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AN UPPER CARDINAL BOUND ON ABSOLUTE E-RINGS

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ABSTRACT. We show that for every abelian group A of cardinality $\geq \kappa(\omega)$ there exists a generic extension of the universe, where A is countable with $2^{\aleph_0}$ injective endomorphisms. As an immediate consequence of this result there are no absolute E-rings of cardinality $\geq \kappa(\omega)$. This paper does not require any specific prior knowledge of forcing or model theory and can be considered accessible also for graduate students.

1. INTRODUCTION

A mathematical object, notion or property is called *absolute* if it is preserved in generic extensions of the universe. An example: Consider an $\aleph_1$-free abelian group A; i.e. every countable subgroup of A is free. Using suitable combinatorial techniques a large $\aleph_1$-free abelian group A with the additional property $\operatorname{End} A \cong \mathbb{Z}$ can easily be constructed. But this property is not absolute as using a suitable generic extension $V[G]$ of the underlying universe V, e.g. the Levy collapse $\operatorname{Levy}(\aleph_0, |A|)$, the constructed group A becomes countable. Thus in $V[G]$ by definition A will be free of countable rank with $|\operatorname{End} A| = 2^{\aleph_0}$, contradicting $|\operatorname{End} A| = |\mathbb{Z}| = \aleph_0$. The root of this astounding effect lies in the construction being non-absolute itself, relying on non-absolute notions such as stationary sets.

Conversely, absolute objects can be considered set-theoretically particularly stable and absolute constructions are highly appreciated. One of the first absolute constructions appeared as part of [10] dealing with *rigid families* of coloured trees and more general structures.

Theorem 1.1.

(1) *Let κ, λ be cardinals and $\{\mathcal{T}_\alpha \,|\, \alpha < \kappa\}$ be a family of λ-coloured trees. If $\kappa \geq \kappa(\omega)$ and $\lambda < \kappa(\omega)$, then there exist $\alpha, \beta < \kappa$ with $\alpha \neq \beta$ and* $\operatorname{Hom}(\mathcal{T}_\alpha, \mathcal{T}_\beta) \neq \emptyset$.
(2) *For $\kappa < \kappa(\omega)$ and $\lambda = \aleph_0$ there exists a family $\{\mathcal{T}_\alpha \,|\, \alpha < \kappa\}$ of λ-coloured trees such that* $\operatorname{Hom}(\mathcal{T}_\alpha, \mathcal{T}_\beta) = \emptyset$ *holds absolutely for all $\alpha, \beta < \kappa$ with $\alpha \neq \beta$.*

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To understand this result let us fix some terminology. For any cardinal μ let ${}^{\omega>}\mu$ denote the set of finite sequences in μ and $[\mu]^{<\aleph_0}$ the set of finite subsets of μ. A *tree* T is a subset of ${}^{\omega>}\mu$ that is closed under taking initial segments of sequences. In particular $\emptyset \in {}^{\omega>}\mu$ and $\emptyset \in T$ hold for the empty sequence. For every $t \in T$ the *height* $\mathrm{ht}(t)$ denotes the length of the sequence t where $\mathrm{ht}(\emptyset) = 0$. For a cardinal λ a *λ-coloured tree* $\mathcal{T}$ is a pair (T, c) consisting of a tree and a colouring function $c : T \longrightarrow \lambda$. A *homomorphism* $f : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ between λ-coloured trees $\mathcal{T}_1$ and $\mathcal{T}_2$ is a mapping $f : T_1 \longrightarrow T_2$ that preserves initial segments, the heights and the colours. Finally $\mathrm{Hom}\,(\mathcal{T}_1, \mathcal{T}_2)$ is the set of all homomorphisms $f : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$.

