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ON SHARP EMBEDDINGS OF BESOV AND TRIEBEL-LIZORKIN SPACES IN THE SUBCRITICAL CASE

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Abstract. We discuss the growth envelopes of Fourier-analytically defined Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ in the limiting case $s=\sigma_p:=n\max(\frac{1}{p}-1,0)$. These results may also be reformulated as optimal embeddings into the scale of Lorentz spaces $L_{p,q}(\mathbb{R}^n)$. We close several open problems outlined already in [H. Triebel, *The structure of functions*, Birkhäuser, Basel, 2001] and explicitly stated in [D. D. Haroske, *Envelopes and sharp embeddings of function spaces*, Chapman & Hall/CRC, Boca Raton, FL, 2007].

1. Introduction and main results

In this paper we prove sharp embedding theorems for Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ in some limiting cases of the range guaranteeing that these spaces consist of locally integrable functions. As proven in [12, Theorem 3.3.2],

$$B^s_{p,q}(\mathbb{R}^n)\hookrightarrow L_1^{\rm loc}(\mathbb{R}^n)\Leftrightarrow\begin{cases}\text{either} & s>\sigma_p:=n\max(\frac{1}{p}-1,0),\\ \text{or} & s=\sigma_p, 1<p\le\infty, 0<q\le\min(p,2),\\ \text{or} & s=\sigma_p, 0<p\le 1, 0<q\le 1\end{cases}\tag{1}$$

and

$$F^s_{p,q}(\mathbb{R}^n)\hookrightarrow L_1^{\rm loc}(\mathbb{R}^n)\Leftrightarrow\begin{cases}\text{either} & s>\sigma_p,\\ \text{or} & s=\sigma_p, 1\le p<\infty, 0<q\le 2,\\ \text{or} & s=\sigma_p, 0<p<1, 0<q\le\infty.\end{cases}\tag{2}$$

The embeddings can be measured quantitatively by the *growth envelope function* of X as defined by D. D. Haroske and H. Triebel (see [5], [6], [16] and the references given there) by

$$\mathcal{E}^X_G(t):=\sup_{||f|X||\le 1}f^*(t),\quad 0<t<1,$$

where f^* denotes the non-increasing rearrangement of f.

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In the case where $\mathcal{E}_G^X(t) \approx t^{-\alpha}$ for $0 < t < 1$ and some $\alpha > 0$ the *growth envelope index* u_X is given as the infimum of all numbers v, $0 < v \le \infty$, such that

$$\left(\int_0^\epsilon \left[\frac{f^*(t)}{\mathcal{E}_G^X(t)}\right]^v \frac{dt}{t}\right)^{1/v} \le c\,||f|X|| \tag{3}$$

(with the usual modification for $v = \infty$) holds for some $\epsilon > 0, c > 0$ and all $f \in X$. The pair $\mathfrak{E}_G(X) = (\mathcal{E}_G^X, u_X)$ is called the *growth envelope* for the function space X.

In the case $\sigma_p < s$, the growth envelopes of $A^s_{p,q}(\mathbb{R}^n)$ are known; cf. [16, Theorem 15.2] and [6, Theorem 8.1]. If $s = \sigma_p$ and (1) or (2) is fulfilled in the B or F case, respectively, then the growth function is given by $t^{-\frac{1}{\max(p,1)}}$, but the known information about the growth index u is not complete; cf. [16, Remarks 12.5, 15.1] and [6, Props. 8.12, 8.14 and Remark 8.15].

The growth index of $B^{\sigma_p}_{p,q}(\mathbb{R}^n)$ satisfies

$$\begin{cases} q \le u \le p & \text{if } 1 \le p < \infty \text{ and } 0 < q \le \min(p,2), \\ q \le u \le 1 & \text{if } 0 < p < 1 \text{ and } 0 < q \le 1. \end{cases} \tag{4}$$

The growth index of $F^{\sigma_p}_{p,q}(\mathbb{R}^n)$ satisfies $p \le u \le 1$ if $0 < p < 1$ and $0 < q \le \infty$ and is equal to p if $1 \le p < \infty$ and $0 < q \le 2$.

