

SUPPRESSION OF UNBOUNDED GRADIENTS IN AN SDE ASSOCIATED WITH THE BURGERS EQUATION

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ABSTRACT. We consider the Langevin equation describing a non-viscous Burgers fluid stochastically perturbed by uniform noise. We introduce a deterministic function that corresponds to the mean of the velocity when we keep the value of the position fixed. We study interrelations between this function and the solution of the non-perturbed Burgers equation. We are especially interested in the property of the solution of the latter equation to develop unbounded gradients within a finite time. We study the question of how the initial distribution of particles for the Langevin equation influences this blowup phenomenon. We show that for a wide class of initial data and initial distributions of particles the unbounded gradients are eliminated. The case of a linear initial velocity is particular. We show that if the initial distribution of particles is uniform, then the mean of the velocity for a given position coincides with the solution of the Burgers equation and, in particular, it does not depend on the constant variance of the stochastic perturbation. Further, for a one space variable we get the following result: if the decay rate of the even power-behaved initial particles distribution at infinity is greater than or equal to $|x|^{-2}$, then the blowup is suppressed; otherwise, the blowup takes place at the same moment of time as in the case of the non-perturbed Burgers equation.

1. INTRODUCTION

It is well known that the non-viscous Burgers equation, the simplest equation that models the non-linear phenomena in a force free mass transfer,

$$(1.1) \quad u_t + (u, \nabla) u = 0,$$

where $u(x, t) = (u_1, \dots, u_n)$ is a vector-function $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$, before the formation of shocks, is equivalent to the system of ODE,

$$(1.2) \quad \dot{x}(t) = u(t, x(t)), \quad \dot{u}(t, x(t)) = 0.$$

The latter system defines a family of characteristic lines $x = x(t)$ that can be interpreted as the Lagrangian coordinates of the particles.

Given initial data

$$(1.3) \quad u(x, 0) = u_0(x),$$

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one can readily get an implicit solution of (1.1), (1.3), namely,

$$u(t, x) = u_0(x - tu(t, x)).$$

For special classes of initial data we can obtain an explicit solution. The simplest case is

$$(1.4) \quad u_0(x) = \alpha x, \quad \alpha \in \mathbb{R},$$

where

$$(1.5) \quad u(t, x) = \frac{\alpha x}{1 + \alpha t}.$$

Thus, if $\alpha < 0$, the solution develops a singularity at the origin as $t \rightarrow T$, $0 < T < \infty$, where

$$(1.6) \quad T = -\frac{1}{\alpha}.$$

In the present paper we consider a $2 \times n$ dimensional Itô stochastic differential system of equations, associated with (1.2), namely

$$\begin{aligned} dX_k(t) &= U_k(t) dt, \\ dU_k(t) &= \sigma d(W_k)_t, \quad k = 1, \dots, n, \\ X(0) &= x, \quad U(0) = u, \quad t \geq 0, \end{aligned}$$

where $(X(t), U(t))$ runs the phase space $\mathbb{R}^n \times \mathbb{R}^n$, $\sigma > 0$ is constant, and $(W_k)_t$, $k = 1, \dots, n$, is the n dimensional Brownian motion.

Our main question is: can a stochastic perturbation suppress the appearance of unbounded gradients?

The stochastically perturbed Burgers equation and the relative Langevin equation were treated in many works (e.g. [1],[2]). The behavior of the space gradient of the velocity was studied earlier in other contexts in [3], [4], but this problem is quite different from the problem considered in the present paper. The analogous problem concerning the behavior of gradients of solutions to the Burgers equation under other types of stochastic perturbations was studied in [5].

Let us consider the mean of the velocity $U(t)$ at time t when we keep the value of $X(t)$ at time t fixed but allow $U(t)$ to take any value it wants, namely

$$(1.7) \quad \hat{u}(t, x) = \frac{\int_{\mathbb{R}^n} u P(t, x, u) du}{\int_{\mathbb{R}^n} P(t, x, u) du}, \quad t \geq 0, x \in \mathbb{R}^n,$$

where $P(t, x, u)$ is the probability density in position and velocity space, so that $\int_{\mathbb{R}^n \times \mathbb{R}^n} P(t, x, u) dx du = 1$.

