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AVERAGE BEHAVIOR OF FOURIER COEFFICIENTS OF CUSP FORMS

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Abstract. Let $a_0(n)$ and $b_0(n)$ be the normalized Fourier coefficients of the two holomorphic Hecke eigenforms $f(z) \in S_{2k}(\Gamma)$ and $\varphi(z) \in S_{2l}(\Gamma)$ respectively. In 1999, Fomenko studied the following average sums of $a_0(n)$ and $b_0(n)$:

$$\sum_{n\le x} a_0(n)^3, \quad \sum_{n\le x} a_0(n)^2 b_0(n), \quad \sum_{n\le x} a_0(n)^2 b_0(n)^2, \quad \sum_{n\le x} a_0(n)^4.$$

In this paper, we are able to improve on Fomenko's results.

1. Introduction and main results

Let $S_{2k}(\Gamma)$ be the space of holomorphic cusp forms of even weight $2k$ for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Suppose that $f(z)$ and $\varphi(z)$ are two eigenfunctions of Hecke operators belonging to $S_{2k}(\Gamma)$ and $S_{2l}(\Gamma)$ respectively. Then the Hecke eigenforms $f(z)$ and $\varphi(z)$ have the following Fourier expansions at cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}, \quad \varphi(z) = \sum_{n=1}^{\infty} b(n) e^{2\pi i n z},$$

where we normalize $f(z)$ and $\varphi(z)$ so that $a(1) = b(1) = 1$. The Fourier coefficients of cusp forms are interesting objects (see [3], [12]). Instead of $a(n)$ and $b(n)$, one often considers the normalized Fourier coefficients

$$a_0(n) = \frac{a(n)}{n^{k-\frac{1}{2}}}, \quad b_0(n) = \frac{b(n)}{n^{l-\frac{1}{2}}}.$$

Hecke's theory (see for example [9]) succeeds in developing the nice properties of the L-functions attached to cusp forms $f(z)$ and $\varphi(z)$, which are defined by

$$L_f(s) = \sum_{n=1}^{\infty} a_0(n) n^{-s} = \prod_p \{(1-\alpha_p p^{-s})(1-\bar{\alpha}_p p^{-s})\}^{-1}, \ |\alpha_p| = 1, \ \alpha_p + \bar{\alpha}_p = a_0(p);$$

$$L_\varphi(s) = \sum_{n=1}^{\infty} b_0(n) n^{-s} = \prod_p \{(1-\beta_p p^{-s})(1-\bar{\beta}_p p^{-s})\}^{-1}, \ |\beta_p| = 1, \ \beta_p + \bar{\beta}_p = b_0(p).$$

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These Hecke L-functions are initially defined in the half-plane $\mathrm{Re}(s) > 1$, and are then extended to entire functions satisfying functional equations of Riemann type (see for example [6], [9], [10]). Nowadays Hecke L-functions and their generalizations, the standard L-functions attached to cuspidal automorphic representations of $GL(n, \mathbb{A}_{\mathbb{Q}})$, have become important tools in number theory under the guidance of the far-reaching Langlands program.

In 1999, Fomenko [4] studied the properties of some L-functions attached to several Hecke eigenforms, and proved the following results.

Theorem 0.1. *For $x \to \infty$, we have*

$$\sum_{n \le x} a_0(n)^3 \ll x^{\frac{5}{6}+\varepsilon}.$$

Theorem 0.2. *For $x \to \infty$, we have*

$$\sum_{n \le x} a_0(n)^2 b_0(n) \ll x^{\frac{5}{6}+\varepsilon}.$$

Theorem 0.3. *Let F_1 be the Gelbart-Jacquet lift (see Gelbart and Jacquet [5]) on $GL(3)$ associated to f, and let F_2 be the Gelbart-Jacquet lift on $GL(3)$ associated to φ. If F_1 and F_2 are distinct, then for $x \to \infty$, we have*

$$\sum_{n \le x} a_0(n)^2 b_0(n)^2 = Cx + O(x^{\frac{9}{10}+\varepsilon}),$$

where $C > 0$.

