ADMISSIBLE LOCAL SYSTEMS
FOR A CLASS OF LINE ARRANGEMENTS

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ABSTRACT. A rank one local system $\mathcal{L}$ on a smooth complex algebraic variety M is admissible if roughly speaking the dimension of the cohomology groups $H^m(M, \mathcal{L})$ can be computed directly from the cohomology algebra $H(M, \mathbb{C})$.

We say that a line arrangement $\mathcal{A}$ is of type $\mathcal{C}_k$ for some $k \geq 0$ if k is the minimal number of lines in $\mathcal{A}$ containing all the points of multiplicity at least 3. We show that if $\mathcal{A}$ is a line arrangement in the classes $\mathcal{C}_k$ for $k \leq 2$, then any rank one local system $\mathcal{L}$ on the line arrangement complement M is admissible. Partial results are obtained for the class $\mathcal{C}_3$.

1. INTRODUCTION

When M is a hyperplane arrangement complement in some projective space $\mathbb{P}^n$, one defines the notion of an *admissible* local system $\mathcal{L}$ on M in terms of some conditions on the residues of an associated logarithmic connection $\nabla(\alpha)$ on a good compactification of M; see for instance [7], [11], [8], [9], [6]. This notion plays a key role in the theory, since for such an admissible local system $\mathcal{L}$ on M one has

$$(1.1) \qquad \dim H^i(M, \mathcal{L}) = \dim H^i(H^*(M, \mathbb{C}), \alpha \wedge)$$

for all $i \in \mathbb{N}$.

Let $\mathcal{A}$ be a line arrangement in the complex projective plane $\mathbb{P}^2$ and denote by M the corresponding arrangement complement. For the case of line arrangements, a good compactification is obtained just by blowing-up the points of multiplicity at least 3 in $\mathcal{A}$. This explains the simple version of the admissibility definition given below in Definition 2.1.

Let k be a non-negative integer. We say that a line arrangement $\mathcal{A}$ is of type $\mathcal{C}_k$ if k is the minimal number of lines in $\mathcal{A}$ containing all the points of multiplicity at least 3. For instance, $k = 0$ corresponds to nodal arrangements, while $k = 1$ corresponds to the case of a nodal *affine* arrangement; see [1]. Note that $k = k(\mathcal{A})$ is combinatorially defined; i.e. it depends only on the intersection lattice $L(\mathcal{A})$.

The main result of this note is the following one, which, as explained above, is new in the case $n = 2$.

Theorem 1.1. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$. If $\mathcal{A}$ belongs to the class $\mathcal{C}_k$ for some $k \leq 2$, then any rank one local system $\mathcal{L}$ on M is admissible.*

This result implies in particular that for a combinatorially defined class of arrangements, the characteristic varieties are determined by $L(\mathcal{A})$; see Theorem 2.2.

In section 2 we first make explicit the admissibility condition in the case of line arrangements. Then we prove our main theorem and derive some consequences for the characteristic varieties of line arrangements in the classes $\mathcal{C}_k$ for $k \leq 2$.

In the final section we find some sufficient conditions for a local system $\mathcal{L}$ on the complement of a line arrangement in the class $\mathcal{C}_3$ to be admissible; see Proposition 3.1.

The deleted B_3-arrangement discovered by A. Suciu shows that Theorem 1.1 does not hold for $k = 3$, so our result is the best possible. A discussion of this example from the point of view of our paper is given in Example 3.2. Further examples conclude the paper.