This function obeys the following Fokker-Planck equation:

$$(1.8) \quad \frac{\partial P(t, x, u)}{\partial t} = \left[-\sum_{k=1}^n u_k \frac{\partial}{\partial x_k} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial u_k^2} \right] P(t, x, u),$$

subject to the initial data

$$P(0, x, u) = P_0(x, u).$$

If we choose

$$(1.9) \quad P_0(x, u) = \delta(u - u_0(x)) f(x) = \prod_{k=1}^n \delta(u_k - (u_0(x))_k) f(x),$$

with an arbitrary sufficiently regular $f(x)$, then $\hat{u}(0, x) = u_0(x)$. The function $f(x)$ has the meaning of a probability density of the particle positions in the space at the initial moment of time, and therefore $f(x)$ has to be chosen such that $\int_{\mathbb{R}^n} f(x) dx = 1$. If the latter integral diverges for a certain choice of $f(x)$,

we consider the domain $\Omega_L := [-L, L]^n$, $L > 0$, and the renormalized density $f_L(x) := \chi(\Omega_L) f(x) \left(\int_{\Omega_L} f(x) dx \right)^{-1}$, where $\chi(\Omega_L)$ is the characteristic function of Ω_L , we denote the respective probability density in velocity and position by $P_L(t, x, u)$ and modify the definition of $\hat{u}(t, x)$ as follows:

$$(1.10) \quad \hat{u}(t, x) = \lim_{L \rightarrow \infty} \frac{\int_{\mathbb{R}^n} u P_L(t, x, u) du}{\int_{\mathbb{R}^n} P_L(t, x, u) du}, \quad t \geq 0, x \in \Omega_L,$$

provided the limit exists.

We apply heuristically the Fourier transform in the variables u and x to (1.8), (1.9) to obtain for $\tilde{P} = \tilde{P}(t, \lambda, \xi)$

$$(1.11) \quad \frac{\partial \tilde{P}}{\partial t} = -\frac{\sigma^2}{2} |\xi|^2 \tilde{P} + \left(\lambda, \frac{\partial \tilde{P}}{\partial \xi} \right),$$

$$(1.12) \quad \tilde{P}(0, \lambda, \xi) = \int_{\mathbb{R}^n} f(s) e^{-i(\xi, u_0(s))} e^{-i(\lambda, s)} ds.$$

Thus, (1.11) and (1.12) give

$$(1.13) \quad \tilde{P}(t, \lambda, \xi) = e^{-\frac{\sigma^2}{6|\lambda|} (|\xi + \lambda t|^3 - |\xi|^3)} \int_{\mathbb{R}^n} f(s) e^{-i((\xi + \lambda t), u_0(s))} e^{-i(\lambda, s)} ds,$$

$$(1.14) \quad \begin{aligned} P(t, x, u) &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \tilde{P}(t, \lambda, \xi) e^{i(\xi, u)} e^{i(\lambda, x)} d\lambda d\xi \\ &= \left(\frac{\sqrt{3}}{\pi \sigma^2 t^2} \right)^n \int_{\mathbb{R}^n} f(s) e^{-\frac{2}{\sigma^2 t^3} (3t^2 (u, u_0(s)) + t^2 |u_0(s) - u|^2 + 3|x - s|^2 + 3t (u + u_0(s), s - x))} ds. \end{aligned}$$

Now we substitute (1.14) in (1.8) or (1.10), integrate with respect to u and obtain the formula

$$(1.15) \quad \hat{u}(t, x) = \frac{1}{2t} \frac{\int_{\mathbb{R}^n} (-u_0(s)t - 3(s - x)) f(s) e^{-\frac{3|u_0(s)t + (s-x)|^2}{2\sigma^2 t^3}} ds}{\int_{\mathbb{R}^n} f(s) e^{-\frac{3|u_0(s)t + (s-x)|^2}{2\sigma^2 t^3}} ds}, \quad t \geq 0, x \in \mathbb{R}^n,$$

provided all integrals exist.

Thus, we can compare $\hat{u}(t, x)$ with the solution $u(t, x)$ of the non-viscous Burgers equation (1.1).