Remarkable for Theorem 1.1 is that it not only comes with an absolute construction in clause (2) but also provides and proves in (1) a sharp bound above which an absolute construction is impossible. This cardinal $\kappa(\omega)$ is called the *first ω-Erdös cardinal* and is defined as the least cardinal κ such that every 2-colouring function $c : [\kappa]^{<\aleph_0} \longrightarrow 2$ admits a countable subset $X \subseteq \kappa$ and some function $c_X : yo \longrightarrow 2$ with $c(Y) = c_X(|Y|)$ for all $Y \in [X]^{<\aleph_0}$. The cardinal $\kappa(\omega)$ (if existent) is quite large and known to be strongly inaccessible; thus for every cardinal $\kappa < \kappa(\omega)$, $2^\kappa < \kappa(\omega)$ also holds.

Weaving Theorem 1.1 into different mathematical structures led subsequently to corresponding results for rigid families of groups [3, 5] and rigid families of coloured graphs [1], showing that $\kappa(\omega)$ again is a sharp upper bound for absolute constructions.

For an abelian group A we denote by $\mathrm{End}\, A$ and $\mathrm{Mon}\, A$ its ring of endomorphisms and its monoid of injective endomorphisms respectively. Now we can formulate the main result of this note as

Theorem 1.2. *For every abelian group A of cardinality $|A| \geq \kappa(\omega)$ there exists a generic extension $V[G]$ of the universe V, where $|A| = \aleph_0$ and $|\mathrm{End}\, A| = |\mathrm{Mon}\, A| = 2^{\aleph_0}$ hold.*

After its proof in Section 2 we will apply this theorem in Section 3 to show that no absolute E-rings of cardinality $\geq \kappa(\omega)$ exist. Together with the absolute construction of E-rings of cardinality $< \kappa(\omega)$ to appear in [6] this will be another incidence of a sharp bound $\kappa(\omega)$.

Our notation is standard (see [2, 7, 8, 9]). For a more extensive survey on absoluteness we refer to [3].

2. Countable groups with large endomorphism rings

In this section we will provide the chain of deductions needed to prove Theorem 1.2. For a start we strengthen [3, Theorem 4].

Lemma 2.1. *Let A be an abelian group of cardinality $|A| \geq \kappa(\omega)$, $B \subseteq A$ be a subgroup of cardinality $|B| < \kappa(\omega)$ and $V[G]$ be a generic extension of the underlying universe V such that $|A| = \aleph_0$ holds in $V[G]$. Then in $V[G]$ there exists some $\varphi \in \mathrm{Mon}\, A$ with $\varphi \restriction B = \mathrm{id}_B$ and $\varphi \neq \mathrm{id}_A$.*

Proof. We start with some preparatory work in the universe V.

Let $s = \langle s_i | \, i < m\rangle$ and $t = \langle t_j | j < n\rangle$ be elements in ${}^{\omega>}A$. We define $B_s := \langle B, s_i | \, i < m\rangle$ to be the induced subgroup of A. Setting $B_m := B \oplus \bigoplus_{i<m} \mathbb{Z}e_i$ we have a canonical projection $\pi_s : B_m \longrightarrow B_s$ induced by $\pi_s \restriction B = \mathrm{id}_B$ and $\pi_s(e_i) = s_i$ for all $i < m$. Next an equivalence relation $\mathcal{E}$ on ${}^{\omega>}A$ is defined by setting $s\,\mathcal{E} t$ iff $m = n$ and Ker π_s = Ker π_t hold. For the induced partition $A/\mathcal{E}$, obviously $|A/\mathcal{E}| \leq \aleph_0 \cdot 2^{|B|+\aleph_0} < \kappa(\omega)$ holds as $\kappa(\omega)$ is strongly inaccessible. Also remarkable is the following easy observation:

We have $s\,\mathcal{E} t$ if and only if $m = n$ and some isomorphism $\psi : B_s \longrightarrow B_t$ exists such that

$$\psi \restriction B = \mathrm{id}_B \text{ and } \psi(s_i) = t_i \text{ for all } i < m. \tag{2.1}$$

