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FROM A RAMANUJAN-SELBERG CONTINUED FRACTION TO A JACOBIAN IDENTITY

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ABSTRACT. Jacobi proved an elegant identity involving eight-fold infinite products. In this paper, we give a new proof of this identity. A key ingredient of our proof is an identity satisfied by a Ramanujan-Selberg continued fraction.

1. INTRODUCTION

For $|q| < 1$, we define

$$T(q) := \frac{1}{1} + \frac{q}{1} + \frac{q+q^2}{1} + \frac{q^3}{1} + \frac{q^2+q^4}{1} + \cdots.$$

Independently, Ramanujan and Selberg studied this interesting continued fraction. Ramanujan asserted in his notebook [25, p. 290] that

$$T(q) = \frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty}. \tag{1.1}$$

Here and in the rest of the paper we follow the customary q-product notation: we set (for $|q| < 1$)

$$(c;q)_\infty := \prod_{k=0}^{\infty}(1 - c\,q^k).$$

It was A. Selberg, unaware of Ramanujan's assertion at the time of writing, who gave the first proof of (1.1) in print [26]. Other proofs were discovered by Ramanathan [24], Andrews, Berndt, Jacobsen and Lamphere [4], and, Zhang [30]. Zhang, building upon his earlier results in [31, 32], gave explicit evaluations of $T(q)$ in terms of class invariants and singular moduli in [33]. See also the paper by Baruah and Saikia [7] for a penetrating study of $T(q)$ (and related continued fractions).

It may not be obvious how $T(q)$ is related to the following amazing identity discovered by Jacobi:

Theorem 1.1 (Jacobi)**.** *For* $|q| < 1$,

$$(q;q^2)_\infty^8 + 16q\,(-q^2;q^2)_\infty^8 = (-q;q^2)_\infty^8, \tag{1.2}$$

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or, in standard notation,

$$\prod_{n=1}^{\infty}\left(1-q^{2n-1}\right)^8+16q\prod_{n=1}^{\infty}\left(1+q^{2n}\right)^8=\prod_{n=1}^{\infty}\left(1+q^{2n-1}\right)^8.$$

Whittaker and Watson note that [29, p. 470] Jacobi was deeply impressed by (1.2). For proofs of (1.2), see the wonderful books by Whittaker and Watson [29] and by the Borweins [11]; see also an elegant paper by Ewell [19].

In this paper, we will give a new proof of (1.2) that is based on the following identity of $T(q)$:

Theorem 1.2. *Let* $x(q)=1/(2\,T^2(q))$. *Then*

$$x(q)-\frac{q^{1/2}}{x(q)}=\frac{(q^{1/2};q^{1/2})_\infty^4}{2(q^2;q^2)_\infty^4}. \tag{1.3}$$

In Section 2, we will prove Theorem 1.2. In Section 3, we will prove the Jacobian Identity (1.2). Section 4 is our concluding remarks.

Before we turn to Section 2, we apply Theorem 1.2 to evaluate $T(q)$. In terms of the eta function $\eta(z)=e^{2\pi iz/24}\prod_{n=1}\left(1-e^{2\pi inz}\right)$, we can write (1.3) as (with $q=e^{2\pi iz}$)

$$x(q)-\frac{q^{1/2}}{x(q)}=\frac{1}{2}q^{1/4}\left(\frac{\eta(z/2)}{\eta(2z)}\right)^4. \tag{1.4}$$

With the well-known identity, $\eta(-1/z)=\sqrt{z/i}\,\eta(z)$, we have, at $z=i$ (i.e., $q=e^{-2\pi}$), $\eta(i/2)/\eta(2i)=\sqrt{2}$. This, with (1.4), implies

$$T(e^{-2\pi})=\sqrt{\frac{e^{\pi/2}}{2\,(\sqrt{2}+1)}}.$$

For a comprehensive theory of the evaluations of $T(q)$, consult the wonderful papers by Zhang [33] and by Baruah and Saikia [7].

