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IN ORBIFOLDS, SMALL ISOPERIMETRIC REGIONS ARE SMALL BALLS

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Abstract. In a compact orbifold, for a small prescribed volume, an isoperimetric region is close to a small metric ball; in a Euclidean orbifold, it is a small metric ball.

1. Introduction

Even in smooth Riemannian manifolds M, there are relatively few examples of explicitly known regions which minimize the perimeter for a prescribed volume (see [M1, 13.2], [HHM]). One general result is that for M compact, for small volume, a perimeter-minimizing region is a nearly round ball where the scalar curvature is large ([K], [MJ, Thm. 2.2], or in 3D [Ros, Thm. 18], with [Dr]).

For singular ambients, one result in general dimensions is that in a smooth cone of positive Ricci curvature, which has just a single singularity at the apex, geodesic balls about the apex minimize the perimeter [MR, Cor. 3.9]. Similarly in the surface of a convex polytope in $\mathbf{R}^n$, with its stratified singular set, for a small prescribed volume, geodesic balls about some vertex minimize the perimeter [M2, Thm. 3.8]. Our Theorem 3.4 proves a similar result for orbifolds, which unlike the previous categories, are not topological manifolds. An orbifold is locally a Riemannian manifold modulo a finite group of isometries (see Definition 2.1). Corollary 3.3 concludes that in a Euclidean orbifold, for small volumes, an isoperimetric region is a metric ball.

The proof. The proof of our main Theorem 3.4 has the following steps, sometimes following [M2] and its predecessors, sometimes needing new arguments special to orbifolds. Let R_α be a sequence of isoperimetric regions in the orbifold O with volumes approaching zero.

(1) After covering O by maps from unit lattice cubes in $\mathbf{R}^n$, Lemma 3.1 obtains a concentration of volume after [M1, 13.7].
(2) After covering O by geodesic balls modulo finite groups and isometrically embedding the balls in some Euclidean space, we obtain a limit of blowups of preimages T_α of the R_α, nonzero by (1).
(3) After Almgren [Alm] and [M2, Lemma 3.5], we carefully show that ∂T_α locally minimizes the area plus $C\Delta V$ among invariant surfaces.

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(4) Using (3), generalizing "monotonicity" as in [M2, Lemma 3.6], we obtain a lower bound on the area of ∂T_α in any unit ball about one of its points and deduce that components of isoperimetric regions have small diameters.
(5) As in [MJ], it follows that for a small prescribed volume, isoperimetric regions are close to being small metric balls.

2. Orbifolds and invariant regions

2.1. **Definitions.** For $n \geq 1$, an nD (smooth Riemannian) *orbifold* O is a connected metric space locally isometric to a smooth nD Riemannian manifold M modulo a finite group G of isometries. Locally, the rectifiable currents of O (the generalized kD surfaces of geometric measure theory) are just the G-invariant rectifiable currents of M. (The locally rectifiable currents of M are defined by isometrically embedding M in some Euclidean space. The space and its topology are independent of the embedding. See [M1].)

The following lemma makes many basic geometric properties of orbifolds obvious.

2.2. **Structure Lemma.** *An nD orbifold is a smooth Riemannian manifold with boundary except for a smoothly stratified $(n-2)D$ singular set.*

We mean that the singular set is an $(n-2)D$ manifold except for a smoothly stratified $(n-3)D$ singular set defined recursively down to isolated singular points.

Proof by induction. The case $n = 1$ is trivial. For $n > 1$, let p be a point in an nD orbifold O, which is locally isometric to a manifold M modulo a finite group of isometries G leaving a corresponding point q fixed. The tangent space modulo G is the cone over the unit sphere modulo G, for which the lemma holds by induction. Therefore the lemma holds for the tangent space modulo G and hence for M/G, hence for O.

2.3. **Observation** ([M2, Remark after 3.3])**.** Let G be a finite subgroup of $\mathbf{O}_n$. Since balls are uniquely perimeter minimizing in $\mathbf{R}^n$, G-invariant balls are uniquely minimizing among invariant regions. In other words, in $\mathbf{R}^n$ modulo a finite subgroup of $\mathbf{O}_n$, a perimeter-minimizing region for fixed volume is a metric ball about the image of a fixed point.

The following regularity theorem of [M3] applies indirectly to orbifolds, since surfaces in orbifolds locally correspond to invariant surfaces in manifolds.