2. ADMISSIBLE RANK ONE LOCAL SYSTEMS

Let $\mathcal{A} = \{L_0, L_1, ..., L_n\}$ be a line arrangement in $\mathbb{P}^2$ and set $M = \mathbb{P}^2 \setminus (L_0 \cup ... \cup L_n)$. Let $\mathbb{T}(M) = \mathrm{Hom}(\pi_1(M), \mathbb{C}^*)$ be the character group of M. This is an algebraic torus $\mathbb{T}(M) \simeq (\mathbb{C}^*)^n$. Consider the exponential mapping

$$(2.1) \qquad \exp : H^1(M, \mathbb{C}) \to H^1(M, \mathbb{C}^*) = \mathbb{T}(M)$$

induced by the usual exponential function $\mathbb{C} \to \mathbb{C}^*$, $t \mapsto \exp(2\pi i t)$, where $i = \sqrt{-1}$. Clearly $\exp(H^1(M, \mathbb{C})) = \mathbb{T}(M)$ and $\exp(H^1(M, \mathbb{Z})) = \{1\}$.

A rank one local system $\mathcal{L} \in \mathbb{T}(M)$ is identified with the corresponding homomorphism $\rho : \pi_1(M) \to \mathbb{C}^*$. This homomorphism is determined by the induced morphism $\tilde{\rho} : H_1(M, \mathbb{Z}) \to \mathbb{C}^*$. The homology group $H_1(M, \mathbb{Z})$ is generated by the elementary loops γ_j around the lines L_j, with a single relation, namely $\gamma_0 + ... + \gamma_n = 0$; see [10]. If we set $\lambda_j = \tilde{\rho}(\gamma_j)$, then $\lambda_0...\lambda_n = 1$. A cohomology class $\alpha \in H^1(M, \mathbb{C})$ is given by

$$(2.2) \qquad \alpha = \sum_{j=0}^{n} a_j \frac{df_j}{f_j},$$

with the residues $a_j \in \mathbb{C}$ satisfying $\sum_{j=0}^{n} a_j = 0$ and $f_j = 0$ being a linear equation for the line L_j. With this notation one has $\exp(\alpha) = \mathcal{L}$ if and only if $\lambda_j = \exp(2\pi i a_j)$ for $j = 0, ..., n$.

Definition 2.1. A local system $\mathcal{L} \in \mathbb{T}(M)$ as above is admissible if there is a cohomology class $\alpha \in H^1(M, \mathbb{C})$ as in (2.2) such that $\exp(\alpha) = \mathcal{L}$, $a_j \notin \mathbb{Z}_{>0}$ and, for any point $p \in L_0 \cup ... \cup L_n$ of multiplicity at least 3, one has

$$a(p) = \sum_{j} a_j \notin \mathbb{Z}_{>0}$$

where the sum is over all j's such that $p \in L_j$.

For an admissible local system the isomorphisms in (1.1) were shown in [7], [11].

Proof of Theorem 1.1. We can assume that, for a line arrangement $\mathcal{A}$ of type $\mathcal{C}_k$, the lines in $\mathcal{A}$ containing all the points of multiplicity at least 3 are $L_0,...,L_{k-1}$. Set $b(p) = \Re(a(p))$, where $\Re$ denotes the real part of a complex number.

Given $\mathcal{L}$, for the residues $a_j \in \mathbb{C}$ we can choose $\Re(a_j) \in [0, 1)$ for all $j \geq 1$. Then the relation $\sum_{j=0}^{n} a_j = 0$ implies that

$$\Re(a_0) = -\sum_{j=1}^{n} \Re(a_j) \le 0.$$

This completes the proof in the case where $\mathcal{A}$ is in the class $\mathcal{C}_0$.

Consider now the case where $\mathcal{A}$ is in the class $\mathcal{C}_1$ and choose $p \in L_0$ as in Definition 2.1. If the set of lines passing through p is $L_0, L_{j_1}, ..., L_{j_s}$ for some $s \ge 2$, then one has

$$b(p) = \sum_{j} \Re(a_j) = -\sum_{s} \Re(a_s) \le 0$$

where the second sum is over all $s > 0$ such that $p \notin L_{j_s}$. This shows that any local system $\mathcal{L} \in \mathbb{T}(M)$ is admissible in this case.

The case where $\mathcal{A}$ is in the class $\mathcal{C}_2$ is much more subtle. Let m_1 be the maximum of the real numbers $b(p)$ where p is a point of multiplicity at least 3 on the line L_1 but not on L_0 and $b(p) \in \mathbb{Z}_{>0}$. By convention we set $\max \emptyset = 0$.