2. EXACT RESULTS

It is natural to begin with the case where the solution to the Burgers equation (1.1) can be obtained explicitly. Let us choose

$$(2.1) \quad u_0(x) = \alpha x, \quad \alpha < 0.$$

One can see from (1.5), (1.6) that the gradient of the solution becomes unbounded as $t \rightarrow T$.

If the initial distribution of particles is either uniform or Gaussian, it is possible to get explicit formulas for \hat{u} . Namely, for the uniform distribution $f(x) = \text{const}$ both integrals in the numerator and the denominator in (1.15) can be taken, and we get

$$\hat{u}(t, x) = \frac{\alpha x}{1 + \alpha t},$$

which coincides with (1.5). Therefore, the gradient becomes unbounded at $T = -\frac{1}{\alpha}$. On the contrary, in the case of a Gaussian distribution, $f(x) = \left(\frac{r}{\sqrt{\pi}}\right)^n e^{-r^2|x|^2}$, $r > 0$, we get another explicit formula:

$$(2.2) \quad \hat{u}(t, x) = \frac{3(\alpha(\alpha t + 1) + r^2\sigma^2t^2)}{3(\alpha t + 1)^2 + 2r^2\sigma^2t^3} x.$$

One can see that the denominator does not vanish for all positive t , and at the critical time T we have $\hat{u}(t, x) = -\frac{3}{2}\alpha x$; that is, the gradient becomes positive and tends to zero as $t \rightarrow +\infty$.

3. 1D CASE, SPECIFIC CLASSES OF INITIAL DISTRIBUTIONS OF PARTICLES AND INITIAL DATA

Our main question is how the decay rate of the function $f(x)$ at infinity relates to the property of \hat{u} to reproduce the behavior of the solution of the non-perturbed Burgers equation at the critical time. For the sake of simplicity we dwell on the case of a one dimensional space; however the results can be extended to the higher dimensional space. Let us consider the class of initial distributions of particles $f(x)$ which are intermediate between Gaussian and uniform. Our aim is to find a threshold rate of decay at infinity that still allows us to preserve the singularity at the origin.

We restrict ourselves to the class of smooth distributions $f(x)$ and initial data $u_0(x)$ satisfying the condition

$$(3.1) \quad \left| \int_{\mathbb{R}} \xi^m (u_0(\xi))^l f(\xi) \exp(-\gamma\xi^2) d\xi \right| < \infty \quad \text{for all } m, l \in \mathbb{N} \cup \{0\}, \gamma > 0.$$

As a representative of such a class of distributions we can consider

$$(3.2) \quad f(x) = \text{const} \cdot (1 + |x|^2)^k, \quad k \in \mathbb{R}.$$

Theorem 3.1. *Let the initial $u_0(x)$ be smooth, and for a certain fixed $\beta < 0$ and all $x \in \mathbb{R}$ (except for maybe a bounded set) $|u_0(x) - \beta x| \geq \gamma > 0$. Moreover, assume that the distribution function $f(x)$ is smooth, nonnegative, and the property (3.1) is satisfied. Then the mean $\hat{u}(t, x)$ has at the origin $x = 0$ at the moment $t_0 = -\frac{1}{\beta}$, $\beta < 0$, a bounded derivative $u'_x(t_0, 0)$.*

We remark that the initial data with a linear initial profile except for $\beta = u'_0(x)$ fall into the class of initial data that we have described above.

Proof. First of all we perform a change of the time variable. Let $\epsilon = t + \frac{1}{\beta}$, $\beta < 0$. We expand $\hat{u}(t(\epsilon), x)$ given by (1.15) into a Taylor series at the point $t = t_0 = -\frac{1}{\beta}$ ($\epsilon = 0$), $x = 0$, taking into account that condition (3.1) guarantees the convergence of the integrals in the coefficients of the expansion. This expansion has the form