2.4. **Theorem.** *In a smooth (C^∞) Riemannian manifold M, among hypersurfaces oriented by the unit normal invariant under a group G of isometries of M, perhaps with given boundary or homology or volume, suppose that S minimizes the area. Then on the interior, S is a smooth constant-mean-curvature hypersurface except for a singular set of codimension at least* 7 *in S.*

Proof. This theorem is stated and proved in [M3, Thm. 4.1], without full details for the case of a volume constraint, which we now provide. There is a $\Upsilon > 0$ such that a hypercone which is area-minimizing without volume constraint among invariant surfaces and which has density less than Υ is a hyperplane [M3, Lemma 4.4]. By Allard's regularity theorem [A, Sect. 8], S is regular ($C^{1,1/2}$) at all such points. By Federer [F, Lemma 2] the Hausdorff dimension of the singular set of S is less than or equal to the Hausdorff dimension of the singular set of some area-minimizing (tangent) hypercone in $\mathbf{R}^n$, which is at most $n-8$, i.e., codimension at least 7 in S. Higher smoothness follows by Schauder theory (see [M4, Prop. 3.5]).

3. Isoperimetric regions in orbifolds

Our main Theorem 3.4 follows from Proposition 3.2 and the methods of [MJ], which treats the easier case of manifolds. First we need a lemma on concentration of volume after [M1, 13.7]. For the lemma we will work in the category of metric spaces O which are smooth nD manifolds except for a set of nD Hausdorff measure 0. Regions and their boundaries are technically locally rectifiable currents in the regular part of O [M1].

3.1. **Lemma.** *For $n \geq 1$, let O be a metric space which is a smooth nD manifold except for a set of nD Hausdorff measure 0, covered by some finite number k_1 of distance-decreasing maps from disjoint open unit lattice cubes in $\mathbf{R}^n$ each of multiplicity bounded by k_2 which shrink $(n-1)D$ or nD Hausdorff measure by at most some constant factor C_1. Then given $C > 0$ there is a constant $\delta > 0$ such that for any region R in O with topological boundary ∂R and with nD and $(n-1)D$ Hausdorff measures satisfying $|\partial R| \leq C|R|^{(n-1)/n}$, the image K of some relevant lattice subcube of volume between $2^{-n}|R|$ and $|R|$ satisfies*

$$|R \cap K| \geq \delta |R|. \tag{1}$$

Proof. Subdivide the unit lattice cubes into congruent subcubes J_i with volume between $|R|$ and $2^{-n}|R|$. Let K_i denote the image of J_i in O and let R_i denote the preimage of R in J_i. We may assume that $\delta < 1/2$ and $|R_i| \leq |J_i|/2$, since otherwise $|R \cap K_i| \geq |J_i|/2C_1 \geq |R|/2^{n+1}C_1$ and we are done. By the relative isoperimetric inequality [M1, 12.3(1)] for open cubes, there is a single constant γ such that

$$|\partial R_i| \geq \gamma |R_i|^{(n-1)/n}.$$

Hence

$$|\partial R \cap K_i| \geq \gamma |R \cap K_i|^{(n-1)/n}/C_1 \geq \gamma |R \cap K_i|/C_1 \max |R \cap K_j|^{1/n}.$$

Since the total multiplicity of the covering is at most $k = k_1 k_2$, summing over i yields

$$k|\partial R| \geq \gamma |R|/C_1 \max |R \cap K_j|^{1/n},$$

$$\max |R \cap K_j|^{1/n} \geq \gamma |R|/kC_1|\partial R| \geq \gamma |R|^{1/n}/kCC_1.$$

Hence some K satisfies

$$|R \cap K| \geq \delta |R|$$

with $\delta = (\gamma/kCC_1)^n$, as desired.

3.2. **Proposition.** *For $n \geq 2$, let O be a compact nD orbifold. For a small prescribed volume, the sum of the diameters of the components of a perimeter-minimizing region is small.*

Proof. Fix finitely many Riemannian geodesic balls M_i modulo finite groups G_i of rotations about the center which cover O. We may assume that the M_i are isometrically embedded in some fixed $\mathbf{R}^N$.

