = a\}$. r is called the **right hollow length** of a, denoted as $r.h.length(a) = r$.

Remark. "Left hollow chain" and "left hollow length" can be defined similarly. For simplicity, "right hollow chain" and "right hollow length" are called in this paper respectively "hollow chain" and "hollow length", and "r.h.length(a)" is written as "h.length(a)".

Lemma 3. *Let R be a ring, $a, x \in R$. The following conditions are equivalent:*
   (1) $aR = (a - axa)R$.
   (2) $a \in (1 - ax)R$.
   (3) $1 - ax \in r.U(R)$.
   (4) $\overline{a}\overline{R} = (\overline{a - axa})\overline{R}$.

Proof. $(1) \Rightarrow (2)$ and $(1) \Rightarrow (4)$ are trivial.

$(2) \Rightarrow (3)$ Set $a = (1 - ax)t$. Then $1 = (1 - ax) + ax = (1 - ax) + (1 - ax)tx = (1 - ax)(1 + tx)$, so $1 - ax \in r.U(R)$.

$(3) \Rightarrow (1)$ Set $(1 - ax)v = 1$. Then $v = 1 + axv$, and so $a = [(1 - ax)v]a = (1 - ax)(1 + axv)a = (a - axa)(1 + xva) \in (a - axa)R$.

$(4) \Rightarrow (3)$ $\overline{a}\overline{R} = (\overline{a - axa})\overline{R}$ shows that there exists $t \in R$ such that $a - (a - axa)t \in J(R)$, so $(1 - ax)(1 + atx) = 1 - [a - (a - axa)t]x \in U(R)$, and so $1 - ax \in r.U(R)$. $\square$

Remark 4. Let M be a module and $End(M)$ its endomorphism ring. For all $f, g \in End(M)$, $(f - fgf)(M) = f(M)$ implies that $1 - fg$ is surjective. In fact, $f(M) = (f - fgf)(M)$ shows $f(M) = (1 - fg)f(M) \subseteq (1 - fg)(M)$. Thus $\forall x \in M$, $fg(x) = (1 - fg)(y)$ for some $y \in M$, so $x = (1 - fg)(x) + fg(x) = (1 - fg)(x) + (1 - fg)(y) = (1 - fg)(x + y) \in (1 - fg)(M)$; i.e., $1 - fg$ is surjective.

Now we present some properties of hollow length.

Lemma 5. *Let R be a ring, $a \in R$.*
  (1) $h.length(a) = 0$ if and only if $a \in J(R)$.
  (2)   *If $h.length(a) = 1$, then $\overline{a}\overline{R}$ is a simple $\overline{R}$-module.*

Proof. (1) $h.length(a) = 0$ if and only if for all $x \in R$, $aR = (a - axa)R$. Lemma 3 shows that this means $1 - ax \in r.U(R)$ for all $x \in R$, i.e., $a \in J(R)$.

  (2)   Suppose that $h.length(a) = 1$. We need to prove that for all $x \in R$, if $0 \neq \overline{ax}\overline{R}$, then $\overline{ax}\overline{R} = \overline{a}\overline{R}$. In fact, $0 \neq \overline{ax}\overline{R}$ gives $ax \notin J(R)$, so there exists $y \in R$ such that $1 - axy \notin r.U(R)$. Lemma 3 shows $aR \supsetneq (a - axya)R$. Since $h.length(a) = 1$, we have $h.length(a - axya) = 0$, so $a - axya \in J(R)$, i.e., $\overline{a} - \overline{axya} = \overline{0} \in \overline{R}$. Thus $\overline{ax}\overline{R} = \overline{a}\overline{R}$, as desired. $\square$

Remark. Essentially, the proof of Lemma 5(2) has its origin in Camps and Dicks's proof of Theorem 1.

We can characterize hollow length as follows.

Proposition 6. *Let R be a ring, $a \in R$ and let $n \geq 0$ be an integer. The following conditions are equivalent:*
  (1) $h.length(a) = n$.
  (2) $\overline{a}\overline{R}$ *is a semisimple $\overline{R}$-module with $dim(\overline{a}\overline{R}) = n$.*

Proof. For $n = 0$, this is Lemma 5(1). We now prove the equivalence for $n \geq 1$.