$$\begin{aligned}
 \hat{u}(t(\epsilon), x) &\sim \frac{1}{2} \frac{\int_{\mathbb{R}} (3s\beta + u_0(s)) f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds}{\int_{\mathbb{R}} f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds} \\
 &- \frac{3\beta}{2\sigma^2} \left(\frac{\int_{\mathbb{R}} (\sigma^2 - 4\beta^2 s u_0(s) + 3s^2 \beta^3 + \beta(u_0(s))^2) f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds}{\int_{\mathbb{R}} f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds} \right. \\
 (3.3) \quad &+ \left. \frac{\int_{\mathbb{R}} (\beta s - u_0(s)) f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds \int_{\mathbb{R}} (3\beta s - u_0(s)) f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds}{\left(\int_{\mathbb{R}} f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds \right)^2} \right) x,
 \end{aligned}$$

as $x \rightarrow 0$, $\epsilon \rightarrow 0-$ (where \sim stands for the quotient of the left and right sides converging to 1).

The theorem follows immediately from the asymptotics in (3.3). □

Let us notice that for even $f(x)$ and odd $u_0(x)$ the expansion (3.3) is less cumbersome; namely,

$$(3.3^*) \quad \hat{u}(t(\epsilon), x) \sim -\frac{3\beta}{2\sigma^2} \frac{\int_0^\infty (\sigma^2 - 4\beta^2 s u_0(s) + 3s^2 \beta^3 + \beta(u_0(s))^2) f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds}{\int_0^\infty f(s) e^{\frac{3\beta^3}{2\sigma^2} \left(\frac{u_0(s)}{\beta} - s\right)^2} ds} x,$$

as $x \rightarrow 0$, $\epsilon \rightarrow 0-$.

It can be readily calculated that if $\beta \rightarrow -\infty$ ($t_0 \rightarrow 0$), then (3.3*) yields $\hat{u}(t, x) \sim \alpha x$, $x \rightarrow 0$, $\epsilon \rightarrow 0-$, where $\alpha = u_x(0)$ (taking account of $\frac{u_0(\xi)}{\xi} \sim \alpha$, $u'_0(\xi) \sim \alpha$, $\xi \rightarrow 0$).

3.1. Power-behaved distribution. Let us consider the specific class of even distributions (3.2) and linear initial data.

The case of a linear initial function $u_0(x) = \alpha x$, $\alpha \neq \beta$, is particular. Indeed, we have from (3.3*) for $x \rightarrow 0$ and for $t \rightarrow t_0 = -\frac{1}{\beta}$, $\beta < 0$, the following asymptotic behavior:

$$\hat{u}(t, x) \sim \Lambda(\beta) x,$$

with

$$\Lambda(\beta) = -\frac{3\beta}{2\sigma^2} \frac{\int_0^\infty (\sigma^2 + \beta s^2(\alpha - \beta)(\alpha - 3\beta)) f(s) e^{\frac{3s^2\beta}{2\sigma^2}(\beta - \alpha)^2} ds}{\int_0^\infty f(s) e^{\frac{3s^2\beta}{2\sigma^2}(\beta - \alpha)^2} ds}.$$

We can see that if $\beta < \alpha$ (before the critical time $T = -\frac{1}{\alpha}$, when the solution of the non-perturbed Burgers equation blows up) or $\beta > \alpha$ (after the time T), both integrals in (3.3*) converge and therefore the derivative $\hat{u}'_x(t, 0)$ remains bounded.

Let us consider now the critical moment of time $t = T$, where $\beta = \alpha$. In this case $\frac{u_0(x)}{x} = \beta$ identically, and we do not have a multiplier that guarantees the convergence of integrals of the form

$$\int_{\mathbb{R}_+} \xi^m f(\xi) d\xi \quad \text{for all } m \in \mathbb{N},$$

which is necessary for the validity of the asymptotics (3.3*).