If $m_1 = 0$, then the initial choice is good, exactly as in the previous situation.

Assume now that $m_1 > 0$ and let $p_1 \in L_1$ be a multiple point with $b(p_1) = m_1$. In general this value m_1 can be attained at several points $p_1, p_1', p_1'', ...$ on L_1. Any such point will be called an *extremal point for $\mathcal{L}$ on L_1*.

Then we replace a_1 by $a_1' = a_1 - m_1$, and hence $\Re(a_1') \le 0$ if $\Re(a_1')$ is an integer. We also replace a_0 by $a_0' = a_0 + m_1$, so that the total sum of the residues is still zero. Note that $\Re(a_0') \le 0$.

After these two changes, it is clear that $b(q) \notin \mathbb{Z}_{>0}$ for any point q which is of multiplicity at least 3 on L_1 but not on L_0. If $O = L_0 \cap L_1$ is a point of multiplicity at least 3, then

$$b(O) = \Re(a_0') + \Re(a_1') + \sum_{m} \Re(a_m) = -\sum_{t} \Re(a_t) \le 0$$

where the first sum is over $m = 2, ..., n$ such that $O \in L_m$, and the second sum is over all $t = 2, ..., n$ such that $O \notin L_t$. Note that O is never an extremal point.

If now $q \neq O$ is a point of multiplicity at least 3 on the line L_0, in the corresponding sum for $b(q)$ we have the following terms:
(i) $-\Re(a_j)$ for all $j > 0$,
(ii) $+\Re(a_{j_t})$ for all the lines different from L_0 passing through q,
(iii) $+\Re(a_{j_s})$ for all the lines passing through p_1.
(In fact the terms of type (i) come from $\Re(a_0)$, and the terms of type (iii) come from adding m_1 to a_0 to get a_0'.)

In the last two families there is at most one line in common, namely the line determined by p_1 and q. It follows that the sum for $b(q)$ involves some negative terms from (i) and possibly one positive term in the interval $[0, 1)$. Therefore $b(q) < 1$, and this concludes the proof of Theorem 1.1. $\qquad\square$

The *characteristic varieties* of M are the jumping loci for the cohomology of M, with coefficients in rank 1 local systems:

$$(2.3) \qquad \mathcal{V}_d^i(M) = \{\rho \in \mathbb{T}(M) \mid \dim H^i(M, \mathcal{L}_\rho) \ge d\}.$$

When $i = 1$, we use the simpler notation $\mathcal{V}_d(M) = \mathcal{V}_d^1(M)$. One has the following result; see [4], Remark 2.9 (ii).

Theorem 2.2. *If any local system in $\mathbb{T}(M)$ is admissible, then for any d there are no translated components in the characteristic variety $\mathcal{V}_d(M)$.*

In particular, for any irreducible component W of some characteristic variety $\mathcal{V}_d(M)$, the dimension of $H^1(M, \mathcal{L})$ is constant for $\mathcal{L} \in W \setminus \{1\}$.

Combining this result and our Theorem 1.1, we get the following.

Corollary 2.3. *If $\mathcal{A}$ is a line arrangement in the classes $\mathcal{C}_k$ for $k \leq 2$, then for any d there are no translated components in the characteristic variety $\mathcal{V}_d(M)$.*

3.   Line arrangements in the class $\mathcal{C}_3$

In this section we discuss only the situation where the 3 lines L_0, L_1 and L_2 containing all the points of multiplicity at least 3 are concurrent. Let $O = L_0 \cap L_1 \cap L_2$. Let $\mathcal{L} \in \mathbb{T}(M)$ be a rank one local system and choose the residues a_j for $0 \leq j \leq n$ as above. Then we have a collection P_1 of extremal points for $\mathcal{L}$ on L_1 and a collection P_2 of extremal points for $\mathcal{L}$ on L_2. By convention P_j is empty if the corresponding maximum $m_j = 0$ (see also the beginning of the proof below for the definition of m_j). Since we start with the a_j's such that $\Re(a_j) \in [0,1)$ for all $j > 0$, we have $O \notin P_1 \cup P_2$ as above. For each point $q \in L_0$ of multiplicity at least 3, we denote by $\mathcal{A}_q$ the set of lines in $\mathcal{A}$ passing through q.