The growth envelopes of $B^0_{\infty,q}$ defined on the torus $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$ with $1 \le q \le 2$ were identified recently by Seeger and Trebels in [10] and are equivalent to $|\log t|^{1/q'}$ for $0 < t \le 1/2$. We fill the remaining gaps for the range $p < \infty$.

Theorem 1.1. (i) *Let* $1 \le p < \infty$ *and* $0 < q \le \min(p,2)$. *Then*

$$\mathfrak{E}_G(B^0_{p,q}) = (t^{-\frac{1}{p}}, p).$$

(ii) *Let* $0 < p < 1$ *and* $0 < q \le 1$. *Then*

$$\mathfrak{E}_G(B^{\sigma_p}_{p,q}) = (t^{-1}, q).$$

(iii) *Let* $0 < p < 1$ *and* $0 < q \le \infty$. *Then*

$$\mathfrak{E}_G(F^{\sigma_p}_{p,q}) = (t^{-1}, p).$$

These results are closely related to optimal embeddings into the scale of Lorentz spaces. In this context, we prove the following.

Theorem 1.2. (i) *Let* $1 \le p < \infty$ *and* $0 < q \le \min(p,2)$. *Then*

$$B^0_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n).$$

(ii) *Let* $0 < p < 1$ *and* $0 < q \le 1$. *Then*

$$B^{\sigma_p}_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,q}(\mathbb{R}^n). \tag{5}$$

(iii) *Let* $0 < p < 1$ *and* $0 < q \le \infty$. *Then*

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,p}(\mathbb{R}^n)$$

and all these embeddings are optimal with respect to the second fine parameter of the scale of the Lorentz spaces.

Remark 1.3. (i) Let us observe that (5) improves [12, Theorem 3.2.1] and [11, Theorem 2.2.3], where the embedding $B^{n(\frac{1}{p}-1)}_{p,q}(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n)$ is proved for all $0 < p < 1$ and $0 < q \le 1$.

(ii) We also mention that growth envelopes for function spaces with minimal smoothness were recently studied in [2]. These authors worked with spaces defined by differences and their results differ from ours in logarithmic factors. This shows indirectly that the Fourier-analytical definition and the classical definition of Besov spaces do not coincide for $s = 0$, an effect observed in detail recently by Schneider [9].

We denote the Lebesgue and Lorentz spaces by $L_p(\mathbb{R}^n)$ and $L_{p,q}(\mathbb{R}^n)$, respectively. The reader may consult [13, Chapter 5, Section 3] or [1, Chapter 4, Section 4]. We shall use the following well-known property of Lorentz spaces $L_{1,q}$. Its proof follows immediately from Hardy's lemma (cf. [1, Chapter 2, Proposition 3.6]).

Lemma 1.4. *Let $0 < q < 1$. Then the $||\cdot|L_{1,q}(\mathbb{R}^n)||$ is the q-norm; it means that*

$$||f_1 + f_2|L_{1,q}(\mathbb{R}^n)||^q \le ||f_1|L_{1,q}(\mathbb{R}^n)||^q + ||f_2|L_{1,q}(\mathbb{R}^n)||^q$$

holds for all $f_1, f_2 \in L_{1,q}(\mathbb{R}^n)$.

We work with Fourier analytically defined Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ as studied for example in [8], [14], [15] and [17]. We shall also use the sequence spaces b^s_{pq} associated to $B^s_{p,q}(\mathbb{R}^n)$ in a way described in [17, Chapters 2 and 3]. This approach goes back to [3] and [4].

All the unimportant constants are denoted by the letter c, whose meaning may differ from one occurrence to another. If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are two sequences of positive real numbers, we write $a_n \lesssim b_n$ if, and only if, there is a positive real number $c > 0$ such that $a_n \le c\, b_n, n \in \mathbb{N}$. Furthermore, $a_n \approx b_n$ means that $a_n \lesssim b_n$ and simultaneously $b_n \lesssim a_n$.