  $(1) \Rightarrow (2)$ We use induction on n. The case $n = 1$ is Lemma 5(2). Let $n > 1$ and suppose that our implication holds for all $x \in R$ with $h.length(x) \leq n - 1$. Let $aR \supsetneq a_1 R \supsetneq \cdots \supsetneq a_{n-1} R \supsetneq a_n R$ be a hollow chain of a, where $a_1 = a - aba$ and $a_{i+1} = a_i - a_i b_i a_i$ for all $i \geq 1$. Then $h.length(a_i) = n - i$ for all $i \in \{0, 1, \cdots, n\}$.

  Since $h.length(a_n) = 0$, Lemma 5(1) gives $a_n \in J(R)$. Therefore $\overline{a_n} = \overline{a_{n-1}} - a_{n-1} b_{n-1} a_{n-1} = \overline{0}$, i.e., $\overline{a_{n-1}} = \overline{a_{n-1} b_{n-1} a_{n-1}}$. Thus $\overline{a_{n-1} b_{n-1}}$ is idempotent and $\overline{R} = \overline{(1 - a_{n-1} b_{n-1})R} \oplus \overline{a_{n-1} b_{n-1} R}$, and so $\overline{a}\overline{R} = \overline{(1 - a_{n-1} b_{n-1})a R} \oplus \overline{a_{n-1} b_{n-1} a R}$. Noting that $a_{n-1} \in aR$, we have $\overline{a_{n-1}} = \overline{a_{n-1} b_{n-1} a_{n-1}} \in \overline{a_{n-1} b_{n-1} a R}$, so $\overline{a_{n-1}}\overline{R} = \overline{a_{n-1} b_{n-1} a R}$. Therefore we have

(∗) $\overline{a}\overline{R} = \overline{(1 - a_{n-1} b_{n-1})a R} \oplus \overline{a_{n-1}}\overline{R}$.

  Note that $h.length(a_{n-1}) = 1$. Lemma 5(2) shows that $\overline{a_{n-1}}\overline{R}$ is a simple $\overline{R}$-module.

  Since $a_{n-1} R \supsetneq a_n R = (a_{n-1} - a_{n-1} b_{n-1} a_{n-1})R$, we know from Lemma 3 that $(1 - a_{n-1} b_{n-1}) \notin r.U(R)$, and so Lemma 3 gives $aR \supsetneq (a - a_{n-1} b_{n-1} a)R$. Thus we have $h.length(a - a_{n-1} b_{n-1} a) < h.length(a) = n$. By induction we know $\overline{(a - a_{n-1} b_{n-1} a)R}$ is a semisimple module with $dim(\overline{(a - a_{n-1} b_{n-1} a)R}) \leq n - 1$. Therefore equation (∗) yields that $\overline{a}\overline{R}$ is a semisimple module with $dim(\overline{a}\overline{R}) \leq n$.

  Now we prove $dim(\overline{a}\overline{R}) \geq n$. Noting that $\overline{a_1}\overline{R} \subseteq \overline{a}\overline{R}$ and $\overline{a}\overline{R}$ is semisimple, we have $\overline{a}\overline{R} = \overline{a_1}\overline{R} \oplus D$ for some $D \subseteq \overline{a}\overline{R}$. Since $h.length(a_1) = n - 1$, by induction, $\overline{a_1}\overline{R}$ is a semisimple module with $dim(\overline{a_1}\overline{R}) = n - 1$. If we can prove $D \neq 0$, then

we have $dim(\overline{a}\overline{R}) = dim(\overline{a_1}\overline{R}) + dim(D) \geq n$. In fact, if $D = 0$, then $\overline{a}\overline{R} = \overline{a_1}\overline{R}$. Lemma 3 gives $aR = a_1R$, a contradiction.

$(2) \Rightarrow (1)$ Suppose that $\overline{a}\overline{R}$ is a semisimple module with $dim(\overline{a}\overline{R}) = n$.

First assume that $h.length(a) = \infty$. Then there is a finite descending chain

$$a_0R \supsetneq a_1R \supsetneq a_2R \supsetneq a_3R \supsetneq \cdots \supsetneq a_mR$$

with $a_0 = a$, $a_{i+1} = a_i - a_ib_ia_i$ $(0 \leq i \leq m-1)$ and $m > n$. By Lemma 3, there is a finite descending chain

$$\overline{a}\overline{R} \supsetneq \overline{a_1}\overline{R} \supsetneq \overline{a_2}\overline{R} \supsetneq \overline{a_3}\overline{R} \supsetneq \cdots \supsetneq \overline{a_m}\overline{R}.$$

Thus $dim(\overline{a}\overline{R}) \geq m > n$, a contradiction.