2. Proof of Theorem 1.2

Our proof is motivated by the recent works [15, 16]. First, we observe that

$$T^2(q)=\frac{1}{2}\left(\frac{\sum q^{j^2+j}}{\sum q^{j^2}}\right), \tag{2.1}$$

where in both sums j runs from $-\infty$ to ∞. To see this, we note that

$$T^2(q)=\frac{1}{2}\left(\frac{2(q^2;q^2)_\infty(-q^2;q^2)_\infty^2}{(q^2;q^2)_\infty(-q;q^2)_\infty^2}\right)=\frac{1}{2}\left(\frac{\sum q^{j^2+j}}{\sum q^{j^2}}\right).$$

The first equation is due to (1.1). The second is from Jacobi's Triple Product Identity (see, e.g., [2, 3, 6, 9])

$$\sum_{n=-\infty}^{\infty}(-z)^nQ^{n^2}=\prod_{j=1}^{\infty}(1-Q^{2j})(1-zQ^{2j-1})(1-z^{-1}Q^{2j-1}). \tag{2.2}$$

This proves (2.1).

For later convenience, we define, for $a = 0$ and 1,

$$G(a) := \sum_{j=-\infty}^{\infty} q^{j^2 + a\, j} \tag{2.3}$$

and

$$J(z) := \prod_{n=1}^{\infty} \left(1 - q^{n/2}\right) \left(1 + z\, q^{(2n-1)/4}\right) \left(1 + z^{-1} q^{(2n-1)/4}\right). \tag{2.4}$$

Note that the dependence of G on q is suppressed in this notation. Also, we can write $x(q) = 1/(2\,T^2(q))$ (as defined in Theorem 1.2) as

$$x(q) = \frac{G(0)}{G(1)}. \tag{2.5}$$

With the above notation understood, we prove the following lemma, which expresses G in terms of $J(\pm 1)$:

Lemma 2.1.

$$G(0) = \frac{1}{2}\left(J(1) + J(-1)\right), \tag{2.6}$$

$$q^{1/4}\, G(1) = \frac{1}{2}\left(J(1) - J(-1)\right). \tag{2.7}$$

Proof. We start with defining, for $a = 0$ and 1,

$$K_n(a) := \sum_{j=-\infty}^{\infty} q^{j^2 + a\, j} \left[\begin{array}{c} 2n \\ n + 2j \end{array} \right]. \tag{2.8}$$

Here, the q-binomial coefficients (or the Gaussian polynomials) are defined as

$$\left[\begin{array}{c} A \\ B \end{array} \right] = \begin{cases} 0 & \text{if } B < 0 \text{ or } B > A \text{ or } A < 0, \\ \dfrac{(1-q^A)(1-q^{A-1})\cdots(1-q^{A-B+1})}{(1-q^B)(1-q^{B-1})\cdots(1-q)} & \text{otherwise.} \end{cases}$$

See, e.g., [2, 6]. We will need the following property of the q-binomial numbers (see, e.g., [6]): for fixed m_1 and m_2, with $R > S$ positive,

$$\lim_{N \to \infty} \left[\begin{array}{c} R\,N + m_1 \\ S\,N + m_2 \end{array} \right] = \frac{1}{(q;q)_\infty}. \tag{2.9}$$

The key idea of our proof is to express (the limit of) $K_n(a)$ in two different ways.

First, we note that, with $K_\infty(a) := \lim_{n\to\infty} K_n(a)$, we have, by (2.8) and (2.9),

$$K_\infty(a) = \frac{G(a)}{(q;q)_\infty}. \tag{2.10}$$

Next, we write $K_\infty(a)$ in a different way. To proceed, we observe that

$$\begin{aligned} K_n(a) &= \sum_{\sigma \equiv 0 \ (\mathrm{mod}\ 2)} q^{\sigma^2/4 + (a/2)\,\sigma} \left[\begin{array}{c} 2n \\ n + \sigma \end{array} \right] \\ &= \frac{1}{2} \sum_{k=0}^{1} \left(\sum_{\sigma=-\infty}^{\infty} q^{\sigma^2/4 + (a/2)\,\sigma} \left[\begin{array}{c} 2n \\ n + \sigma \end{array} \right] (-1)^{k\sigma} \right) \\ &:= \frac{1}{2} \sum_{k=0}^{1} C_n(\,(-1)^k, a\,). \end{aligned} \tag{2.11}$$