However, fortunately, due to the relative simplicity of $f(x)$ we can compute $\hat{u}(t, x)$ in the vicinity of the origin directly, using the formula (1.15), which in this case takes the form

$$(3.4) \quad \hat{u}(t, x) = \frac{1}{2t} \frac{\int_{\mathbb{R}_+} (-\alpha st - 3(s-x)) (1+s^2)^k e^{-3|\alpha st+(s-x)|^2} ds}{\int_{\mathbb{R}_+} (1+s^2)^k e^{-3|\alpha st+(s-x)|^2} ds}, \quad t \geq 0, x \in \mathbb{R}.$$

Computations show that for $k \neq \frac{m}{2}$, $m \in \mathbb{Z}$, the asymptotic behavior of (3.4) as $x \rightarrow 0$, $\epsilon \rightarrow 0-$, where $\epsilon = t + \frac{1}{\alpha}$, can be expressed through the Gamma function and the generalized Laguerre functions $L(\nu_1, \nu_2, \nu_3)$; see [6]. It has the form

$$(3.5) \quad \begin{aligned} \hat{u}(t, x) &\sim \frac{F_1(\epsilon, k, \alpha, \sigma)}{F_2(\epsilon, k, \alpha, \sigma)} x, \\ F_1(\epsilon, k, \alpha, \sigma) &= A_1(k) \epsilon^{-2k-2} + o(\epsilon^{-2k-2}) + A_2(k) \epsilon^0 + o(\epsilon^0), \\ F_2(\epsilon, k, \alpha, \sigma) &= A_3(k) \epsilon^{-2k-1} + o(\epsilon^{-2k-1}) + A_4(k) \epsilon^0 + o(\epsilon^0), \end{aligned}$$

where the coefficients $A_i(k)$, $i = 1, \dots, 4$, are as follows:

$$\begin{aligned} A_1(k) &= \frac{\pi^2 2^{k+2} \sigma^{2k} (4k^2 - 1)}{3^k |\alpha|^{5k+1} \cos \pi k} \Gamma(k+1) L(k, -k + \frac{1}{2}, 0), \\ A_2(k) &= \frac{3\sqrt{6} |\pi \alpha|^{\frac{5}{2}}}{2\sigma(k+1)} \tan(\pi k) L(\frac{1}{2}, k + \frac{1}{2}, 0), \\ A_3(k) &= \frac{\pi^2 2^{k+1} \sigma^{2k} (2k-1)}{3^k |\alpha|^{5k+1} (k+1)(k+2) \cos \pi k} \Gamma(k+3) L(k, -k + \frac{1}{2}, 0), \\ A_4(k) &= \frac{\sqrt{6} \pi^2 |\alpha|^{\frac{3}{2}} (2k+3) \Gamma(k+3)}{\sigma(k+1)(k+2) \Gamma(k + \frac{5}{2})} \tan(\pi k). \end{aligned}$$

Thus, if $k < -1$, then the leading term of the numerator and denominator in (3.5) as $\epsilon \rightarrow 0-$ is $A_2 \epsilon^0$ and (3.5) can be written as

$$(3.6) \quad \hat{u}(t, x) \sim \frac{A_2(k) \epsilon^0 + o(\epsilon^0)}{A_4(k) \epsilon^0 + o(\epsilon^0)} x \sim (B_1(k) + o(\epsilon^0)) x, \quad x \rightarrow 0,$$

where $B_1(k) = \frac{A_2(k)}{A_4(k)}$.

This signifies that the derivative $\hat{u}'_x(t, 0)$ tends to a finite limit as $\epsilon \rightarrow 0-$.

If $-\frac{1}{2} > k > -1$, then the leading term of the denominator is $A_4(k)\epsilon^0$. Otherwise, if $k > -\frac{1}{2}$, then this leading term is $A_3(k)\epsilon^{-2k-1}$. Thus we have for $-\frac{1}{2} > k > -1$

$$\hat{u}(t, x) \sim \frac{A_1(k)\epsilon^{-2k-2} + o(\epsilon^{-2k-2})}{A_4(k)\epsilon^0 + o(\epsilon^0)}$$

and

$$(3.7) \quad \hat{u}'_x(t, 0) \sim B_2(k) \cdot \frac{1}{\epsilon^{2k+2}}, \quad B_2(k) = \frac{A_1}{A_4}, \quad x \rightarrow 0, \epsilon \rightarrow 0-.$$