Next choose in V a list $\langle u_\alpha |\, \alpha < \kappa(\omega)\rangle$ of pairwise distinct elements $u_\alpha \in A$. Let T_α for $\alpha < \kappa(\omega)$ be the tree generated by all finite sequences in ${}^{\omega>}A$ with starting element u_α. Furthermore let $\mathcal{T}_\alpha = (T_\alpha, c_\alpha)$ be the $|A/\mathcal{E}|$-coloured tree, where the colouring function c_α is defined by setting $c_\alpha(t) := t/\mathcal{E}$ for every $t \in {}^{\omega>}A$. Using Theorem 1.1(1) we find that there exist $\alpha, \beta < \kappa$ with $\alpha \neq \beta$ and $\mathrm{Hom}\,(\mathcal{T}_\alpha, \mathcal{T}_\beta) \neq \emptyset$. Memorize some homomorphism $f : T_\alpha \longrightarrow T_\beta$. Switching now to $V[G]$ this map f will remain a homomorphism.

In $V[G]$ the group A is countable, and we can choose a list $\langle a_i |\, i < \omega\rangle$ of A. As f preserves initial segments and heights there exists a sequence $\langle a'_i |\, i < \omega\rangle$ in A with

$$f(\langle u_\alpha, a_0, a_1, \ldots, a_i\rangle) = \langle u_\beta, a'_0, a'_1, \ldots, a'_i\rangle$$

for all $i < \omega$. As f also preserves colours we can make use of (1) to define a monomorphism $\varphi \in \mathrm{Mon}\, A$ by setting $\varphi(a_i) := a'_i$ $(i < \omega)$. From (1) follows particularly $\varphi \restriction B = \mathrm{id}_B$ and $\varphi(u_\alpha) = u_\beta \neq u_\alpha$; thus $\varphi \neq \mathrm{id}_A$. □

We go on giving an elementary argument for a large set of injective endomorphisms.

Lemma 2.2. *Let A be a countable abelian group such that for every finite set $S \subseteq A$ there exists some $\varphi \in \mathrm{Mon}\, A$ with $\varphi \restriction S = \mathrm{id}_S$ and $\varphi \neq \mathrm{id}_A$. Then $|\mathrm{End}\, A| = |\mathrm{Mon}\, A| = 2^{\aleph_0}$ holds.*

Proof. $|\mathrm{Mon}\, A| \leq |\mathrm{End}\, A| \leq 2^{\aleph_0}$ is obvious.

Given an element $a \in A$ and a sequence $\langle \varphi_i |\, i < yo\rangle$ in $\mathrm{End}\, A$, we define a^η for $\eta \in {}^{\omega>}2$ by recursively setting $a^\emptyset := a$, $a^\eta := a^\theta$ for $\eta = \theta^\wedge 0$ and $a^\eta := \varphi_i(a^\theta)$ for $\eta = \theta^\wedge 1$ where $\theta \in {}^i 2$. Furthermore choose a list $\langle a_i |\, i < \omega\rangle$ of A.

Next we specify the sequence $\langle \varphi_i |\, i < \omega\rangle$: for $i < \omega$ we define recursively a tuple $(S_i, \varphi_i, b_i, c_i)$ consisting of a finite set $S_i \subseteq A$, some $\varphi_i \in \mathrm{Mon}\, A$ and elements $b_i, c_i \in A$. Set $S_0 := \emptyset$. Given S_i choose $\varphi_i \in \mathrm{Mon}\, A$ such that $\varphi_i \restriction S_i = \mathrm{id}_{S_i}$ while $\varphi_i(b_i) = c_i \neq b_i$ for suitable $b_i, c_i \in A$. Set $S_{i+1} := S_i \cup \{b_i, c_i\} \cup \{a_i^\eta |\, \eta \in {}^{i+1}2\}$.

For every $a \in A$, $\eta \in {}^\omega 2$ we now have the sequence $\langle a^{\eta\restriction n} |\, n < \omega\rangle$ in A. This sequence always becomes stationary in A. More precisely, $\langle a^{\eta\restriction n} |\, n < \omega\rangle$ is for $a = a_i$ a sequence in S_{i+1} with $a^{\eta\restriction n} = a^{\eta\restriction(i+1)}$ for $n \geq i+1$. Thus, setting

$$\varphi_\eta(a) := a^{\eta\restriction n} \text{ for large } n,$$

we have a well-defined endomorphism $\varphi_\eta \in \mathrm{Mon}\, A$.