Theorem 0.4. *For $x \to \infty$, we have*

$$\sum_{n \le x} a_0(n)^4 = C_1 x \log x + C_2 x + O(x^{\frac{9}{10}+\varepsilon}),$$

where $C_1 > 0$.

In this paper we are able to prove the following results.

Theorem 1.1. *For $x \to \infty$, we have*

$$\sum_{n \le x} a_0(n)^3 \ll x^{\frac{3}{4}+\varepsilon}.$$

Theorem 1.2. *For $x \to \infty$, we have*

$$\sum_{n \le x} a_0(n)^2 b_0(n) \ll x^{\frac{3}{4}+\varepsilon}.$$

Theorem 1.3. *If f and φ are distinct, then for $x \to \infty$, we have*

$$\sum_{n \le x} a_0(n)^2 b_0(n)^2 = Cx + O(x^{\frac{7}{8}+\varepsilon}),$$

where $C > 0$.

Theorem 1.4. *For $x \to \infty$, we have*

$$\sum_{n \le x} a_0(n)^4 = C_1 x \log x + C_2 x + O(x^{\frac{7}{8}+\varepsilon}),$$

where $C_1 > 0$.

2. Proof of Theorems 1.1 and 1.2

The L-function attached to cusp forms f, f and φ is defined, for $\mathrm{Re}(s) > 1$, by

(2.1)
$$L_{f,f,\varphi}(s) = \prod_p \{(1-\alpha_p^2\beta_p p^{-s})(1-\alpha_p^2\bar{\beta}_p p^{-s})(1-\bar{\alpha}_p^2\beta_p p^{-s})(1-\bar{\alpha}_p^2\bar{\beta}_p p^{-s})\}^{-1} \times \prod_p \{(1-\beta_p p^{-s})(1-\bar{\beta}_p p^{-s})\}^{-2} = L_\varphi(s)^2 L_3(s),$$

where $L_3(s)$ is given by

$$L_3(s) = \prod_p \{(1-\alpha_p^2\beta_p p^{-s})(1-\alpha_p^2\bar{\beta}_p p^{-s})(1-\bar{\alpha}_p^2\beta_p p^{-s})(1-\bar{\alpha}_p^2\bar{\beta}_p p^{-s})\}^{-1}.$$

It was shown in [4] that for $\mathrm{Re}(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{a_0(n)^2 b_0(n)}{n^s} = L_\varphi(s)^2 L_3(s) \prod_p M_p(p^{-s}) = L_{f,f,\varphi}(s) \prod_p M_p(p^{-s}),$$

where

$$M_p(p^{-s}) = 1 + \frac{3 - 2a_0(p)^2 - b_0(p)^2}{p^{2s}} + \cdots + \frac{1}{p^{6s}}.$$

Obviously $\prod_p M_p(p^{-s})$ converges uniformly in the half-plane $\mathrm{Re}(s) \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$.

It is easy to show that $L_\varphi(s)L_3(s) = L(F \times \varphi, s)$, where F is the Gelbart-Jacquet lift on GL(3) associated to f. Therefore Shahidi's work [13] shows that $L_{f,f,\varphi}(s)$ is holomorphic on the whole complex plane and satisfies a functional equation of Riemann type.

In the special case of $f = \varphi$, we have that for $\mathrm{Re}(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{a_0(n)^3}{n^s} = L_f(s)^2 L_3(s) \prod_p M_p(p^{-s}) = L_{f,f,f}(s) \prod_p M_p(p^{-s}). \tag{2.2}$$

It follows from Shahidi's work [13] that $L_f(s)^2 L_3(s) = L_{f,f,f}(s)$ is holomorphic on the whole plane and satisfies a functional equation of Riemann type.