Note that the second line is due to the fact that

$$\frac{1}{2}\sum_{j=0}^{1}(-1)^{j\sigma} = \begin{cases} 1 & \text{if } \sigma \equiv 0 \,(\mathrm{mod}\, 2), \\ 0 & \text{otherwise.} \end{cases}$$

Let $C_\infty(z,a) := \lim_{n\to\infty} C_n(z,a)$; then (2.11) implies

$$K_\infty(a) = \frac{1}{2}\sum_{k=0}^{1} C_\infty(\,(-1)^k, a\,). \tag{2.12}$$

To finish proving the second way of expressing $K_\infty(a)$, we need to write $C_\infty(z,a\,)$ in terms of $J(z)$. We claim that

$$C_\infty(z,0) = \frac{J(z)}{(q;q)_\infty}, \tag{2.13}$$

$$C_\infty(z,1) = \frac{1}{z\,q^{1/4}}\left(\frac{J(z)}{(q;q)_\infty}\right). \tag{2.14}$$

To prove (2.14), we observe that

$$\begin{aligned} C_\infty(z,1) &= \frac{1}{(q;q)_\infty}\sum_{\sigma=-\infty}^{\infty} z^\sigma q^{\sigma^2/4+\sigma/2} \\ &= \frac{1}{(q;q)_\infty}\prod_{n=1}^{\infty}\left(1-q^{n/2}\right)\left(1+zq^{(2n+1)/4}\right)\left(1+z^{-1}q^{(2n-3)/4}\right) \\ &= \frac{1}{z\,q^{1/4}}\left(\frac{J(z)}{(q;q)_\infty}\right). \end{aligned}$$

Note that the first line follows from the definition of $C_n(z,a)$ (cf. (2.11)) and the property of the q-binomial numbers (cf. (2.9)). The second line is due to Jacobi's Triple Product Identity (cf. (2.2)). For the last line, we recall the definition of $J(z)$ in (2.4). The proof of (2.13) is similar and will be omitted. Note that (2.12)-(2.14) give us a second way of writing $K_\infty(a)$.

By putting the above together, we prove the lemma. Indeed,

$$\begin{aligned} \frac{G(1)}{(q;q)_\infty} &= \frac{1}{2}\sum_{k=0}^{1} C_\infty(\,(-1)^k, 1\,) && \text{by (2.10) and (2.12)} \\ &= \frac{1}{2\,q^{1/4}\,(q;q)_\infty}\,(J(1)-J(-1)) && \text{by (2.14).} \end{aligned}$$

This proves (2.7). The proof of (2.6) follows the same type of calculation and will be omitted here. □

Proof of Theorem 1.2. First, we write the two identities in Lemma 2.1 as

$$\begin{aligned} G(0)+q^{1/4}\,G(1) &= J(1), \\ G(0)-q^{1/4}\,G(1) &= J(-1). \end{aligned}$$

Next, we multiply them together to get

$$G^2(0)-q^{1/2}\,G^2(1) = J(1)J(-1).$$

By dividing the last equation by $G(0)G(1)$ we obtain

$$x(q) - \frac{q^{1/2}}{x(q)} = \frac{J(1)J(-1)}{G(0)G(1)} \tag{2.15}$$

(recall that $x = G(0)/G(1)$ by (2.5)). Our last task is to simplify the right-hand side of (2.15). To this end, we observe that

