CURVATURE INVARIANTS, KILLING VECTOR FIELDS, CONNECTIONS AND COHOMOGENEITY

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ABSTRACT. A direct, bundle-theoretic method for defining and extending local isometries out of curvature data is developed. As a by-product, conceptual direct proofs of a classical result of Singer and a recent result of the authors are derived.

A classical result of I. M. Singer [11] states that a Riemannian manifold is locally homogeneous if and only if its Riemannian curvature tensor together with its covariant derivatives up to some index $k+1$ are independent of the point (the integer k is called the Singer invariant). More precisely,

Theorem 1 (Singer [11]). *Let M be a Riemannian manifold. Then M is a locally homogeneous if and only if for any $p, q \in M$ there is a linear isometry $F : T_pM \to T_qM$ such that*

$$F^*\nabla^s R_q = \nabla^s R_p,$$

for any $s \leq k+1$.

An alternate proof with a more direct approach was given in [8].

Out of the curvature tensor and its covariant derivatives one can construct scalar invariants, like for instance the scalar curvature. In general, any polynomial function in the components of the curvature tensor and its covariant derivatives which does not depend on the choice of the orthonormal basis at the tangent space of each point is a *scalar Weyl invariant* or a *scalar curvature invariant*. By the Weyl theory of invariants, a scalar Weyl invariant is a linear combination of complete traces of tensors $\langle \nabla^{m_1} R, \rangle \ldots \langle \nabla^{m_\ell} R, \rangle$ $(m_1, \ldots m_s \geq 0, \nabla^0 R = R)$.

Prüfer, Tricerri and Vanhecke studied the interplay among local homogeneity and these curvature invariants. Using Singer's Theorem, they got the following:

Theorem 2 (Prüfer, Tricerri and Vanhecke [10]). *Let M be an n-dimensional Riemannian manifold. Then M is locally homogeneous if and only if all scalar Weyl invariants of order s with $s \leq \frac{n(n-1)}{2}$ are constant.*

Furthermore, in general, for a non-homogeneous Riemannian manifold one can look at the regular level sets of scalar Weyl invariants. In a recent paper [4], we proved the following theorem, which generalizes the above results.

Theorem 3 ([4]). *The cohomogeneity of a Riemannian manifold M (with respect to the full isometry group) coincides locally with the codimension of the foliation by regular level sets of the scalar Weyl invariants.*

The key point in the proof was extending the Killing vector fields in any level set to M (at least locally). In the present paper we use a direct method, identifying Killing fields with parallel sections of a bundle on M (namely the one whose fiber at p is $E_p = T_pM \oplus \mathfrak{so}(T_pM)$, where $\mathfrak{so}(T_pM)$ are the skew-symmetric endomorphisms of T_pM) endowed with a connection $\tilde{\nabla}$ (induced by the Levi-Civita connection of M). This local characterization of Killing vector fields as parallel sections of a vector bundle with connection goes back to Kostant [6] and it is used in theoretical physics (see for instance [2]). In the setting of Theorem 3, it turns out that Killing fields in level sets correspond to a subbundle $\bar{E}$ of E, which we prove to be parallel and flat. This yields a conceptual and direct proof of Theorem 3 and Singer's Theorem, which does not use homogeneous structures.

1. THE CONNECTION

Let M be a Riemannian manifold and let $E \xrightarrow{\pi} M$ be the metric bundle over M whose fibers are $E_p = T_pM \oplus \mathfrak{so}(T_pM)$. The bundle E is canonically isomorphic with $TM \oplus \Lambda^2(M)$. We endow E with the connection $\tilde{\nabla}$, induced by the Levi-Civita connection ∇ on M. Namely, if (v, B) is a section of E (i.e., v is a vector field and B is skew-symmetric tensor field of type $(1,1)$ on M), then

$$\tilde{\nabla}_X(v, B) = (\nabla_X v - B.X, \nabla_X B - R_{X,v}),$$

where R is the curvature tensor on M and, as usual, $(\nabla_X B).Y = \nabla_X(B.Y) - B.\nabla_X Y$.

The *canonical lift* of the vector field Z on M is the section $\hat{Z}$ of E given by

$$\hat{Z}(p) = (Z(p), [(\nabla Z)_p]^{\text{skew}}),$$

where the upper script "skew" denotes the skew-symmetric part of the endomorphisms $(\nabla Z)_p$ of T_pM.

The following result is well known and elementary to show. For the sake of self-completeness we include the proof (cf. also [2, Section 3.5.2]).

Proposition 1. *The canonical lift gives an isomorphism between the set $\mathcal{K}(M)$ of Killing fields on M and the parallel sections of E with respect to $\tilde{\nabla}$.*

Proof. A vector field Z on M is a Killing field if and only if $(\nabla Z)_p$ is skew-symmetric. Observe that in this case Z satisfies the affine Jacobi equation

$$\nabla_X(\nabla Z) - R_{X,Z} = 0$$

for all X. This equation is derived from the fact that the associated flow to Z preserves the Levi-Civita connection (by making use of the fact that ∇ is torsion-free and the first Bianchi identity).