Let R_α be a sequence of perimeter-minimizing regions in O with volume $|R_\alpha|$ small and approaching 0. Comparison with small geodesic balls about a regular point of O shows that $|\partial R_\alpha| \leq C|R_\alpha|^{(n-1)/n}$. For each R_α use Lemma 3.1 to choose K_α with $2^{-n}|R_\alpha| \leq |K_\alpha| \leq |R_\alpha|$ and

$$|R_\alpha \cap K_\alpha| \geq \delta_0 |R_\alpha| \geq \delta_0 |K_\alpha|.$$

By taking a subsequence, we may assume that each K_α is contained well inside one fixed M_i/G_i, which we will now call M/G. Rescale K_α and M up to $1 = |R_\alpha| \geq |K_\alpha| \geq 2^{-n}$; denote the rescaling of M by M_α, with extrinsic curvature going to zero. We may assume that M_α is tangent to $\mathbf{R}^n \times \{0\}$ at the origin at a point p_α in the preimage of K_α.

By compactness [M1, 9.1], we may assume that the preimages T_α of R_α in M_α converge weakly to a region T with $|G| \geq |T| \geq 2^{-n}\delta_0 > 0$, lying in $\mathbf{R}^n \times \{0\}$.

Consider any g in G, and let q_α be its fixed point in M_α closest to the origin of $\mathbf{R}^N$. Unless the sequence q_α diverges to infinity, by taking a subsequence we may assume that q_α converges to a point q in $\mathbf{R}^n \times \{0\}$ and that the action of g on M_α converges to an isometry g of $\mathbf{R}^n \times \{0\}$ with fixed point q, in the sense that if points x_α in M_α converge to x in $\mathbf{R}^n \times \{0\}$, then $g(x_\alpha)$ converges to $g(x)$, uniformly on compacts. Thus we obtain a limit action of some subgroup G_1 of G on $\mathbf{R}^n \times \{0\}$, which extends trivially to $\mathbf{R}^N$.

There are many points of ∂T not on the cone of fixed points of G_1. Choose a small positive δ_1 so that there are two δ_1-balls B_1, B_2 in $\mathbf{R}^n \times \{0\}$ about such points of ∂T with disjoint images under G_1 such that any third δ_1-ball B is disjoint from the image under G_1 of one of them, say B_1. By the Gauss-Green-De Giorgi-Federer Theorem [M1, 12.2], at almost all points of the current boundary ∂T, T has a measure-theoretic exterior normal and the approximate tangent cone is a half-space of $\mathbf{R}^n \times \{0\}$. As in Almgren [Alm, V1.2(3)], gently pushing in that normal direction yields a smooth family Φ_t of diffeomorphisms for $|t|$ small supported in a shrunken ball $B_1'' \subset B_1' \subset B_1$ such that Φ_0 is the identity and the initial rate of change of the volume of T satisfies $dV/dt|_0 = 1$; let $A_0 = dA/dt|_0$ be the initial rate of change of the boundary area. Choose $t_0 > 0$ so small that for all $-t_0 \leq t \leq t_0$, we have $.9 \leq dV/dt \leq 1.1$ and $|dA/dt| \leq |A_0| + 1$. View the Φ_t as a smooth family of diffeomorphisms of $\mathbf{R}^N$ supported in $B_1'' \times \mathbf{R}^{N-n}$.

(2) For α large, for all $-.9t_0 \leq \Delta V \leq .9t_0$, such diffeomorphisms of T_α alter the volume by ΔV and increase the area of ∂T by at most $|\Delta V|(|A_0|+1)/.9 = C_1|\Delta V|$.

In M_a, let $B_\alpha = M_\alpha \cap (B_1' \times \mathbf{R}^{N-n})$ and let Φ_t^α denote the G symmetrization of the Φ_t, supported in M_α in $G(B_\alpha)$, which consists of the disjoint images of B_α under G_1 and distant copies thereof under other elements of G. Since the T_α are G-invariant, (2) still holds. We may assume that any $\delta_1/2$-ball in M_α is disjoint from the support of the Φ_t^α. Choose $\delta < \delta_1/2$ such that a δ-ball in any M_α has volume less than $.9t_0$. We claim that for some $C_2 > (1/2\delta)$ (independent of α)

(3) in any δ-ball B in $M_\alpha, \partial T_\alpha$ minimizes the area plus $C_2|S - T_\alpha|$ in comparison with the restriction of other G-invariant surfaces ∂S to B with the same boundary in ∂B.