$$\begin{aligned}
\frac{J(1)J(-1)}{G(0)G(1)} &= \frac{1}{2}\prod_{n=1} \frac{(1-q^{n/2})^2(1+q^{(2n-1)/4})^2(1-q^{(2n-1)/4})^2}{(1-q^{2n})^2(1+q^{2n})^2(1+q^{2n-1})^2} \\
&= \frac{1}{2}\prod_{n=1} \frac{(1-q^{n/2})^2}{(1-q^{2n})^2}\cdot\frac{(1-q^{(2n-1)/2})^2}{(1+q^n)^2} \\
&= \frac{1}{2}\prod_{n=1} \frac{(1-q^{n/2})^4}{(1-q^{2n})^4},
\end{aligned}$$

which is the right-hand side of (1.3). Note that in the first equality we used Jacobi's Triple Identity to write $G(a)$ as infinite products. This completes our proof of Theorem 1.2. □

3. A new proof of the Jacobian Identity

The first part of our proof mimics a proof for Ramanujan's "Most Beautiful Identity" (see [3, 9, 18, 22, 23]; see also [12] for a similar identity for Ramanujan's cubic continued fraction).

By replacing q by q^2 in (1.3), we obtain

$$x(q^2) - \frac{q}{x(q^2)} = \frac{(q;q)_\infty^4}{2(q^4;q^4)_\infty^4}. \tag{3.1}$$

Replacing q by $-q$ in (3.1) gives

$$x(q^2) + \frac{q}{x(q^2)} = \frac{(q^2;q^2)_\infty^4(-q;q^2)_\infty^4}{2(q^4;q^4)_\infty^4}. \tag{3.2}$$

Now, square (3.2) and subtract the square of (3.1). Precisely, the left-hand side gives

$$\left(x(q^2) + \frac{q}{x(q^2)}\right)^2 - \left(x(q^2) - \frac{q}{x(q^2)}\right)^2 = 4q.$$

The other side of the same expression gives

$$\frac{(q^2;q^2)_\infty^8(-q;q^2)_\infty^8 - (q;q)_\infty^8}{4(q^4;q^4)_\infty^8}.$$

Equating the last two expressions gives

$$(q;q)_\infty^8 + 16\,q\,(q^4;q^4)_\infty^8 = (q^2;q^2)_\infty^8(-q;q^2)_\infty^8.$$

Dividing both sides of the last equation by $(q^2;q^2)_\infty^8$ gives (1.2). □

4. Concluding remarks

Two remarks are in order. First, we note that Theorem 1.2 could be compared with the following results (i.e., equations (4.1) and (4.2) below).

Let $R(q)$ denote the celebrated Rogers-Ramanujan continued fraction:

$$R(q) := \frac{q^{1/5}}{1} \; + \; \frac{q}{1} \; + \; \frac{q^2}{1} \; + \; \frac{q^3}{1} \; + \; \cdots .$$

It is known that $R(q)$ satisfies the following remarkable equation:

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}, \tag{4.1}$$

where $f(-q) := (q;q)_\infty$; cf. [27, 28]. See also the following excellent introductions [2, 5, 8, 10].

Recently, the following was discovered. Denote the Ramanujan cubic continued fraction by

$$v := \frac{q^{1/3}}{1} \; + \; \frac{q+q^2}{1} \; + \; \frac{q^2+q^4}{1} \; + \; \cdots .$$

It can be shown that

$$\frac{1}{x_v(q)} - q^{1/3} - 2q^{2/3} x_v(q) = \frac{(q^{1/3};q^{1/3})_\infty (q^{2/3};q^{2/3})_\infty}{(q^3;q^3)_\infty (q^6;q^6)_\infty}, \tag{4.2}$$

where $x_v(q) = q^{-1/3} v$ (cf. Theorem 2 in [12]; see also its implications in [13, 14]).

Second, we remark that the strategy used in this paper can also be applied to the Ramanujan-Göllnitz-Gordon continued fraction (see p. 229 of Ramanujan's second notebook [25]; see also [1, 20, 21]; and also the paper by Chan and Huang [17], in which the authors provided a comprehensive theory of the Ramanujan-Göllnitz-Gordon continued fraction) and proved certain q-identities. We will address these results on other occasions.

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