So, if Z is Killing, then $\hat{Z}$ is a parallel section of E, since ∇Z is skew-symmetric. Conversely, if (v, B) is a parallel section of E, then the first component of $\tilde{\nabla}(v, B) = 0$ implies that ∇v is skew-symmetric and hence v is a Killing field on M. $\qquad\square$

It is straightforward to compute the curvature tensor of E, by making use of the first and the second Bianchi identity (for the first and the second component respectively).

Proposition 2. *The curvature tensor of E, with respect to the connection $\tilde{\nabla}$ is given by*

$$\tilde{R}_{X,Y}(v, B) = (0, (\nabla_v R)_{X,Y} - (B.R)_{X,Y}),$$

where B acts on R as a derivation.

Proposition 3. *Let T be a given tensor on M and let (v, B) be a section of E that satisfies the equation*

$$\nabla_v T = B.T.$$

We have that $\tilde{\nabla}_X(v, B)$ satisfies this equation, for all vector fields X on M, if and only if (v, B) also satisfies the following equation:

$$\nabla_v(\nabla T) = B.(\nabla T).$$

Proof. It is straightforward and makes use of the so-called Ricci identity $\nabla_{X,Y} T - \nabla_{Y,X} T = R_{X,Y}.T$, where $R_{X,Y}$ acts as a derivation. $\qquad\square$

2. EXTENSION OF KILLING VECTOR FIELDS

Let M be a Riemannian manifold. By making M possibly smaller we may assume that M is foliated by the regular level sets of the scalar Weyl invariants (cf. [4, Section 3]). Given p, q in the same level set, let us say F, then there exists a linear isometry $h : T_p M \to T_q M$ which maps any covariant derivative $(\nabla^k R)_q$ to the same object at p, for any $k \geq 0$.

Let $c(t)$ be a curve in F with $c(0) = p$ and let $\tau_t : T_p M \to T_{c(t)} M$ be the parallel transport along $c(t)$. Since the parallel transport is a linear isometry (from the corresponding tangent spaces) and by the previous paragraph, one has that $\tau_t^{-1}(\nabla^k R)_{c(t)}$ lies in the same $O(T_p M)$ orbit of $(\nabla^k R)_p$, for all t. Differentiating this condition at $t = 0$ yields that there exists $B \in \mathfrak{so}(T_p M)$ such that

$$(\nabla_v(\nabla^k R))_p = B.(\nabla^k R)_p,$$

where $v = c'(0) \in T_p F$. Observe that v is arbitrary in $T_p F$, since $c(t)$ is an arbitrary curve in F.

Let, for $q \in M$, E_q^k be the subspace of E_q which consists of all the pairs (v, B) such that

$$(\nabla_v(\nabla^i R))_q - B.(\nabla^i R)_q = 0$$

for all $0 \leq i \leq k$. Notice that the projection to the first component maps E_q^k onto $T_q F(q)$, where $F(q)$ is the level set of the scalar Weyl invariants by q. So, $\dim E_q^k \geq r$, where r is the dimension of $F(q)$. It is clear that there exists $j(q) \leq \dim E_q = n + \frac{1}{2}n(n-1)$ such that $\dim E^{j(q)} = \dim E_q^{j(q)+1}$.

By making possibly M smaller we may assume that $\dim E_q^{j(q)}$ does not depend on q. This gives rise to a subbundle $\bar{E}$ of E whose fibers are $E_q^{j(q)}$. By Proposition 3 we have that $\bar{E}$ is a parallel subbundle of E and by Proposition 2 one has that it is flat. Therefore any $(v, B) \in \bar{E}_p$, p fixed in M, gives rise to a parallel section $(\tilde{v}, \tilde{B})$ of $\bar{E}$ and so to a parallel section of E (we may assume that M is simply connected). By Proposition 1, this parallel section corresponds to a Killing field on M, whose value at p is v, which is arbitrary in $T_p F(p)$. This Killing field must be

always tangent to any level set, because the scalar Weyl invariants are preserved by isometries.

This finishes a shorter, conceptual proof of Theorem 3 (and Singer's Theorem).

Remark 1 (On the pseudo-Riemannian case). Singer's Theorem generalizes to the pseudo-Riemannian case (Podestà and Spiro, [9]). Our proof can be applied to this setting and more generally to any affine connection without torsion.

For pseudo-Riemannian manifolds, the behavior of the curvature tensor and its covariant derivatives differs from the one of scalar curvature invariants.

Indeed, there are examples of (not locally homogeneous) Lorentzian manifolds whose scalar curvature invariants vanish (see e.g. [1, 3, 7] and the references therein). Therefore, in this case, having constant scalar curvature invariants does not imply local homogeneity.

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