Suppose $h.length(a) = m < \infty$. Then by $(i) \Rightarrow (ii)$, we know that $\overline{a}\overline{R}$ is a semisimple module with $dim(\overline{a}\overline{R}) = m$. Therefore $m = n$. $\square$

Corollary 7. *Let R be a ring. If $a, b \in R$ and $aR \subseteq bR$, then $h.length(a) \leq h.length(b)$. In particular, for all $a \in R$, $h.length(a) \leq h.length(1_R)$.*

Proof. Suppose that $h.length(b) = n < \infty$. By Proposition 6, $\overline{b}\overline{R}$ is a semisimple $\overline{R}$-module with $dim(\overline{b}\overline{R}) = n$, so $\overline{a}\overline{R}$ is a semisimple $\overline{R}$-module with $dim(\overline{a}\overline{R}) \leq n$. Therefore $h.length(a) = dim(\overline{a}\overline{R}) \leq n = h.length(b)$. $\square$

Proposition 8. *Let R be a semilocal ring, $a, b \in R$. Then*
   (1) $h.length(a) = h.dim(R) - h.dim(R/aR)$.
   (2) $h.length(a) + h.length(1 - ab) - h.length(a - aba) = h.dim(R)$.
   (3) $h.length(a) = h.length(a - aba) + 1$ if and only if $aR/(a - aba)R$ is local.

Proof. (1)   Since $(aR + J(R))/aR$ is small in R/aR, applying [2], Proposition 2.42 (d) to the exact sequence

$$(aR + J(R))/aR \hookrightarrow R/aR \twoheadrightarrow (R/aR)/(aR + J(R)/aR) \cong R/(aR + J(R)),$$

we have

$$h.dim(R/aR) = h.dim(R/(aR + J(R))) = h.dim(\overline{R}/\overline{a}\overline{R})$$
$$= h.dim(\overline{R}) - h.dim(\overline{a}\overline{R}) = h.dim(R) - h.dim(\overline{a}\overline{R}).$$

By Proposition 6, we get $h.dim(R/aR) = h.dim(R) - h.length(a)$.

(2)   This result comes directly from (1) and the fact that $h.dim(R/(a-aba)R) = h.dim(R/aR \oplus R/(1-ab)R) = h.dim(R/aR) + h.dim(R/(1-ab)R)$.

(3)   By (2), $h.length(a) = h.length(a - aba) + 1$ if and only if $h.length(1 - ab) = h.dim(R) - 1$, i.e., $h.dim(R) - h.dim(R/(1-ab)R) = h.dim(R) - 1$; this means that $h.dim(R/(1-ab)R) = 1$. By [2], Proposition 2.42 (b), this means that $R/(1-ab)R$ is hollow. Since $R/(1-ab)R$ is cyclic, [5, 1.1.5 (b)] shows that $R/(1-ab)R = (aR + (1-ab)R)/(1-ab)R \cong aR/(a-aba)R$ is local. $\square$

We now give our characterizations of semilocal rings.

Theorem 9. *The following conditions are equivalent for a ring R and an integer $n \geq 0$:*
   (1)   *R is a semilocal ring and $dim(R/J(R)) = n$.*
   (2)   *$h.length(1_R) = n$.*
   (3)   *There exists a descending chain of principal right ideals of R*

$$R = a_0R \supsetneq a_1R \supsetneq a_2R \supsetneq \cdots \supsetneq a_nR \text{ with } a_{i+1} = a_i - a_ib_ia_i$$

such that

      (i) $a_n \in J(R)$;

      (ii) $a_i R/a_{i+1} R$ *is local for all* $0 \leq i \leq n-1$.

    (4)   *There exist* $d, d_1, \cdots, d_n \in R$ *with* $dR = \bigcap_{i=1}^{n} d_i R \subseteq J(R)$ *such that* $R/d_i R$ *is local for all* $i \in \{1, \cdots, n\}$ *and* $R/dR \cong \bigoplus_{i=1}^{n} R/d_i R$.