We shall use the above properties to prove Theorems 1.1 and 1.2. Since the proofs of Theorems 1.1 and 1.2 are the same in essence, we shall give only the proof of Theorem 1.1 as an example.

To begin with, we recall two folklore results (see [2] and [10]).

Lemma 2.1. *Let $L(f,s)$ be a Dirichlet series with Euler product of degree $m \geq 2$, which means that*

$$L(f,s) = \sum_{n=1}^{\infty} l_f(n) n^{-s} = \prod_{p<\infty} \prod_{j=1}^{m} \left(1 - \frac{\alpha_f(p,j)}{p^s}\right)^{-1},$$

where $\alpha_f(p,j), j = 1, \cdots, m$, are the local parameters of $L(f,s)$ at prime p and $l_f(n) \ll n^\varepsilon$. Assume that this series and its Euler product are absolutely convergent for $Re(s) > 1$. Assume also that it admits a meromorphic continuation to the whole

complex plane $\mathbb{C}$ *and satisfies a functional equation of Riemann type. Then we have that for* $T \geq 1$,

$$\int_T^{2T} |L(f, 1/2+\varepsilon+it)|^2 \, dt \ll T^{\frac{m}{2}+\varepsilon}.$$

Lemma 2.2. *Let* $L(f,s)$ *be the same as in Lemma* 2.1. *Then for* $0 \leq \sigma \leq 1$, *we have*

$$L(f, \sigma+it) \ll (|t|+1)^{\frac{m}{2}(1-\sigma)+\varepsilon}.$$

It should be remarked that in Lemmas 2.1 and 2.2 we consider only the t-aspect in the analytic conductor introduced by Iwaniec and Kowalski [10]. Therefore, like Fomenko's results, all constants in this paper depend on the weights of the corresponding cusp forms.

Proof of Theorem 1.1. By (2.2) and Perron's formula (see Proposition 5.54 in [10]), we have

$$\sum_{n \leq x} a_0(n)^3 = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L_{f,f,f}(s) \prod_p M_p(p^{-s}) \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{2.3}$$

where $b = 1+\varepsilon$ and $1 \leq T \leq x$ is a parameter to be chosen later. Here we have used the Ramanujan-Petersson conjecture that $|a_0(n)| \leq d(n) = \sum_{d|n} 1$. This famous result was proved by Deligne [3].

Next we move the integration to the parallel segment with $\mathrm{Re}(s) = \frac{1}{2}+\varepsilon$. By Cauchy's theorem, we have

$$\begin{aligned} \sum_{n \leq x} a_0(n)^3 &= \frac{1}{2\pi i} \left\{ \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} + \int_{\frac{1}{2}+\varepsilon+iT}^{b+iT} + \int_{b-iT}^{\frac{1}{2}+\varepsilon-iT} \right\} L_{f,f,f}(s) \prod_p M_p(p^{-s}) \frac{x^s}{s} ds \\ &\quad + O\left(\frac{x^{1+\varepsilon}}{T}\right) \\ &=: I_1 + I_2 + I_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right). \end{aligned} \tag{2.4}$$

To proceed further, we recall that $L_{f,f,f}(s) = L_f(s)^2 L_3(s)$ is a Riemann-type nice L-function with Euler product of degree $m = 8$.

For I_1, by (2.1) and Lemma 2.2 we have

$$\begin{aligned} I_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T \left| L_{f,f,f}(1/2+\varepsilon+it) \prod_p M_p(p^{-\frac{1}{2}-it}) \right| t^{-1} dt \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T |L_{f,f,f}(1/2+\varepsilon+it)| \, t^{-1} dt \\ &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T \left| L_f(1/2+\varepsilon+it)^2 L_3(1/2+\varepsilon+it) \right| t^{-1} dt. \end{aligned} \tag{2.5}$$

