# BETTI NUMBERS AND INJECTIVITY RADII

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*Dedicated to José María Montesinos on the occasion of his 65th birthday*

ABSTRACT. We give lower bounds on the maximal injectivity radius for a
closed hyperbolic 3-manifold with first Betti number 2 under some additional
topological hypotheses.

The theme of this paper is the connection between topological properties of a
closed orientable hyperbolic 3-manifold $M$ and the maximal injectivity radius of
$M$. In [4] we showed that if the first Betti number of $M$ is at least 3, then the
maximal injectivity radius of $M$ is at least $\log 3$. By contrast, the best known lower
bound for the maximal injectivity radius of $M$ with no topological restriction on
$M$ is the lower bound of $\operatorname{arcsinh}(\frac{1}{4}) = 0.24746\ldots$ due to Przeworski [7]. One of the
results of this paper, Corollary 4, gives a lower bound of 0.32798 for the case where
the first Betti number of $M$ is 2 and $M$ does not contain a "fibroid" (see below).
Our main result, Theorem 3, is somewhat stronger than this.

The proofs of our results combine a result due to Andrew Przeworski [7] with
results from [5] and [6].

The results of [5] and [6] were motivated by applications to the study of hyper-
bolic volume, and these applications were superseded by the results of [2]. The
applications presented in the present paper do not seem to be accessible by other
methods.

As in [5], we define a *book of I-bundles* to be a compact, connected, orientable
topological 3-manifold (with boundary) $W$ which has the form $W = \mathcal{P} \cup \mathcal{B}$, where

- $\mathcal{P}$ is an $I$-bundle over a non-empty compact 2-manifold-with-boundary,
- each component of $\mathcal{B}$ is homeomorphic to $D^2 \times S^1$,
- the set $\mathcal{A} = \mathcal{P} \cap \mathcal{B}$ is the vertical boundary of the $I$-bundle $\mathcal{P}$, and
- each component of $\mathcal{A}$ is an annulus in $\partial\mathcal{B}$ which is homotopically non-trivial
  in $\mathcal{B}$.

(Note that this terminology differs slightly from that of [1], where it is the triple
$(W, \mathcal{B}, \mathcal{P})$ that is called a book of $I$-bundles.)

As in [5], we define a *fibroid* in a closed, connected, orientable 3-manifold $M$ to
be a connected incompressible surface $F$ with the property that each component
of the compact manifold obtained by cutting $M$ along $F$ is a book of $I$-bundles.

Note that in defining a fibroid to be connected, we are following the convention of [5] rather than that of [6].

We define a function $R(x)$ for $0 < x < \log 3$ by

$$(1) \qquad R(x) = \frac{1}{2} \operatorname{arccosh}\left( \frac{e^{2x} + 2e^x + 5}{(\cosh \frac{x}{2})(e^x - 1)(e^x + 3)} \right).$$

As in [3, Section 10], we define a function $V(x)$ for $0 < x < \log 3$ by

$$(2) \qquad V(x) = \pi x \sinh^2 R(x) = \frac{\pi x}{e^x - 1}\left( \frac{e^{2x} + 2e^x + 5}{2(e^x + 3)\cosh(x/2)} \right) - \frac{\pi x}{2}.$$

Thus, in a closed hyperbolic 3-manifold, if a geodesic of length $\ell$ is the core of an embedded tube of radius $R(\ell)$, then this tube has volume $V(\ell)$.

The following result is implicit in [6], but we will supply a proof for the sake of comprehensibility.