    (5)   *There exists* $d \in J(R)$ *such that* $R/dR \cong \bigoplus_{i=1}^{n} A_i$, *where* $\{A_i\}_{i \in \{1, \cdots, n\}}$ *are local modules.*

    (2*), (3*), (4*), (5*). *The left-right duals of* (2), (3), (4) *and* (5).

Proof. (1) $\Leftrightarrow$ (2) comes directly from Proposition 6 by setting $a = 1_R$.

    (2) $\Rightarrow$ (3)   Let $R = a_0 R \supsetneq a_1 R \supsetneq a_2 R \supsetneq \cdots \supsetneq a_n R$ with $a_{i+1} = a_i - a_i b_i a_i$ a hollow chain of $a_0 = 1_R$. Then (i) and (ii) come respectively from Lemma 5(1) and Proposition 8(3).

    (3) $\Rightarrow$ (4)   Assume that (3) holds. Let $d = a_n$, $d_1 = a_1$ and $d_{i+1} = (1 - a_i b_i)$ for all $i \in \{1, \cdots, n-1\}$. Then $dR = \bigcap_{i=1}^{n} d_i R \subseteq J(R)$ and $R/d_1 R$ is local. Obviously, for all $i \in \{2, \cdots, n\}$, $R/d_i R \cong a_{i-1} R/a_i R$ is also local. Moreover,

$$
\begin{aligned}
R/dR = R/a_n R &= R/(a_{n-1} - a_{n-1} b_{n-1} a_{n-1})R \cong R/a_{n-1} R \oplus R/(1 - a_{n-1} b_{n-1})R \\
&= R/(a_{n-2} - a_{n-2} b_{n-2} a_{n-2})R \oplus R/(1 - a_{n-1} b_{n-1})R \\
&\cong R/a_{n-2} R \oplus R/(1 - a_{n-2} b_{n-2})R \oplus R/(1 - a_{n-1} b_{n-1})R \\
&\cong \cdots \\
&\cong R/a_1 R \oplus R/(1 - a_1 b_1)R \oplus \cdots \oplus R/(1 - a_{n-2} b_{n-2})R \oplus R/(1 - a_{n-1} b_{n-1})R \\
&= R/d_1 R \oplus R/d_2 R \oplus \cdots \oplus R/d_{n-1} R \oplus R/d_n R.
\end{aligned}
$$

    (4) $\Rightarrow$ (5) is trivial.

    (5) $\Rightarrow$ (1) $d \in J(R)$ shows that dR is small in R, and so

$$
h.dim(R) = h.dim(R/dR) = \sum_{i=1}^{n} h.dim(A_i)
$$

$$
= \sum_{i=1}^{n} 1 = n. \qquad \square
$$

Corollary 10. *The following conditions are equivalent for a ring R:*

    (1) R *is a semilocal ring.*

    (2)   *Every right hollow chain*

$$
a_0 R \supsetneq a_1 R \supsetneq a_2 R \supsetneq \cdots \supsetneq a_n R \supsetneq \cdots \text{ with } a_{i+1} = a_i - a_i b_i a_i
$$

eventually terminates.

    (3)   *There exists a partial order* $\leq$ *on the set* R *such that*

      (i) $(R, \leq)$ *is an Artinian partial order set;*

      (ii)   *if* $a, b \in R$ *and* $1 - ab \notin r.U(R)$, *then* $a - aba < a$.

    (4)   *Every descending chain of principal right ideals of* R

$$
a_0 R \supseteq a_1 R \supseteq a_2 R \supseteq \cdots \supseteq a_n R \supseteq \cdots \text{ with } a_{i+1} = a_i - a_i b_i a_i
$$

eventually terminates.

    (2*), (3*), (4*). *The left-right duals of* (2), (3) *and* (4).

Proof. $(4) \Rightarrow (2)$ is trivial.

$(1) \Rightarrow (2)$ comes from Theorem 9(2) and Corollary 7.