Proposition 3.1. *With the above notation, assume that one of the following conditions holds:*
   (i) $P_1 = \emptyset$;
   (ii) $P_2 = \emptyset$;
   (iii) *for each point $q \in L_0$ of multiplicity at least 3, one has either*
   (a) $P_1 \setminus Y_q$ *and* $P_2 \setminus Y_q$ *are both non-empty, where Y_q is the union of the lines in $\mathcal{A}_q$, or*
   (b) *one of the sets $P_1 \setminus Y_q$ and $P_2 \setminus Y_q$ is non-empty, say $p_1 \in P_1 \setminus Y_q$, and there is an extremal point $p_2 \in P_2$ such that the line determined by p_1 and p_2 is not in $\mathcal{A}$.*
   Then the local system $\mathcal{L}$ is admissible.

Proof. Let m_j, for $j = 1, 2$, be the maximum of the real numbers $b(p)$ where $p \neq O$ is a point of multiplicity at least 3 on the line L_j such that $b(p) \in \mathbb{Z}_{>0}$. The only essentially new case is when $m_1 > 0$ and $m_2 > 0$, i.e. case (iii) above. Then we set $a'_j = a_j - m_j$ for $j = 1, 2$ and $a'_0 = a_0 + m_1 + m_2$. Exactly as in the proof of Theorem 1.1, this settles the case of multiple points on L_1 and L_2 distinct from O. At the point O we get the sum

$$b(O) = \Re(a'_0 + a'_1 + a'_2) + \sum_m \Re(a_m) = -\sum_t \Re(a_t) \leq 0$$

where the first sum is over $m = 3, ..., n$ such that $O \in L_m$, and the second sum is over all $t = 3, ..., n$ such that $O \notin L_t$.

If now $q \neq O$ is a point of multiplicity at least 3 on the line L_0, in the corresponding sum for $b(q)$ we have the following terms:
   (i) $-\Re(a_j)$ for all $j > 0$;
   (ii) $+\Re(a_{j_s})$ for all the lines different from L_0 passing through q;
   (iii) $+\Re(a_{j_m})$ for all the lines passing through p_1, for a fixed choice $p_1 \in P_1$;
   (iv) $+\Re(a_{j_r})$ for all the lines passing through p_2, for a fixed choice $p_2 \in P_2$.
   (In fact the terms of type (i) come from $\Re(a_0)$, the terms of type (iii) come from adding m_1, and the terms of type (iv) come from adding m_2.)

Our assumption (iii) says exactly that there is a choice of p_1 and p_2 such that in the last 3 families of positive terms there is at most one line occurring twice (and none occurring three times). It follows that the sum for $b(q)$ involves some negative terms from (i) and possibly one positive term in the interval $[0, 1)$. Therefore $b(q) < 1$, and this concludes the proof. $\square$

Example 3.2. Here we analyze from the point of view of the above proof the non-admissible local systems in the case of Suciu's arrangement [12].

Choose: $L_0 : x = 0$, $L_1 : x = y$, $L_2 : y = 0$, $L_3 : x = z$, $L_4 : y = z$, $L_5 : x - y + z = 0$, $L_6 : x - y - z = 0$ and $L_7 : z = 0$ (this last one is the line at infinity).