**Proposition 1.** *Let $M$ be a closed hyperbolic 3-manifold. Suppose that there is an infinite subset $\mathcal{N}$ of $H_2(M; \mathbb{Z})$ such that every element of $\mathcal{N}$ is represented by some connected, incompressible surface which is not a fibroid. Let $\lambda$ be a positive number less than $\log 3$. Then either the maximal injectivity radius of $M$ is at least $\lambda/2$ or $M$ contains a closed geodesic $C$ of length at most $\lambda$ such that the maximal tube about $C$ has radius at least $R(\lambda)$ and volume at least $V(\lambda)$, where $R(\lambda)$ and $V(\lambda)$ are defined by (1) and (2).*

*Proof.* The hypothesis implies in particular that $H_2(M; \mathbb{Z})$ has infinitely many primitive elements, and so the first Betti number $\beta_1(M)$ is at least 2. If $\beta_1(M) \geq 3$, then according to [4, Corollary 10.4], the maximal injectivity radius is at least $\frac{1}{2} \log 3 > \lambda/2$. We may therefore assume that $\beta_1(M) = 2$. Hence the quotient of $H_1(M, \mathbb{Z})$ by its torsion subgroup is a free abelian group $L$ of rank 2. We let $h \colon \pi_1(M) \to L$ denote the natural homomorphism.

We distinguish two cases. First consider the case in which $M$ contains a non-trivial closed geodesic $C$ of some length $\ell < \lambda$ such that the conjugacy class represented by $C$ is contained in the kernel of $h$. Let $T$ denote the maximal embedded tube about $C$. According to [3, Corollary 10.5] we have $\operatorname{Vol} T \geq V(\lambda)$. If $\rho$ denotes the radius of $T$, this gives

$$\pi \ell \sinh^2 \rho = \operatorname{Vol} T \geq V(\lambda) = \pi \lambda \sinh^2 R(\lambda) > \pi \ell \sinh^2 R(\lambda),$$

and hence $\rho > R(\lambda)$.

Thus the second alternative of the proposition holds in this case.

We now turn to the case in which no non-trivial closed geodesic of length $< \lambda$ represents a conjugacy class contained in the kernel of $h$. Since $M$ is closed, there are only a finite number $n \geq 0$ of conjugacy classes in $\pi_1(M)$ that are represented by closed geodesics of length $< \lambda$. Let $\gamma_1, \ldots, \gamma_n$ be elements belonging to these $n$ conjugacy classes. Then $\bar{\gamma}_i = h(\gamma_i)$ is a non-trivial element of $L$ for $i = 1, \ldots, n$. Since $L$ is a free abelian group of rank 2, there exists, for each $i \in \{1, \ldots, n\}$, a homomorphism $\phi_i$ of $L$ onto $\mathbb{Z}$ such that $\phi_i(\bar{\gamma}_i) = 0$. Because $\bar{\gamma}_i \neq 0$, the epimorphism $\phi_i$ is unique up to sign.

The epimorphism $\phi_i \circ h \colon \pi_1(M) \to \mathbb{Z}$ corresponds to a primitive element of $H^1(M; \mathbb{Z})$, whose Poincaré dual in $H_2(M; \mathbb{Z})$ we shall denote by $c_i$. Since the set $\mathcal{N} \subset H_2(M; \mathbb{Z})$ given by the hypothesis of the theorem is infinite, there is an element $c$ of $\mathcal{N}$ which is distinct from $\pm c_i$ for $i = 1, \ldots, n$. Since $c \in \mathcal{N}$ it follows

from the hypothesis that there is a connected incompressible surface $S \subset M$ which represents the homology class $c$ and is not a fibroid.

We now apply Theorem A of [5], which asserts that if $S$ is a connected non-fibroid incompressible surface in a closed, orientable hyperbolic 3-manifold $M$, and if $\lambda$ is any positive number, then either (i) $M$ contains a non-trivial closed geodesic of length $< \lambda$ which is homotopic in $M$ to a closed curve in $M - S$ or (ii) $M$ contains a hyperbolic ball of radius $\lambda/2$. In the present situation, with $\lambda$ chosen as above, we claim that alternative (i) of the conclusion of Theorem A of [5] cannot hold.