$(2) \Rightarrow (3)$ If (2) holds, define an order $\leq$ on R via

$$b \leq a \text{ if } a = b \text{ or } aR \supsetneq bR \text{ and } b = a - ada \text{ for some } d \in R.$$

If we can prove that $\leq$ is a partial order, then (3) is easily verified. We need only to prove the transitivity of "$\leq$". In fact, suppose that $a_3 < a_2$ and $a_2 < a_1$. Set $a_{i+1} = a_i - a_i b_i a_i$ $(i = 1, 2)$, where $1 - a_i b_i \notin r.U(R)$. Write $d = b_1 + (1 - b_1 a_1) b_2 (1 - a_1 b_1)$. Then

$$a_3 = a_2 - a_2 b_2 a_2 = (a_1 - a_1 b_1 a_1) - (a_1 - a_1 b_1 a_1) b_2 (a_1 - a_1 b_1 a_1)$$
$$= a_1 - a_1 [b_1 + (1 - b_1 a_1) b_2 (1 - a_1 b_1)] a_1 = a_1 - a_1 d a_1.$$

Noting that $1 - a_1 d = 1 - a_1 [b_1 + (1 - b_1 a_1) b_2 (1 - a_1 b_1)] = (1 - a_1 b_1)[1 - a_1 b_2 (1 - a_1 b_1)]$ and $1 - a_1 b_1 \notin r.U(R)$, we get $1 - a_1 d \notin r.U(R)$, and so $a_3 < a_1$, as desired.

$(3) \Rightarrow (1)$ Note that $\forall a \in R$, $a \in J(R)$ if and only if $1 - ab \in r.U(R)$ for all $b \in R$. Then using the same proof of Camps and Dicks' Theorem 1 (see $(c) \Rightarrow (a)$ of [2, Theorem 4.2] or $(f) \Rightarrow (a)$ of [1, Theorem 1]), we can prove (1).

$(1), (2) \Rightarrow (4)$ We prove first the following result.

Claim. For all descending chains

$$a_1 R \supseteq a_2 R \supseteq a_3 R \supseteq \cdots \supseteq a_n R \supseteq \cdots \text{ with } a_{n+1} = a_n - a_n b_n a_n.$$

If $a_1 R = a_2 R$, then we have a descending chain

$$a_1 R \supseteq a_3' R \supseteq \cdots \supseteq a_n' R \supseteq \cdots \text{ with } a_3' = a_1 - a_1 d a_1, a_{n+1}' = a_n' - a_n' d_n a_n'$$

for some $d, d_n \in R$ $(n \geq 3)$, and $a_n' R = a_{n+1}' R$ if and only if $a_n R = a_{n+1} R$. $a_3' R = a_1 R$ if and only if $a_3 R = a_2 R$.

In fact, $a_1 R = a_2 R$ with $a_2 = a_1 - a_1 b_1 a_1$ implies that $1 - a_1 b_1 \in r.U(R) = U(R)$, i.e., $1 - b_1 a_1 \in U(R)$.

For $n \geq 3$, set $a_n' = a_n (1 - b_1 a_1)^{-1}$. Then $a_n' R = a_n R$, so $a_n' R = a_{n+1}' R$ if and only if $a_n R = a_{n+1} R$. In addition, $a_{n+1}' = (a_n - a_n b_n a_n)(1 - b_1 a_1)^{-1} = a_n' - a_n' (1 - b_1 a_1) b_n a_n'$.

$a_3' = a_3 (1 - b_1 a_1)^{-1} = (a_2 - a_2 b_2 a_2)(1 - b_1 a_1)^{-1} = [a_1 - a_1 (1 - b_1 a_1) b_2 a_1]$, and $a_3' R = a_1 R$ means that $a_3 (1 - b_1 a_1)^{-1} R = a_1 R$, that is, $a_3 R = a_1 R = a_2 R$.

Assume that there exists a descending chain

$$a_0 R \supseteq a_1 R \supseteq a_2 R \supseteq \cdots \supseteq a_n R \supseteq \cdots \text{ with } a_{i+1} = a_i - a_i b_i a_i$$

which never terminates. Then, by the claim, we can obtain an infinite hollow chain

$$d_0 R \supsetneq d_1 R \supsetneq d_2 R \supsetneq \cdots \supsetneq d_n R \supsetneq \cdots \text{ with } d_{i+1} = d_i - d_i c_i d_i,$$

a contradiction.   □

As an application of Corollary 10, we can characterize modules with a semilocal endomorphism ring as follows.