We may assume that S coincides with ∂T_α outside $G(B)$. Choose a ball B_α as above with $G(B_\alpha)$ disjoint from B and hence disjoint from $G(B)$. By (2), obtain S' by altering S in $G(B_\alpha)$ so that it bounds a net volume 0 with ∂T_α and such that its area satisfies

$$|S'| \le |S| + C_1|S - T_\alpha|_{G(B)} \le |S| + |G|C_1|S - T_\alpha|_B.$$

Since T_α is isoperimetric among G-invariant regions,

$$|\partial T_\alpha| \le |S'| \le |S| + |G|C_1|S - T_\alpha|_B,$$

proving the claim (3). Next we claim that

(4) there is a positive lower bound (independent of α) on the area of ∂T_α in any unit ball about one of its points in M_α.

Consider a unit ball about a point p of ∂T_α. For $r \le \delta$, let $T_r = \partial T_\alpha \cap B(p, r)$ and let $g(r)$ denote the $(n-1)D$ area of T_r. For almost all r, the boundary of T_r has $(n-2)D$ area at most $g'(r)$ (see [M1, Chapt. 9]). By an isoperimetric inequality for the manifold M [M1, Sect. 12.3], which is invariant under scaling of M in $\mathbf{R}^N$ and hence holds for each M_α, this same boundary bounds a surface T_r' of area

$$|T_r'| \le \gamma g'(r)^{(n-1)/(n-2)},$$

for some isoperimetric constant γ; together with T_r, T_r' bounds a region of mass or volume at most

$$\beta[g(r) + \gamma g'(r)^{(n-1)/(n-2)}]^{n/(n-1)} \le C_3 g(r)^{n/(n-1)} + C_3 g'(r)^{n/(n-2)}$$

for some isoperimetric constant β and constant C_3. By claim (3),

$$\begin{aligned}|T_r| &\le |T_r'| + C_4 g(r)^{n/(n-1)} + C_4 g'(r)^{n/(n-2)} \\ &\le \gamma g'(r)^{(n-1)/(n-2)} + C_4 g(r)^{n/(n-1)} + C_4 g'(r)^{n/(n-2)}.\end{aligned}$$

Hence for some $C_5 > 0$,

$$g(r)(1 - C_5 g(r)^{1/(n-1)}) \le C_5[g'(r)^{(n-1)/(n-2)} + g'(r)^{n/(n-2)}].$$

We may assume that the coefficient of $g(r)$ for any $r \le 1/(2C_2) \le \delta$ is at least $1/2$, since otherwise we have a lower bound as desired for $g(r)$. Let

$$\Omega = \{0 < r < \delta : g'(r) \le 1\}.$$

We may assume that $|\Omega| \ge \delta/2$, since otherwise we immediately obtain a lower bounded as desired for $g(r)$. On Ω,

$$g(r) \le C_6 g'(r)^{(n-1)/(n-2)}.$$

Since g is monotonically increasing, integration yields

$$g(r) \ge C_7 r^{n-1},$$

again yielding the desired lower bound for $g(1/(2C_2))$, proving the claim (4).

By claim (4), there is a bound on the sums of the diameters of the components of ∂R_α after our scaling, so for the original ∂R_α before scaling, the sum of the diameters of the components of ∂R_α and hence of R_α goes to zero.

3.3. **Corollary.** *In a compact Euclidean orbifold, for small prescribed volume, an isoperimetric region is a small metric ball.*

Proof. By Proposition 3.2, a component of an isoperimetric region has small diameter. By Observation 2.3, it is a metric ball about the image of a fixed point. For such balls of volume V, the perimeter is proportional to $V^{(n-1)/n}$. Since these functions are concave, a single ball is best.

Remark. Corollary 3.3 and its proof also apply to orbifolds modeled on the sphere or on hyperbolic space.

3.4. **Theorem.** *In a compact orbifold O, for small prescribed volume, an isoperimetric region R is C^0 close to a small metric ball about a point where the group action has highest effective order. More specifically, if locally O is the quotient of a Riemannian geodesic ball M by a finite group G fixing its center, the preimage of R is C^∞ close to a geodesic ball in M, and the order of G is the maximum over all such groups associated with O.*

Remark. In particular, in a smooth compact Riemannian manifold, for small volume, an isoperimetric region among regions invariant under a group G of isometries is the orbit of a nearly round ball.