There are 7 points of multiplicity at least 3, namely, $O = L_0 \cap L_1 \cap L_2 = (0 : 0 : 1)$, $p_1 = L_1 \cap L_3 \cap L_4 = (1 : 1 : 1)$, $p_1' = L_1 \cap L_5 \cap L_6 \cap L_7 = (1 : 1 : 0)$, $p_2 = L_2 \cap L_3 \cap L_6 = (1 : 0 : 1)$, $p_2' = L_2 \cap L_4 \cap L_7 = (1 : 0 : 0)$, $q_1 = L_0 \cap L_4 \cap L_5$ and $q_2 = L_0 \cap L_3 \cap L_7 = (0 : 1 : 0)$. It is clear that L_0, L_1, L_2 contain all the points of multiplicity at least 3, even those at infinity.

Consider the local system $\mathcal{L}_\rho$ which is used to define the translated component
$$W = \rho \cdot \{(t, 1, t^{-1}, t^{-1}, t, t^{-2}, t^2, 1) \mid t \in \mathbb{C}^*\},$$
where $\rho = (1, -1, -1, -1, 1, 1, 1, -1)$. The corresponding residues are $a_1 = a_2 = a_3 = a_7 = 1/2$, $a_4 = a_5 = a_6 = 0$ and $a_0 = -2$. Using these residues we find out that $m_1 = m_2 = 1$, $P_1 = \{p_1, p_1'\}$ and $P_2 = \{p_2, p_2'\}$.

For $q = q_2$, both sets $P_1 \setminus Y_q$ and $P_2 \setminus Y_q$ are empty. Thus condition (iii) of Proposition 3.1 is not fulfilled.
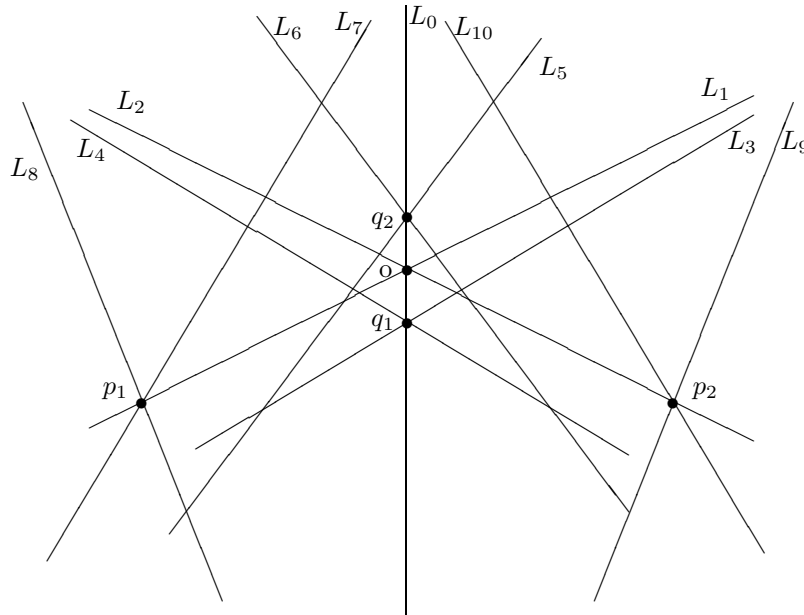

FIGURE 1

Example 3.3. Here is an example of an arrangement in $\mathcal{C}_3$ of the type discussed above for which all local systems are admissible. Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^2$ consisting of the following 10 lines: $L_0 : x = 0$, $L_1 : y = 1/2(x + 3)$, $L_2 : y = -1/2(x - 3)$, $L_3 : y = x + 1$, $L_4 : y = -(x - 1)$, $L_5 : y = 2(x + 1)$, $L_6 : y = -2(x - 1)$, $L_7 : y = 3/2(x + 3)$, $L_8 : y = -3/2(x + 3)$, $L_9 : y = 5/2(x - 3)$ and $L_{10} : y = -5/2(x - 3)$.