Theorem 11. *The following conditions are equivalent for a module M:*

(1)   *The ring $End(M)$ is semilocal.*

(2)   *For every $f_0 \in End(M)$ and every sequence $g_0, g_1, g_2, \cdots$ of elements of $End(M)$, if we set $f_{n+1} = f_n - f_n g_n f_n$ for every $n \geq 0$, then the chains*

(i) $f_0(M) \supseteq f_1(M) \supseteq \cdots \supseteq f_n(M) \supseteq \cdots$

(ii) $\ker f_0 \subseteq \ker f_1 \subseteq \cdots \subseteq \ker f_n \subseteq \cdots$

of submodules of M both eventually terminate.

Proof. $(1) \Rightarrow (2)$ Set $E = End(M)$. Corollary 10 shows that

$$f_0 E \supseteq f_1 E \supseteq f_2 E \supseteq \cdots$$

of E eventually terminates, so there exists $m \in \mathbb{N}$ such that $\forall n > m$, $f_{n+1}E = (f_n - f_n g_n f_n)E$, i.e., $1 - f_n g_n \in r.U(E) = U(E)$. Thus $1 - g_n f_n \in U(E)$. So $\forall n > m$, $f_{n+1}(M) = (f_n - f_n g_n f_n)(M) = f_n(1 - g_n f_n)(M) = f_n(M)$, and $\ker f_{n+1} = \ker f_n \oplus \ker(1 - g_n f_n) = \ker f_n \oplus 0 = \ker f_n$. Therefore the two chains (i) and (ii) eventually terminate.

$(2) \Rightarrow (1)$ Suppose that $\forall n > m$, $\ker f_{n+1} = \ker f_n \oplus \ker(1 - g_n f_n) = \ker f_n$ and $f_{n+1}(M) = (f_n - f_n g_n f_n)(M) = f_n(M)$.

Then $\ker(1 - g_n f_n) = 0$. By [2, Lemma 4.1 (a)], $\ker(1 - f_n g_n) \cong \ker(1 - g_n f_n) = 0$. so $1 - f_n g_n$ is injective.

By Remark 4, $f_{n+1}(M) = (f_n - f_n g_n f_n)(M) = f_n(M)$ implies that $1 - f_n g_n$ is surjective.

Therefore $\forall n > m$, $1 - f_n g_n$ is bijective. Lemma 3 shows that the descending chain of E

$$E = f_0 E \supseteq f_1 E \supseteq f_2 E \supseteq \cdots$$

with $f_{n+1} = f_n - f_n g_n f_n$ eventually terminates. Corollary 10 yields that $End(M)$ is semilocal. $\square$

We extend Theorem 3 (2) of [4] as follows.

Corollary 12. *The following conditions are equivalent for a module M for which every epimorphism $M \to M$ splits:*

(i) *The ring $End(M)$ is semilocal.*

(ii) *For every $f_0 \in End(M)$ and every sequence $g_0, g_1, g_2, \cdots$ of elements of $End(M)$, if we set $f_{n+1} = f_n - f_n g_n f_n$ for every $n \geq 0$, then the chain $f_0(M) \supseteq f_1(M) \supseteq \cdots \supseteq f_n(M) \supseteq \cdots$ of submodules of M eventually terminates.*

Moreover, $h.dim(End(M)) \leq h.dim(M)$.

Proof. By Theorem 11, we need only to prove $(ii) \Rightarrow (i)$. Set $E = End(M)$. Given a descending chain $f_0 E \supseteq f_1 E \supseteq \cdots \supseteq f_n E \supseteq \cdots$ with $f_{n+1} = f_n - f_n g_n f_n$, by (ii), there exists $m \in \mathbb{N}$ such that $\forall n > m$, $f_{n+1}(M) = (f_n - f_n g_n f_n)(M) = f_n(M)$. By Remark 4, $1 - f_n g_n$ is surjective, so $1 - f_n g_n \in r.U(E)$. Lemma 3 and Corollary 10 yield that $End(M)$ is semilocal.