2. Proofs of the main results

2.1. **Proof of Theorem 1.1 (i).** In view of (4), it is enough to prove that for $1 \le p < \infty$ and $0 < q \le \min(p, 2)$ the index u associated to $B^0_{p,q}(\mathbb{R}^n)$ is greater than or equal to p.

We assume to the contrary that (3) is fulfilled for some $0 < v < p$, $\epsilon > 0$, $c > 0$ and all $f \in B^0_{p,q}(\mathbb{R}^n)$. Let ψ be a non-vanishing C^∞ function in $\mathbb{R}^n$ supported in $[0,1]^n$ with $\int_{\mathbb{R}^n} \psi(x)dx = 0$.

Let $J \in \mathbb{N}$ be such that $2^{-Jn} < \epsilon$ and consider the function

$$f_j = \sum_{m=1}^{2^{(j-J)n}} \lambda_{jm} \psi(2^j(x - (m, 0, \ldots, 0))), \quad j > J, \tag{6}$$

where

$$\lambda_{jm} = \frac{1}{m^{\frac{1}{p}} \log^{\frac{1}{v}}(m+1)}, \quad m = 1, \ldots, 2^{(j-J)n}.$$

Then (6) represents an atomic decomposition of f in the space $B^0_{p,q}(\mathbb{R}^n)$ according to [17, Chapter 1.5], and we obtain (recall that $v < p$)

$$\begin{aligned} ||f_j|B^0_{p,q}(\mathbb{R}^n)|| &\lesssim 2^{-j\frac{n}{p}} \left(\sum_{m=1}^{2^{(j-J)n}} \lambda^p_{jm} \right)^{1/p} \\ &\le 2^{-j\frac{n}{p}} \left(\sum_{m=1}^{\infty} m^{-1} (\log(m+1))^{-\frac{p}{v}} \right)^{1/p} \lesssim 2^{-j\frac{n}{p}}. \end{aligned} \tag{7}$$

On the other hand,

$$\left(\int_0^\epsilon \left[f_j^*(t)t^{\frac{1}{p}}\right]^v \frac{dt}{t}\right)^{1/v} \geq \left(\int_0^{2^{-Jn}} f_j^*(t)^v t^{v/p-1} dt\right)^{1/v}$$

$$\gtrsim \left(\sum_{m=1}^{2^{(j-J)n}} \lambda_{jm}^v \int_{c\,2^{-jn}(m-1)}^{c\,2^{-jn}m} t^{v/p-1} dt\right)^{1/v} \gtrsim \left(\sum_{m=1}^{2^{(j-J)n}} \lambda_{jm}^v 2^{-jnv/p} m^{v/p-1}\right)^{1/v}$$

$$= 2^{-j\frac{n}{p}} \left(\sum_{m=1}^{2^{(j-J)n}} \frac{1}{m\log(m+1)}\right)^{1/v}.$$

As the last series is divergent for $j \to \infty$, this is in contradiction with (7), and (3) cannot hold for all $f_j, j > J$.

Remark 2.1. Observe that Theorem 1.2 (i) is a direct consequence of Theorem 1.1 (i). The embeddings $B^0_{1,q}(\mathbb{R}^n) \hookrightarrow B^0_{1,1}(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n)$ if $p = 1$ and $B^0_{p,q}(\mathbb{R}^n) \hookrightarrow F^0_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ if $1 < p < \infty$ show that $B^0_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$. Theorem 1.1 (i) implies that if $B^0_{p,q}(\mathbb{R}^n) \hookrightarrow L_{p,v}(\mathbb{R}^n)$ for some $0 < v < \infty$, then $p \leq v$. This proves the optimality of Theorem 1.2 (i) in the frame of the scale of Lorentz spaces.