Then by Lemma 2.1, we have

$$
\begin{aligned}
I_1 \ll & x^{\frac{1}{2}+\varepsilon} \log T \\
& \cdot \max_{T_1 \le T} \left\{ \frac{1}{T_1} \left(\int_{T_1/2}^{T_1} |L_f(1/2+\varepsilon+it)|^2 \, dt \right)^{\frac{1}{2}} \right. \\
& \qquad \left. \left(\int_{T_1/2}^{T_1} |L_f(1/2+\varepsilon+it) L_3(1/2+\varepsilon+it)|^2 \, dt \right)^{\frac{1}{2}} \right\} \\
\ll & x^{\frac{1}{2}+\varepsilon} T^{1+\varepsilon},
\end{aligned} \tag{2.6}
$$

where we have used

$$\int_{T_1/2}^{T_1} |L_f(1/2+\varepsilon+it)|^2 \, dt \ll T_1^{1+\varepsilon}$$

and

$$\int_{T_1/2}^{T_1} |L_f(1/2+\varepsilon+it) L_3(1/2+\varepsilon+it)|^2 \, dt \ll T_1^{3+\varepsilon}.$$

For the integral over the horizontal segments, we use Lemma 2.2 with $m = 8$ to get

$$
\begin{aligned}
I_2 + I_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{b} x^{\sigma} \, |L_{f,f,f}(\sigma+iT)| \, T^{-1} d\sigma \\
&\ll \max_{\frac{1}{2}+\varepsilon \le \sigma \le b} x^{\sigma} T^{4(1-\sigma)+\varepsilon} T^{-1} = \max_{\frac{1}{2}+\varepsilon \le \sigma \le b} \left(\frac{x}{T^4} \right)^{\sigma} T^{3+\varepsilon} \\
&\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon} T^{1+\varepsilon}.
\end{aligned} \tag{2.7}
$$

From (2.4), (2.6) and (2.7), we have

$$\sum_{n \le x} a_0(n)^3 \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon} T^{1+\varepsilon}. \tag{2.8}$$

On taking $T = x^{\frac{1}{4}}$ in (2.8), we have

$$\sum_{n \le x} a_0(n)^3 \ll x^{\frac{3}{4}+\varepsilon}. \tag{2.9}$$

This completes the proof of Theorem 1.1. □

3. Proof of Theorems 1.3 and 1.4

If $f(z) \neq \varphi(z)$, it was shown in [4] that for $\mathrm{Re}(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{a_0(n)^2 b_0(n)^2}{n^s} = \zeta(s) L(s, F_1) L(s, F_2) L(s, F_1 \times F_2) \prod_p Q_p(p^{-s}),$$

where

$$Q_p(p^{-s}) = 1 - \frac{\gamma_2(p)}{p^{2s}} + \cdots + \frac{1}{p^{14s}}.$$

Here F_1 is the Gelbart-Jacquet lift (see Gelbart and Jacquet [5]) on GL(3) associated to f, F_2 is the Gelbart-Jacquet lift on GL(3) associated to φ, and $F_1 \times F_2$ is the

Rankin-Selberg convolution of F_1 and F_2. The L-functions $L(s,F_1)$, $L(s,F_2)$ and $L(s,F_1\times F_2)$ are defined by

$$L(s,F_1)=\prod_p\left(1-\frac{\alpha_p^2}{p^s}\right)^{-1}\left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{\bar\alpha_p^2}{p^s}\right)^{-1},$$

$$L(s,F_2)=\prod_p\left(1-\frac{\beta_p^2}{p^s}\right)^{-1}\left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{\bar\beta_p^2}{p^s}\right)^{-1},$$

$$L(s,F_1\times F_2)=\prod_p\left(1-\frac{\alpha_p^2\beta_p^2}{p^s}\right)^{-1}\left(1-\frac{\alpha_p^2}{p^s}\right)^{-1}\left(1-\frac{\alpha_p^2\bar\beta_p^2}{p^s}\right)^{-1}\left(1-\frac{\beta_p^2}{p^s}\right)^{-1}$$
$$\times\left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{\bar\beta_p^2}{p^s}\right)^{-1}\left(1-\frac{\bar\alpha_p^2\beta_p^2}{p^s}\right)^{-1}\left(1-\frac{\bar\alpha_p^2}{p^s}\right)^{-1}\left(1-\frac{\bar\alpha_p^2\bar\beta_p^2}{p^s}\right)^{-1}.$$