Assume that $h.dim(M) = m < \infty$ and $h.dim(End(M)) > m$. Then by Theorem 9 there exists a hollow chain in E,

$$E = f_0 E \supsetneq f_1 E \supsetneq \cdots \supsetneq f_m E \supsetneq f_{m+1}E,$$

with $f_{i+1} = f_i - f_i g_i f_i \in E$ for all $i \in \{0, 1, \cdots, m\}$. Then for all $i \in \{0, 1, \cdots, m\}$, $1 - f_i g_i \notin r.U(E)$, and so $1 - f_i g_i$ is not surjective. By Remark 4, $f_0(M) \supsetneq f_1(M) \supsetneq \cdots \supsetneq f_m(M) \supsetneq f_{m+1}(M)$ is a strictly descending chain. Write $N_1 = f_1(M) \subsetneq M$ and $N_{i+1} = (1_M - f_i g_i)(M) \subsetneq M$ for all $i \in \{1, \cdots, m\}$. Then for all $i \geq 1$, $(N_1 \cap N_2 \cap \cdots \cap N_i) + N_{i+1} = f_i(M) + (1_M - f_i g_i)(M) = M$, and so $\{N_i | i = 1, \cdots, m + 1\}$ is a coindependent set of proper submodules of M. Thus $h.dim(M) > m$, a contradiction. $\square$

Similarly, we extend Theorem 5 of [1] and Theorem 3 (1) of [4] as follows.

Corollary 13. *The following conditions are equivalent for a module M for which every monomorphism $M \to M$ splits:*

(i)   *The ring $End(M)$ is semilocal.*

(ii)   *For every $f_0 \in End(M)$ and every sequence $g_0, g_1, g_2, \cdots$ of elements of $End(M)$, if we set $f_{n+1} = f_n - f_n g_n f_n$ for every $n \geq 0$, then the chain $\ker f_0 \subseteq \ker f_1 \subseteq \cdots \subseteq \ker f_n \subseteq \cdots$ of submodules of M eventually terminates.*

Moreover, $h.dim(End(M)) \leq dim(M)$.

Proof. Set $E = End(M)$. We need only to prove (ii) $\Rightarrow$ (i). For this, it is enough to show that every descending chain $Ef_0 \supseteq Ef_1 \supseteq \cdots \supseteq Ef_n \supseteq \cdots$ of left ideals of E with $f_{n+1} = f_n - f_n g_n f_n$ eventually terminates. Given such a chain, by (ii) there exists $m \in \mathbb{N}$ such that $\forall n > m$, $\ker f_{n+1} = \ker(f_n - f_n g_n f_n) = \ker f_n$, i.e., $\ker f_n \oplus \ker(1 - g_n f_n) = \ker f_n$, so $\ker(1 - g_n f_n) = 0$, i.e., $1 - g_n f_n$ is injective, and so $1 - g_n f_n \in l.U(E)$ for all $n > m$. It follows that $Ef_0 \subseteq Ef_1 \subseteq \cdots \subseteq Ef_n \subseteq \cdots$ eventually terminates, so $End(M)$ is semilocal. $h.dim(End(M)) \leq dim(M)$ comes from Theorem 9(2) and the fact that $\forall i \geq 1$, $\ker f_{i+1} = \ker(f_i - f_i g_i f_i) = \ker f_i \oplus \ker(1 - g_i f_i)$. $\qquad\square$

References

[1] R. Camps and W. Dicks, On semilocal rings, Israel J. Math. 81, 203-211, 1993. MR1231187 (94m:16027)

[2] A. Facchini, Module Theory: Endomorphism Rings and Direct Sum Decompositions in Some Classes of Modules, Progress in Math., Vol. 167, Birkhäuser Boston, 1998. MR1634015 (99h:16004)

[3] A. Facchini, D. Herbera, L. Levy and P. Vámos, Krull-Schmidt fails for Artinian modules, Proc. Amer. Math. Soc. 123, 3587-3592, 1995. MR1277109 (96b:16020)

[4] D. Herbera and A. Shamsuddin, Modules with semi-local endomorphism ring, Proc. Amer. Math. Soc. 123, 3593-3600, 1995. MR1277114 (96b:16014)

[5] C. Lomp, On dual Goldie dimension, Diplomarbeit, Heinrich Heine Universität, Düsseldorf, 1996.

[6] K. Varadarajan, Dual Goldie dimension, Comm. Algebra 7, 565-610, 1979. MR524269 (80d:16014)

School of Physics and Mathematics, Jiangsu Polytechnic University, Changzhou, Jiangsu 213016, People's Republic of China

E-mail address: hbzhang1212@yahoo.com.cn