2.2. **Proof of Theorem 1.1 (ii) and Theorem 1.2 (ii).** Let $0 < p < 1$, $0 < q \leq 1$ and $s = \sigma_p = n\left(\frac{1}{p} - 1\right)$. We first prove Theorem 1.2 (ii); i.e. we show that

$$B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,q}(\mathbb{R}^n),$$

or, equivalently, that

$$\left(\int_0^\infty [tf^*(t)]^q \frac{dt}{t}\right)^{1/q} \leq c\,||f|B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n)||, \qquad f \in B^{\frac{n}{p}-n}_{pq}(\mathbb{R}^n).$$

Let

$$f = \sum_{j=0}^\infty f_j = \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm}$$

be the optimal atomic decomposition of an $f \in B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n)$, again in the sense of [17, Chapter 1.5]. Then

$$||f|B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n)|| \approx \left(\sum_{j=0}^\infty 2^{-jqn}\left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p\right)^{q/p}\right)^{1/q} \tag{8}$$

and by Lemma 1.4,

$$||f|L_{1,q}(\mathbb{R}^n)|| = ||\sum_{j=0}^\infty f_j|L_{1,q}(\mathbb{R}^n)|| \leq \left(\sum_{j=0}^\infty ||f_j|L_{1,q}(\mathbb{R}^n)||^q\right)^{1/q}. \tag{9}$$

We shall need only one property of the atoms a_{jm}, namely, that their support is contained in the cube $\tilde{Q}_{jm}$, a cube centred at the point $2^{-j}m$ with sides parallel to the coordinate axes and side length $\alpha 2^{-j}$, where $\alpha > 1$ is fixed and independent of f. We denote by $\tilde{\chi}_{jm}(x)$ the characteristic functions of $\tilde{Q}_{jm}$ and by χ_{jl} the characteristic function of the interval $(l2^{-jn}, (l+1)2^{-jn})$. Hence

$$f_j(x) \le c \sum_{m\in\mathbb{Z}^n} |\lambda_{jm}| \tilde{\chi}_{jm}(x), \quad x \in \mathbb{R}^n$$

and

$$\begin{aligned} \|f_j|L_{1,q}(\mathbb{R}^n)\| &\lesssim \left(\int_0^\infty \sum_{l=0}^\infty [(\lambda_j)^*_l \chi_{jl}(t)]^q \, t^{q-1} dt \right)^{1/q} \\ &\le \left(\sum_{l=0}^\infty [(\lambda_j)^*_l]^q \int_{2^{-jn}l}^{2^{-jn}(l+1)} t^{q-1} dt \right)^{1/q} \\ &\lesssim 2^{-jn} \left(\sum_{l=0}^\infty [(\lambda_j)^*_l]^q \, (l+1)^{q-1} \right)^{1/q} \lesssim 2^{-jn} \|\lambda_j|\ell_p\|. \end{aligned} \tag{10}$$

The last inequality follows by $(l+1)^{q-1} \le 1$ and $\ell_p \hookrightarrow \ell_q$ if $p \le q$. If $p > q$, the same follows by Hölder's inequality with respect to the indices $\alpha = \frac{p}{q}$ and $\alpha' = \frac{p}{p-q}$:

$$\begin{aligned} &\left(\sum_{l=0}^\infty [(\lambda_j)^*_l]^q \, (l+1)^{q-1} \right)^{1/q} \\ &\le \left(\sum_{l=0}^\infty [(\lambda_j)^*_l]^{q\cdot\frac{p}{q}} \right)^{\frac{1}{q}\cdot\frac{q}{p}} \cdot \left(\sum_{l=0}^\infty (l+1)^{(q-1)\cdot\frac{p}{p-q}} \right)^{\frac{1}{q}\cdot\frac{p-q}{p}} \le c \, \|\lambda_j|\ell_p\|. \end{aligned}$$

Here, we used that for $0 < q < p < 1$ the exponent $\frac{(q-1)p}{p-q} = -1 + \frac{(p-1)q}{p-q}$ is strictly smaller than -1.