Lastly, for $k > -\frac{1}{2}$ we have

$$\hat{u}(t, x) \sim \frac{A_1(k)\epsilon^{-2k-2} + o(\epsilon^{-2k-2})}{A_3(k)\epsilon^{-2k-1} + o(\epsilon^{-2k-1})} x, \quad x \rightarrow 0, \epsilon \rightarrow 0-,$$

and

$$(3.8) \quad \hat{u}'_x(t, 0) \sim B_3(k) \cdot \epsilon^{-1}, \quad B_3(k) = \frac{A_1(k)}{A_3(k)} = 2k + 1.$$

If $k \in \mathbb{Z}$, then the numerator and the denominator in the leading term in the expansion of (3.5) as $x \rightarrow 0$ are expressed either through rational functions ($k \geq 0$) or through a Gaussian distribution function ($k < 0$). For $k = \frac{2l+1}{2}$, $l \in \mathbb{Z}$, the coefficient of the leading term is expressed through a ratio of series consisting of the digamma functions. Anyway, the asymptotics in (3.6) takes place also for $k = \frac{1}{2}$, $l \in \mathbb{Z}$, $k \neq -\frac{1}{2}$, and it can also be found as a limit $\kappa \rightarrow k$. For $k < -1$ the function $\hat{u}(t, x)$ behaves as in (3.6), where the coefficient $B_1(k)$ can be calculated either independently or as $\lim_{\kappa \rightarrow k} \frac{A_2(\kappa)}{A_4(\kappa)}$. Since for $k = -1$ the degrees in ϵ^{-2k-2} and ϵ^0 coincide, then

$$\hat{u}(t, x) \sim \lim_{\kappa \rightarrow -1} \frac{(A_1(\kappa) + A_2(\kappa))\epsilon^0 + o(\epsilon^0)}{A_4(\kappa)\epsilon^0 + o(\epsilon^0)} x \sim (B_4 + o(\epsilon^0)) x, \quad x \rightarrow 0, \epsilon \rightarrow 0-,$$

where $B_4 = \lim_{\kappa \rightarrow -1} \left(B_1(\kappa) + \frac{A_1(k)}{A_4(k)} \right) = \frac{3|\alpha|}{2} - \frac{\sqrt{6}|\alpha|^{\frac{5}{2}}}{\sigma\sqrt{\pi}}$. For $k \geq 0$ the function $\hat{u}(t, x)$ has the asymptotics in (3.8) with the same value $B_3(k)$. An exceptional case is $k = -\frac{1}{2}$, where

$$F_1(\epsilon, -1/2, \alpha, \sigma) = \bar{A}_1 \epsilon^{-1} + o(\epsilon^{-1}), \quad \bar{A}_1 = \lim_{k \rightarrow -1/2} A_1 = \frac{4\sqrt{6}|\pi\alpha|^{\frac{3}{2}}}{\sigma},$$

$$F_2(\epsilon, -1/2, \alpha, \sigma) = A_5 \ln(-\epsilon) + o(\ln(-\epsilon)), \quad A_5 = -\bar{A}_1, \quad \epsilon \rightarrow 0-.$$

Thus, for $k = -\frac{1}{2}$ we have

$$(3.9) \quad \hat{u}'_x(t, 0) \sim -\frac{1}{\epsilon \ln(-\epsilon)} + o\left(\frac{1}{\epsilon \ln(-\epsilon)}\right), \quad \epsilon \rightarrow 0-.$$

The following theorem summarizes our results:

Theorem 3.2. *Assume that in the case $n = 1$ the initial distribution function is $f(x) = \text{const} \cdot (1 + |x|^2)^k$, $k \in \mathbb{R}$, and the initial velocity has the form $u_0(x) = \alpha x$, $\alpha < 0$. Then the space derivative of the mean $\hat{u}(t, x)$ at the origin $x = 0$ is bounded for all $t > 0$ except for the critical time $T = -\frac{1}{\alpha}$. At that critical time the behavior of the derivative depends on k . Namely, for $k > -1$ the mean $\hat{u}(t, x)$ keeps the property of solutions to the non-perturbed Burgers equation to blow up at the critical time T at $x = 0$. The rates of the blowup for $-\frac{1}{2} > k > -1$, $k > -\frac{1}{2}$ and*

$k = -\frac{1}{2}$ are indicated in (3.7), (3.8) and (3.9), respectively. Otherwise, if $k \leq -1$, the derivative $\hat{u}'_x(t, 0)$ at the critical time remains bounded; i.e. the singularity disappears.