Indeed, suppose that $C$ is a non-trivial closed geodesic of length $< \lambda$ with the properties stated in (i). Since $C$ has length $< \lambda$, the conjugacy class represented by $C$ contains $\gamma_i$ for some $i \in \{1, \ldots, n\}$. Since $C$ is homotopic to a closed curve in $M - S$, it follows that the image of $\gamma_i$ in $H_1(M; \mathbb{Z})$ has homological intersection number 0 with $c$. Thus if $\psi \colon \pi_1(M) \to \mathbb{Z}$ is the homomorphism corresponding to the Poincaré dual of $c$, we have $\psi(\gamma_i) = 0$. Now since $L$ is the quotient of $H_1(M)$ by its torsion subgroup, $\psi$ factors as $\phi \circ h$, where $\phi$ is some homomorphism from $L$ to $\mathbb{Z}$. Since $c$ is primitive, $\psi$ is surjective, and hence so is $\phi$. But we have $\phi(\bar{\gamma}_i) = \psi(\gamma_i) = 0$. In view of the uniqueness that we observed above for $\phi_i$, it follows that $\phi = \pm \phi_i$, so that $\psi = \pm \phi_i \circ h$ and hence $c = \pm c_i$. This contradicts our choice of $c$.

Hence (ii) must hold. This means that the maximal injectivity radius of $M$ is at least $\lambda/2$. Thus the first alternative of the proposition holds in this case. $\square$

**Proposition 2.** *Let $M$ be a closed hyperbolic 3-manifold. Suppose that there is an infinite subset $\mathcal{N}$ of $H_2(M; \mathbb{Z})$ such that every element of $\mathcal{N}$ is represented by some connected, incompressible surface which is not a fibroid. Then the maximal injectivity radius of $M$ exceeds $0.32798$.*

*Proof.* We set $\lambda = 2 \times 0.32798 = 0.65596$.

According to Proposition 1, either the maximal injectivity radius of $M$ is at least $\lambda/2$ — so that the conclusion of the theorem holds — or $M$ contains a closed geodesic $C$ of length at most $\lambda$ such that the maximal tube about $C$ has volume at least $V(\lambda)$, where $V(\lambda)$ is defined by (2). In the latter case, if $R$ denotes the radius of $T$, we have

$$R \geq R(\lambda) = 0.806787 \ldots.$$

Now according to [7, Proposition 4.1], the maximal injectivity radius of $M$ is bounded below by

$$\operatorname{arcsinh}\left(\frac{\tanh R}{2}\right) > \operatorname{arcsinh}\left(\frac{\tanh 0.806787}{2}\right) > 0.32799.$$

This gives the conclusion of the theorem in this case. $\square$

**Theorem 3.** *Let $M$ be a closed hyperbolic 3-manifold. Suppose that there is an infinite set $\mathcal{N}$ of primitive elements of $H_2(M; \mathbb{Z})$ such that no element of $\mathcal{N}$ is represented by a (connected) fibroid. Then the maximal injectivity radius of $M$ exceeds $0.32798$.*

*Proof.* If $\pi_1(M)$ has a non-abelian free quotient, then by [6, Theorem 1.3], the maximal injectivity radius of $M$ is at least $\frac{1}{2}\log 3 = 0.549\ldots$. Now suppose that $\pi_1(M)$ has no non-abelian free quotient. If $\mathcal{N}$ is the set given by the hypothesis of Theorem 3, it now follows from [6, Proposition 2.1] that every element of $\mathcal{N}$ is represented by a connected incompressible surface, which by hypothesis cannot

be a fibroid. Thus $\mathcal{N}$ has the properties stated in the hypothesis of Proposition 2. The latter result therefore implies that the maximal injectivity radius of $M$ exceeds 0.32798. $\qquad\square$

If $M$ is a 3-manifold whose first Betti number is at least 2, then $H_2(M;\mathbb{Z})$ has infinitely many primitive elements. If a non-trivial element of $H_2(M;\mathbb{Z})$ is represented by a connected surface, it must be primitive, since it has intersection number 1 with a class in $H_1(M;\mathbb{Z})$. Hence Theorem 3 implies:

**Corollary 4.** *Let $M$ be a closed hyperbolic 3-manifold. Suppose that the first Betti number of $M$ is at least 2 and that $M$ contains no non-separating fibroid. Then the maximal injectivity radius of $M$ exceeds 0.32798.*

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