The proof now follows by (8), (9) and (10):

$$\begin{aligned} \|f|L_{1,q}(\mathbb{R}^n)\| &\le \left(\sum_{j=0}^\infty \|f_j|L_{1,q}(\mathbb{R}^n)\|^q \right)^{1/q} \le c \left(\sum_{j=0}^\infty 2^{-jnq} \|\lambda_j|\ell_p\|^q \right)^{1/q} \\ &\le c \, \|f|B^{\sigma_p}_{p,q}(\mathbb{R}^n)\|. \end{aligned}$$

Remark 2.2. We actually proved that (3) holds for $X = B^{\frac{n}{p}-n}_{pq}(\mathbb{R}^n)$, $v = q$ and $\epsilon = \infty$. This, together with (4), implies immediately Theorem 1.1 (ii).

2.3. **Proof of Theorem 1.1 (iii) and Theorem 1.2 (iii).** Let $0 < p < 1$ and $0 < q \le \infty$. By the Jawerth embedding (cf. [7] or [18]) and Theorem 1.1 (ii) we get for any $0 < p < \tilde{p} < 1$,

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{\sigma_{\tilde{p}}}_{\tilde{p},p}(\mathbb{R}^n) \hookrightarrow L_{1,p}(\mathbb{R}^n).$$

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References

[1] C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, San Diego, 1988. MR928802 (89e:46001)

[2] A. M. Caetano, A. Gogatishvili and B. Opic, *Sharp embeddings of Besov spaces involving only logarithmic smoothness*, J. Appr. Theory 152 (2008), 188-214. MR2422148

[3] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. 34 (1985), 777-799. MR808825 (87h:46083)

[4] M. Frazier and B. Jawerth, *A discrete transform and decomposition of distribution spaces*, J. Funct. Anal. 93 (1990), 34-170. MR1070037 (92a:46042)

[5] D. D. Haroske, *Limiting embeddings, entropy numbers and envelopes in function spaces*, Habilitationsschrift, Friedrich-Schiller-Universität Jena, Germany, 2002.

[6] D. D. Haroske, *Envelopes and sharp embeddings of function spaces*, Chapman & Hall/CRC, Boca Raton, FL, 2007. MR2262450 (2007i:46031)

[7] B. Jawerth, *Some observations on Besov and Lizorkin-Triebel spaces*, Math. Scand. 40 (1977), 94-104. MR0454618 (56:12867)

[8] J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Math. Series, Durham, NC, 1976. MR0461123 (57:1108)

[9] C. Schneider, *On dilation operators in Besov spaces*, Rev. Mat. Complut. 22 (2009), 111–128. MR2499328

[10] A. Seeger and W. Trebels, *Low regularity classes and entropy numbers*, Archiv der Mathematik (Basel) 92 (2009), 147–157. MR2481510

[11] W. Sickel and T. Runst, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, 3, Walter de Gruyter & Co., Berlin, 1996. MR1419319 (98a:47071)

[12] W. Sickel and H. Triebel, *Hölder inequalities and sharp embeddings in function spaces of B^s_{pq} and F^s_{pq} type*, Z. Anal. Anwend. 14 (1995), 105-140. MR1327495 (96h:46042)

[13] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, NJ, 1971. MR0304972 (46:4102)

[14] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel, 1983. MR781540 (86j:46026)

[15] H. Triebel, *Theory of function spaces. II*, Birkhäuser, Basel, 1992. MR1163193 (93f:46029)

[16] H. Triebel, *The structure of functions*, Birkhäuser, Basel, 2001. MR1851996 (2002k:46087)

[17] H. Triebel, *Theory of function spaces. III*, Birkhäuser, Basel, 2006. MR2250142 (2007k:46058)

[18] J. Vybíral, *A new proof of the Jawerth-Franke embedding*, Rev. Mat. Complut. 21 (2008), 75-82. MR2408037 (2009d:46068)

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