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ON A PROBLEM OF BORSUK AND ULAM

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Abstract. Borsuk and Ulam posed the following problem: For an arbitrary closed subset C of the d-dimensional sphere, does there exist a sequence of homeomorphisms of the sphere such that the sequence of images of every point of the sphere converges to a point of C and each point of C is the limit of such a sequence? The answer is known to be positive, but the existing proof is complicated. We give a simple proof that extends to some other manifolds including the d-dimensional torus.

Among the topological problems that S. Ulam included in his book [5], was the following question originally posed in [2]:

Given an arbitrary closed subset $C \neq \emptyset$ of the d-dimensional sphere S^d, does there exist a sequence of homeomorphisms $H_n\colon S^d \to S^d$, such that
(a) for any $p \in S^d$, there exists a limit of $H_n(p)$ as $n \to \infty$;
(b) this limit belongs to C;
(c) every point of C is such a limit.

M. K. Fort [4] gave an affirmative answer to this question. He proved the following statement:

Theorem 1. *For any closed set $C \subset S^d$ $(C \neq \emptyset)$, there exists a sequence of homeomorphisms H_n of S^d $(n = 1, 2, \ldots)$ such that*
(i) $H_n(p) = p$ for any $p \in C$ and all n;
(ii) for any $p \in S^d \setminus C$ the sequence $H_n(p)$ converges to some point of C.

However, the proof in [4], based on the concept of an admissible polyhedron and on an earlier result [3], is not highly intuitive. Below we give an alternative proof of Theorem 1, which is simpler and gives more (see the discussion following the proof).

Proof. The stereographic projection R provides a homeomorphism of S^d onto the one-point compactification $\overline{\mathbf{R}^d}$ of $\mathbf{R}^d$. It suffices, therefore, to solve the problem for $\overline{\mathbf{R}^d}$ and its compact subset $C' = R(C)$. We may assume that the point $R^{-1}(\infty)$ belongs to C, so that $\infty \in C'$.

We define a function $Q\colon \mathbf{R}^d \to \mathbf{R}^d$ by $Q(x) = f(x)\,u$, where u is a nonzero vector in $\mathbf{R}^d$ and $f(x) = \min(\operatorname{dist}(x, C'), 1)$. Here $\operatorname{dist}(x, C')$ is the Euclidean distance from x to the closed subset $C' \cap \mathbf{R}^d$ of $\mathbf{R}^d$ if that subset is nonempty;

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otherwise $\operatorname{dist}(x, C') = +\infty$. The Lipschitzian function $Q(x)$ defines a system of ordinary differential equations

$$\frac{dx}{dt} = Q(x), \tag{1}$$

to which the existence and uniqueness theorem [1] applies; since $Q(x)$ is bounded, the solutions $x(t)$ are defined for all $t \in \mathbf{R}$. In view of the continuous dependence of a solution on the initial data [1], the mapping G_t that transforms $x(0)$ into $x(t)$ is a homeomorphism of $\mathbf{R}^d$ onto itself. Moreover, letting $G_t(\infty) = \infty$ we obtain a homeomorphism of $\overline{\mathbf{R}^d}$ onto itself.

Clearly, every integral curve $x(t)$ of the system (1) lies on a line parallel to the vector u and, as $t \to \infty$, converges to a limit, which may be infinite or finite; in both cases it belongs to C'. Finally, if $x \in C'$, then $G_t(x) = x$ for all $t \in \mathbf{R}$. Therefore, the sequence of homeomorphisms G_n $(n = 1, 2, \ldots)$ of $\overline{\mathbf{R}^d}$ is such that (i′) $G_n(x) = x$ for any $x \in C'$ and all n; (ii′) for any $x \in \overline{\mathbf{R}^d} \setminus C'$ the sequence $G_n(x)$ converges to some point of C'. It follows that the sequence of homeomorphisms $H_n = R^{-1} G_n R$ of S^d has properties (i) and (ii). □

Remark 1. It is a by-product of the above construction that the sequence of inverse homeomorphisms H_n^{-1} also solves the Borsuk–Ulam problem for S^d and C.

Remark 2. Since $G_{s+t} = G_s\, G_t$ for all $s, t \in \mathbf{R}$, the sequence of homeomorphisms H_n constructed above consists of *powers* of the homeomorphism H_1.

Remark 3. The construction (and its consequences just indicated) extends naturally to an arbitrary closed smooth manifold $\mathcal{M}$ with the following property:

(A) There exists a locally Lipschitzian vector field $V(p)$ on $\mathcal{M} \setminus \{p^*\}$ (with some $p^* \in \mathcal{M}$) whose every integral curve $p(t)$ is defined for all $t \in \mathbf{R}$ and converges to p^* as $t \to \pm\infty$.

It is obvious that the sphere S^d has this property (see above). Another example of a manifold with property (A) is the d-dimensional torus $\mathbf{T}^d$. (Indeed, we can represent $\mathbf{T}^d$ as the quotient space $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$ with coordinates $x_1, x_2, \ldots, x_d$ (x_j mod 1) and consider the vector field with components $V_j(x_1, \ldots, x_d) = \sin^2(\pi x_j)$, $j = 1, 2, \ldots, d$.) Therefore, the Borsuk–Ulam problem for $\mathbf{T}^d$ has a positive solution. The same, and for a similar reason, is true for the Klein bottle.

Problem 1. *Does there exist a connected closed smooth manifold* without *property (A)?*

The following property of a manifold $\mathcal{M}$ is formally weaker than (A), but still is sufficient for the Borsuk–Ulam problem to have a positive solution.

(A_+) There exists a locally Lipschitzian vector field $V(p)$ on $\mathcal{M} \setminus \{p^*\}$ (with some $p^* \in \mathcal{M}$) whose every integral curve $p(t)$ is defined for all $t \in \mathbf{R}$ and converges to p^* as $t \to +\infty$.

Problem 2. *Are properties (A) and (A_+) of a closed smooth manifold equivalent?*

Note that a negative answer to Problem 1 would imply a trivial positive answer to Problem 2.

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References

[1] Arnold, V. I., Ordinary differential equations, 3rd ed., Springer, Berlin, 1992. MR1162307 (93b:34001)

[2] K. Borsuk, S. M. Ulam, Über gewisse Invarianten der ε-Abbildungen, Math. Ann. **108** (1933), 311-318. MR1512851

[3] M. K. Fort, Jr., A theorem about topological n-cells, Proc. Amer. Math. Soc. **5** (1954), 456-459. MR0062434 (15:978d)

[4] M. K. Fort, Jr., Sequences of homeomorphisms on the n-sphere, Proc. Amer. Math. Soc. **12** (1961), 361-363. MR0132538 (24:A2378)

[5] S. M. Ulam, A collection of mathematical problems, Interscience Publishers, New York, 1960. MR0120127 (22:10884)

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