Proof. The orbifold O is covered by quotients M_i/G_i of Riemannian geodesic balls; we may assume that they still cover when shrunken by a factor of two. By Proposition 3.2, R has components of small diameter, each lying in the center of some M_i/G_i. Their preimages R_i in M_i are perimeter-minimizing among G_i-invariant regions. The arguments of [MJ, Thm. 2.2] apply and show that each R_i is C^∞ close to a metric ball. Indeed, a limit of such R_i with volume approaching 0 must be a minimizer in Euclidean space among G_i'-invariant regions, for some subgroup G_i' of G_i, as in the proof of Proposition 3.2. By Observation 2.3, such a minimizer must be a round ball, and R_i consists of an orbit under G_i of G_i'-invariant balls. Easy estimates show that a single G_i-invariant ball is best. In particular, each component of R is C^0 close to a small metric ball (which is topologically a quotient of a ball). If there were more than one R_i, the second variational formula [MJ, 2.1] would imply instability, by uniformly expanding R_1 and shrinking R_2 (preserving the enclosed volume). The final statement holds because for a small metric ball, the boundary area and volume satisfy

$$A \approx \beta V^{(n-1)/n}/|G_i|^{1/n},$$

where $A = \beta V^{(n-1)/n}$ in $\mathbf{R}^n$.

Isolated singularities in orbifolds are modeled on $\mathbf{R}^n$ modulo a finite subgroup of the orthogonal group with a single fixed point. The following consequence of [MR] applies to manifolds with more general isolated conical singularities.

3.5. **Proposition.** *Let M^n be a smooth compact submanifold of $\mathbf{R}^N$ with a finite, positive number of conical singularities, where it is locally equivalent by a diffeomorphism of $\mathbf{R}^N$ to a cone with nonnegative Ricci curvature over a connected submanifold of $\mathbf{S}^{N-1}$. Then for small prescribed volume, an isoperimetric region is C^∞ close to a metric ball about a singularity.*

Proof. Choose $0 < V_0 < 1$ by Bérard-Meyer [MR, Thm. 2.1] for the complement in M of unit balls about its singularities. A sequence of minimizers with volume $V_i \to 0$, scaled up by homothety about the k^{th} singularity to volume V_0, converges to an isoperimetric region in the cone C_k at that singularity. The region is nonempty for at most one such cone C. By [MR, Cor. 3.9], such an isoperimetric region in C must be a ball about the vertex. As in the proof of [MR, Thm. 2.2], no volume is lost to infinity. By [M1, Lemma 3.5], the (rescaled) sequence of minimizers has uniformly, weakly bounded mean curvature. By Allard's regularity theorem, the convergence to the ball is $C^{1,\alpha}$. By Schauder theory (see [M4, 3.3 and 3.5]), the convergence is C^∞.

General singularities in orbifolds are modeled on $\mathbf{R}^n$ modulo a finite subgroup of the orthogonal group, where isoperimetric regions are metric balls (Cor. 3.3). The following consequence of [MR] extends Corollary 3.3.

3.6. **Proposition.** *Let C^n be a cone with nonnegative Ricci curvature over a connected submanifold of the sphere in $\mathbf{R}^N$. Then isoperimetric regions in $C \times \mathbf{R}^m$ are metric balls about points in the product of the vertex with $\mathbf{R}^m$.*

Proof. By [MR, Cor. 3.9], isoperimetric regions in C are balls about the vertex. As noted in [MR, Cor. 3.12], by symmetrization, horizontal slices in C of an isoperimetric region in $C \times \mathbf{R}^m$ are metric balls. Similarly, vertical slices in $\mathbf{R}^m$ are round balls. The minimizer in $C \times \mathbf{R}^m$ is determined by a generating curve in the first quadrant of the plane. It solves the same planar problem as the generator of the $\mathbf{O}(n) \times \mathbf{O}(m)$-invariant minimizer in $\mathbf{R}^{n+m}$, which is of course a sphere. (Namely, it minimizes $\int x^{n-1}y^{m-1}\,ds$ given $\int x^{n-1}y^{m-1}\,dA$.) Therefore the generating curve is a circle and the isoperimetric region is a metric ball.

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