4. NUMERICAL ILLUSTRATIONS

Numerics, performed directly according to formula (1.15) for several classes of initial data $u_0(x)$, illustrate our analytical results. We take $\sigma = 0.5$. Figures 1 and

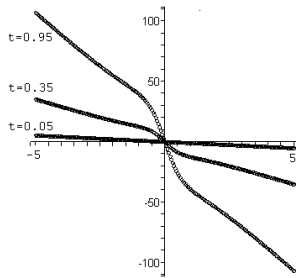


FIGURE 1. $u_0 = -x, k = 1$, before the critical time $T = 1$

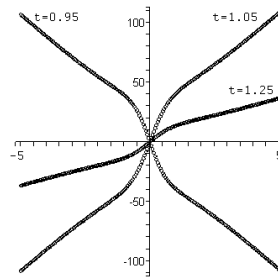


FIGURE 2. $u_0 = -x, k = 1$, pass through the critical time $T = 1$

2 show the behavior of $\hat{u}(t, x)$ near the origin $x = 0$ in the case of linear initial data $u_0 = -x$ and $k = 1$. This is the case where at the critical time $T = 1$ the mean $\hat{u}(t, x)$ keeps the property of the solution of the unperturbed Burgers equation to have an unbounded gradient. Figure 2 illustrates the pass through the critical time. Figures 3 and 4 correspond to the same initial data for $k = -1$. We can see

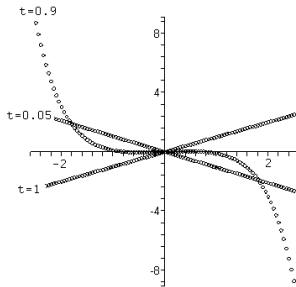


FIGURE 3. $u_0 = -x, k = -1$, before the critical time $T = 1$

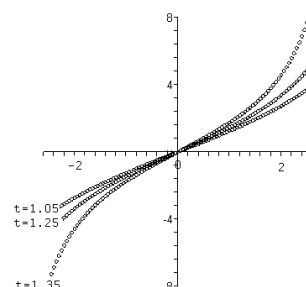


FIGURE 4. $u_0 = -x, k = -1$, after the critical time $T = 1$

that the sign of the derivative at the point $x = 0$ changes before the critical time $T = 1$ and the derivative remains bounded. Figures 5 and 6 illustrate the behavior near the critical time $T = 1$ of the mean $\hat{u}(t, x)$ in the case of bounded initial data $u_0(x) = -\arctan x$ for $k = 1$. We can see that the negative sign of the derivative

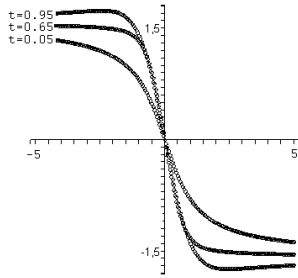


FIGURE 5. $u_0 = -\arctan x$, $k = 1$, before the critical time $T = 1$

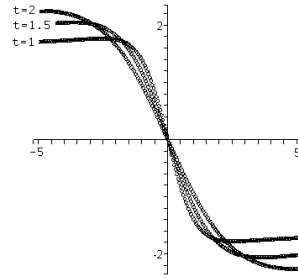


FIGURE 6. $u_0 = -\arctan x$, $k = 1$, after the critical time $T = 1$

does not change, the absolute value of $\hat{u}_x(t, 0)$ increases as $t \rightarrow 1$, then decreases again, and the unbounded gradient is suppressed. Figures 7 and 8 relate to the

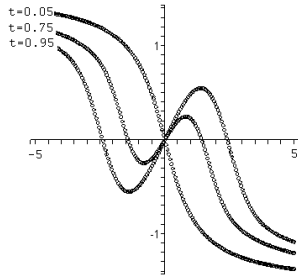


FIGURE 7. $u_0 = -\arctan x$, $k = -1$, before the critical time $T = 1$

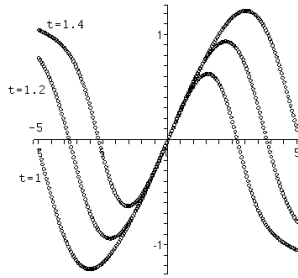


FIGURE 8. $u_0 = -\arctan x$, $k = -1$, after the critical time $T = 1$

case of $u_0(x) = -\arctan x$ for $k = -1$. We see the sign of the derivative $\hat{u}(t, x)$ at the origin $x = 0$. Thus, the unbounded gradient is suppressed again.