For $\eta_0, \eta_1 \in {}^\omega 2$ with $\eta_0 \neq \eta_1$ choose i minimal such that $\eta_0(i) \neq \eta_1(i)$. Without loss of generality let $\eta_0(i) = 0$ and $\eta_1(i) = 1$. Then $\varphi_{\eta_0}(b_i) = b_i \neq c_i = \varphi_{\eta_1}(b_i)$ and $\varphi_{\eta_0} \neq \varphi_{\eta_1}$ follows. This gives testimony of $2^{\aleph_0}$ different elements in $\mathrm{Mon}\, A$. □

Now the proof of Theorem 1.2 is quite immediate.

Proof. Starting from our universe V with $|A| \geq \kappa(\omega)$ we can easily derive a generic extension $V[G]$, where $|A| = \aleph_0$ holds, e.g. by using again the Levy collapse $\mathrm{Levy}(\aleph_0, |A|)$. As the notions of finite and infinite sets are absolute, we can derive from Lemma 2.1 that A accomplishes in $V[G]$ the prerequisites of Lemma 2.2. $\square$

3. Consequences and conclusion

The proofs of Lemma 2.1, Lemma 2.2 and Theorem 1.2 can easily be formulated entirely model-theoretically. We make a note of the resulting generalization of Theorem 1.2.

Theorem 3.1. *For every language $\mathcal{L}$ of cardinality $|\mathcal{L}| < \kappa(\omega)$ and every $\mathcal{L}$-structure $\mathcal{M}$ of cardinality $|\mathcal{M}| \geq \kappa(\omega)$ there exists a generic extension $V[G]$ of the universe V, where $|\mathcal{M}| = \aleph_0$ and $|\mathrm{End}\,\mathcal{M}| = |\mathrm{Mon}\,\mathcal{M}| = 2^{\aleph_0}$.*

Remark 3.2. The restriction $|\mathcal{L}| < \kappa(\omega)$ is merely virtual. Theorem 3.1 remains true in general by replacing $\kappa(\omega)$ by the least Erdös cardinal greater than $|\mathcal{L}|$.

We give a direct application of Theorem 1.2 and Theorem 3.1 to the construction of E-rings and $\mathrm{E}(R)$-algebras respectively. For this let R be a commutative ring and denote for an R-module M by $\mathrm{End}_R M$ its endomorphism R-algebra. Recall that an R-algebra E is an $\mathrm{E}(R)$-*algebra* if the canonical map $\delta : \mathrm{End}_R E \to E$ via $\varphi \mapsto \varphi(1)$ is an R-algebra isomorphism. An R-algebra E is called a *generalized* $\mathrm{E}(R)$-*algebra* if it is isomorphic to $\mathrm{End}_R E$ by an arbitrary isomorphism. These notions are generalizations of E-rings, i.e. $\mathrm{E}(\mathbb{Z})$-algebras. For an extensive history on $\mathrm{E}(R)$-algebras and their applications we refer to [4].

Corollary 3.3. *For every commutative ring R of cardinality $|R| < \kappa(\omega)$ there exist no absolute (generalized) $\mathrm{E}(R)$-algebras E of cardinality $|E| \geq \kappa(\omega)$.*

Proof. Assume E to be an absolute (generalized) $\mathrm{E}(R)$-algebra of cardinality $|E| \geq \kappa(\omega)$. Applying Theorem 3.1 to the language of R-modules there exists a generic extension $V[G]$ of the universe with $|\mathrm{End}_R E| = 2^{\aleph_0} > \aleph_0 = |E|$, contradicting $\mathrm{End}_R E \cong E$. $\square$

We end this paper with a list of related open questions.

Question 3.4. Let A be an abelian group of cardinality $|A| \geq \kappa(\omega)$. Does there always exist a generic extension $V[G]$ of the universe V such that

$Q1$). A admits a non-trivial surjective endomorphism in $V[G]$?
$Q2$). A admits a non-trivial automorphism in $V[G]$?
$Q3$). A is decomposable in $V[G]$?

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