In fact $L(s,F_i)$, $i=1,2$, are entire functions and $L(1,F_i)\neq 0$. Furthermore, from the assumption that f and φ are distinct, one can easily show that F_1 and F_2 are distinct. Therefore $L(s,F_1\times F_2)$ is also an entire function and $L(1,F_1\times F_2)\neq 0$. All these L-functions satisfy functional equations of Riemann type (see [5], [6], [7], [8] and [10]). Therefore $\zeta(s)L(s,F_1)L(s,F_2)L(s,F_1\times F_2)$ only has a simple pole at the point $s=1$ and satisfies a functional equation of Riemann type. In addition the product $\prod_p Q_p(p^{-s})$ converges uniformly for any $\varepsilon>0$ in the half-plane $\mathrm{Re}(s)\geq\frac12+\varepsilon$.

If $f(z)=\varphi(z)$, it was shown by Moreno and Shahidi [11] that for $\mathrm{Re}(s)>1$,

$$\sum_{n=1}^{\infty}\frac{a_0(n)^4}{n^s}=\zeta(s)L(s,F)^2L(s,F\times F)\prod_p T_p(p^{-s}), \tag{3.1}$$

where

$$T_p(p^{-s})=1-\frac{\delta_2(p)}{p^{2s}}+\cdots+\frac{1}{p^{14s}}.$$

Here F is the Gelbart-Jacquet lift on GL(3) associated to f. By the above Euler product of $L(s,F_1\times F_1)$ with $F_1=F_2=F$, we have

$$\zeta(s)L(s,F)^2L(s,F\times F)=\zeta(s)L(s,F)^3L(s,F,\vee^2), \tag{3.2}$$

where

$$\begin{aligned}L(s,F,\vee^2,s)\;=\;&\prod_p\left(1-\frac{\alpha_p^4}{p^s}\right)^{-1}\left(1-\frac{\alpha_p^2}{p^s}\right)^{-1}\left(1-\frac{1}{p^s}\right)^{-1}\\&\times\left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{\bar\alpha_p^2}{p^s}\right)^{-1}\left(1-\frac{\bar\alpha_p^4}{p^s}\right)^{-1}\end{aligned}$$

is the symmetric square L-function of the form F. Bump and Ginzburg [1] proved that the function $L(s,F,\vee^2)=\zeta(s)L(\mathrm{Sym}^4 f,s)$ has a simple pole at $s=1$ and is analytic everywhere else. Therefore $\zeta(s)L(s,F)^2L(s,F\times F)$ has a double pole at $s=1$ and is analytic at other points. The product $\prod_p T_p(p^{-s})$ converges uniformly for any $\varepsilon>0$ in the half-plane $\mathrm{Re}(s)\geq\frac12+\varepsilon$ and has no zeros in this region.

Now we use these properties of L-functions to complete the proofs of Theorems 1.3 and 1.4. The proofs of Theorems 1.3 and 1.4 are very similar in essence; therefore we give the proof only of Theorem 1.4.

Proof of Theorem 1.4. By (3.1) and Perron's formula, we have

$$\sum_{n\le x} a_0(n)^4 = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \zeta(s)L(s,F)^2L(s,F\times F)\prod_p T_p(p^{-s})\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.3}$$

where $b = 1+\varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later.