There are 5 triple points as shown in Fig. 1; 3 are on the line L_0, namely $O = L_1 \cap L_2 \cap L_0 = (0, \frac{3}{2})$, $q_1 = L_0 \cap L_3 \cap L_4 = (0, 1)$ and $q_2 = L_0 \cap L_5 \cap L_6 = (0, 2)$; and the other 2 are $p_1 = L_1 \cap L_7 \cap L_8 = (-3, 0)$ and $p_2 = L_2 \cap L_9 \cap L_{10} = (3, 0)$. Any local system is admissible by Proposition 3.1. Indeed, if we suppose that P_1 and P_2 are both non-empty, then both sets $P_1 \setminus Y_q = P_1$ and $P_2 \setminus Y_q = P_2$ are again non-empty, for $q = q_1$ and $q = q_2$.

Example 3.4. Now we present an example of an arrangement in $\mathcal{C}_3$ of the type discussed above for which Proposition 3.1 does not apply, but where nevertheless most (perhaps all) local systems are admissible. Let $\mathcal{A}$ be the arrangement in $\mathbb{C}^2$ consisting of the following 7 lines: $L_0 : x = 0$, $L_1 : y = -2(x-1)$, $L_2 : y = 2(x+1)$, $L_3 : y = -(x - 1)$, $L_4 : y = x + 1$, $L_5 : y = -1/3(x + 1)$ and $L_6 : y = 1/3(x - 1)$. The line at infinity is denoted by L_7.
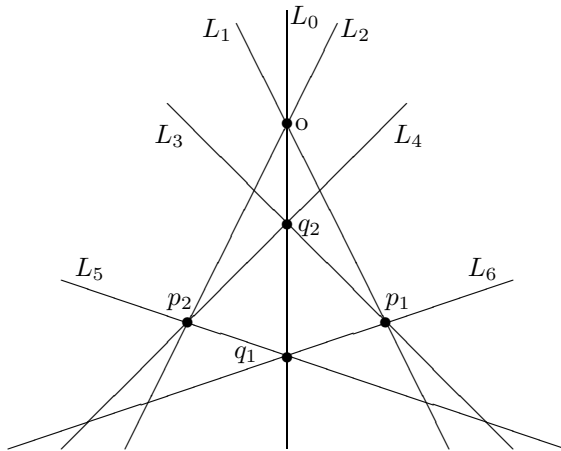

FIGURE 2

There are 5 points of multiplicity at least 3; see Fig. 2. These are $O = L_0 \cap L_1 \cap L_2 = (0, 2)$, $p_1 = L_1 \cap L_3 \cap L_6 = (0, 1)$, $p_2 = L_2 \cap L_4 \cap L_5 = (-1, 0)$, $q_1 = L_0 \cap L_5 \cap L_6 = (0, -\frac{1}{3})$ and $q_2 = L_0 \cap L_3 \cap L_4 = (0, 1)$. In fact all these points have multiplicity 3 and lie on either L_0, L_1 or L_2.

Assume that we have a local system $\mathcal{L}$ such that P_1 and P_2 are both non-empty. Then $m_1 = \Re(a_2 + a_4 + a_5) \in \{1, 2\}$ and $m_2 = \Re(a_1 + a_3 + a_6) \in \{1, 2\}$. In this case, the new residue along L_0 is

$$a_0' = a_0 + m_1 + m_2 = -\sum_{j=1}^{7} a_j + m_1 + m_2 = -a_7 - i \cdot \Im(\sum_{j=1}^{6} a_j)$$

where $\Im$ denotes the imaginary part of a complex number. When we compute the residue along the exceptional curve E_1 created by blowing-up the point q_1, we get

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$$r_1 = a_0' + a_5 + a_6 = a_5 + a_6 - a_7 - i \cdot \Im(\sum_{j=1}^{6} a_j).$$

Similarly we get

$$r_2 = a_3 + a_4 - a_7 - i \cdot \Im(\sum_{j=1}^{6} a_j)$$

for the point q_2. In this case Proposition 3.1 does not apply, since for $q = q_1, q_2$ both sets $P_1 \setminus Y_q$ and $P_2 \setminus Y_q$ are empty. However, whenever $\Re(r_1)$ and $\Re(r_2)$ do not belong to $\mathbb{Z}_{>0}$, the corresponding local system $\mathcal{L}$ is admissible.

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