5. PRESSURELESS GAS DYNAMICS MODEL AND A LIMIT CASE AT VANISHING NOISE

Let us consider the pair $(\rho_t, u(\cdot, t))$, $t \geq 0$, where ρ_t is the probability distribution of the random variable X_t , governed by the SDE $X_t = X_0 + \int_0^t \mathbb{E}[u_0(X_0)|X_s] ds$, $t \geq 0$, with a given random variable X_0 with values in \mathbb{R} and function $u_0(x)$, and $u(t, x) = \mathbb{E}[u_0(X_0)|X_t = x]$. According to [7], $(\rho_t, u(\cdot, t))$, $t \geq 0$ is a weak solution to the pressureless gas dynamics model

$$(5.1) \quad \partial_t \rho + \partial_x(\rho u) = 0, \quad \partial_t(\rho u) + \partial_x(\rho u^2) = 0.$$

Therefore it is natural to expect that the limit as $\sigma \rightarrow 0$ of the mean \hat{u} (denoted by $v(t, x)$) takes part in the solution to (5.1).

For smooth $u_0(x)$ and $f(x)$ this can be readily shown. First of all, let us introduce the function $\rho(t, x) = \int_{\mathbb{R}^n} P(t, x, u) du$ and notice that it satisfies the continuity equation $\partial_t \rho + \partial_x(\rho \hat{u}) = 0$. Further, we check that the function $v(t, x)$ satisfies the Burgers equation (1.1). Indeed, $\frac{t^{3/2}}{\sigma\sqrt{6\pi}} \exp\left(-\frac{3(u_0(s)t + (s-x)^2)}{2\sigma^2 t^3}\right) \rightarrow \delta(s - s(t, x))$, as $\sigma \rightarrow 0$ in \mathcal{D}' , where $s(t, x)$ is a unique solution to equation $u_0(s) + \frac{s-x}{t} = 0$, given in implicit form. This function exists and it is differentiable provided t is less than the moment of time t_* when the solution to the Burgers equation blows up. Thus, $\hat{u}(t, x) \rightarrow u_0(s(t, x))$ as $\sigma \rightarrow 0$. Now it is sufficient to substitute $u_0(s(t, x))$ into (1.1) and compute the derivatives of $s(t, x)$ by means of the implicit function theorem.

Thus, for smooth initial data $(f(x), u_0(x))$, $t < t_*$, the pair (ρ, \hat{u}) is a solution to the system

$$\partial_t \rho + \partial_x(\rho \hat{u}) = 0, \quad \partial_t(\rho \hat{u}) + \partial_x(\rho \hat{u}^2) = \Lambda, \quad \Lambda = - \int_{\mathbb{R}} P_x(t, x, u) (u - \hat{u})^2 du,$$

where $\Lambda \rightarrow 0$ as $\sigma \rightarrow 0$. The integral relaxation term Λ can be used instead of the traditional viscosity [8]. After the blow up time Λ does not vanish as $\sigma \rightarrow 0$.

It is interesting to consider the Fokker-Plank equation (1.8) as a kinetic equation

$$\frac{\partial P(t, x, u)}{\partial t} + \sum_{k=1}^n \left(u_k \frac{\partial P(t, x, u)}{\partial x_k} + \frac{\partial \dot{u}_k P(t, x, u)}{\partial u_k} \right) = 0$$

(e.g. [9]), where the acceleration \dot{u} of the particles is due to external forces and the interaction forces with other particles. It can be readily calculated that in our case $\dot{u} = \frac{2}{t}(\hat{u} - u)$.

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