Then we move the integration to the parallel segment with $\mathrm{Re}(s) = \frac{1}{2}+\varepsilon$. By Cauchy's theorem, we have

$$\begin{aligned}
&\sum_{n\le x} a_0(n)^4 \\
&\quad = \frac{1}{2\pi i}\left\{\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} + \int_{\frac{1}{2}+\varepsilon+iT}^{b+iT} + \int_{b-iT}^{\frac{1}{2}+\varepsilon-iT}\right\} \\
&\qquad\qquad \zeta(s)L(s,F)^2L(s,F\times F)\prod_p T_p(p^{-s})\frac{x^s}{s}ds \\
&\qquad + \mathrm{Res}_{s=1}\left\{\zeta(s)L(s,F)^2L(s,F\times F)\prod_p T_p(p^{-s})\frac{x^s}{s}\right\} + O\left(\frac{x^{1+\varepsilon}}{T}\right) \\
&\quad =: C_1 x\log x + C_2 x + J_1 + J_2 + J_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right). \qquad (3.4)
\end{aligned}$$

To go further, we recall that $\zeta(s)L(s,F)^2L(s,F\times F)$ is a Riemann-type nice L-function with Euler product of degree $m = 16$.

For J_1, by (3.1) and Lemma 2.2 we have

$$\begin{aligned}
J_1 \ll x^{\frac{1}{2}+\varepsilon}\int_1^T &\left|L\left(1/2+\varepsilon+it,F\right)^2 L\left(1/2+\varepsilon+it,F\times F\right)\zeta\left(1/2+\varepsilon+it\right)\right| t^{-1}dt \\
&+ x^{\frac{1}{2}+\varepsilon}.
\end{aligned}$$

Then by Lemma 2.1, we have

$$\begin{aligned}
J_1 &\ll x^{1/2+\varepsilon}\log T \max_{T_1\le T}\left\{\frac{1}{T_1}\left(\int_{T_1/2}^{T_1}\left|L\left(1/2+\varepsilon+it,F\right)^2\right|^2 dt\right)^{\frac{1}{2}}\right. \\
&\qquad \left.\times\left(\int_{T_1/2}^{T_1}\left|L\left(1/2+\varepsilon+it,F\times F\right)\zeta\left(1/2+\varepsilon+it\right)\right|^2 dt\right)^{\frac{1}{2}}\right\} \\
&\ll x^{\frac{1}{2}+\varepsilon}T^{3+\varepsilon}, \qquad (3.5)
\end{aligned}$$

where we have used

$$\int_{T_1/2}^{T_1}\left|L\left(1/2+\varepsilon+it,F\right)^2\right|^2 dt \ll T^{3+\varepsilon},$$

$$\int_{T_1/2}^{T_1}\left|L\left(1/2+\varepsilon+it,F\times F\right)\zeta\left(1/2+\varepsilon+it\right)\right|^2 dt \ll T^{5+\varepsilon}.$$

For the integral over the horizontal segments, we use Lemma 2.2 with $m = 16$ to get

$$\begin{aligned} J_2 + J_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{b} x^{\sigma} \left|\zeta(\sigma+iT)L(\sigma+iT,F)^2 L(\sigma+iT,F\times F)\right| T^{-1} d\sigma \\ &\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le b} x^{\sigma} T^{8(1-\sigma)+\varepsilon} T^{-1} = \max_{\frac{1}{2}+\varepsilon\le\sigma\le b} \left(\frac{x}{T^8}\right)^{\sigma} T^{7+\varepsilon} \qquad (3.6) \\ &\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon} T^{3+\varepsilon}. \end{aligned}$$

From (3.4), (3.5) and (3.6), we obtain

$$\sum_{n\le x} a_0(n)^4 = C_1 x\log x + C_2 x + O\left(\frac{x^{1+\varepsilon}}{T}\right) + O(x^{\frac{1}{2}+\varepsilon}T^{3+\varepsilon}). \qquad (3.7)$$

On taking $T = x^{\frac{1}{8}}$ in (3.7), we have

$$\sum_{n\le x} a_0(n)^4 = C_1 x\log x + C_2 x + O(x^{\frac{7}{8}+\varepsilon}). \qquad (3.8)$$

This completes the proof of Theorem 1.4. □

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