

CHERN SUBRINGS

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ABSTRACT. Let p be an odd prime. We show that for a simply connected semisimple complex linear algebraic group, if its integral homology has p -torsion, the Chern classes do not generate the Chow ring of its classifying space.

1. INTRODUCTION

Let p be an odd prime. Let $h^*(-)$ be one of the mod p cohomology $H\mathbb{Z}/p$, the cohomology $H\mathbb{Z}_{(p)}$ with coefficient $\mathbb{Z}_{(p)}$ and the Brown-Peterson cohomology BP with $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$. Let G be a compact connected Lie group and $G(\mathbb{C})$ its complexification; that is, $G(\mathbb{C})$ is a complex linear algebraic group which is homotopy equivalent to the compact connected Lie group G . Considering a finite dimensional complex representation $\rho : G \rightarrow GL_m(\mathbb{C})$, we have Chern classes $c_i(\rho)$ in the cohomology $h^*(BG)$ of classifying space and the Chern subring $Ch_h(G) \subset h^*(BG)$, a subalgebra over h_* generated by Chern classes, where ρ ranges over all finite dimensional representations. If G is one of the classical groups $SU(n)$, $Spin(n)$ and $Sp(n)$, the cohomology $h^*(BG)$ is generated by Chern classes and $h^*(BG) = Ch_h(G)$ for an arbitrary odd prime p .

The case of the Brown-Peterson cohomology is particularly interesting in conjunction with the study of Chow rings of classifying spaces of complex linear algebraic groups defined by Totaro. In [To], Totaro considered the classifying space of the linear algebraic group $G(\mathbb{C})$ as a limit of algebraic varieties, defined the Chow ring for it and showed that the cycle map factors through the Brown-Peterson cohomology,

$$CH^*(BG(\mathbb{C}))_{(p)} \rightarrow BP^*(BG) \otimes_{BP_*} \mathbb{Z}_{(p)} \rightarrow H^{even}(BG; \mathbb{Z}_{(p)}),$$

where $H^{even}(BG; \mathbb{Z}_{(p)})$ is the direct sum of $H^{2i}(BG; \mathbb{Z}_{(p)})$ ($i \geq 0$). He also conjectured that the left homomorphism $CH^*(BG(\mathbb{C}))_{(p)} \rightarrow BP^*(BG) \otimes_{BP_*} \mathbb{Z}_{(p)}$ is an isomorphism. We may consider a Chern subring for the Chow ring $CH^*(BG(\mathbb{C}))$ as in the case of the above $Ch_h(G)$.

In [Ka-Ya] and [Vi], the Chow ring $CH^*(BPGL_p(\mathbb{C}))_{(p)}$ of the complex linear algebraic group $PGL_p(\mathbb{C})$, which is the complexification of the projective unitary

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group $PU(p)$, and related cohomology theories were computed, and it was shown that

$$CH^*(BPGL_p(\mathbb{C}))_{(p)} = BP^*(BPU(p)) \otimes_{BP_*} \mathbb{Z}_{(p)} = H^{even}(BPU(p); \mathbb{Z}_{(p)})$$

through the cycle map above. In [Ka-Ya, Proposition 5.7], we showed similar results for $(p, G) = (3, F_4), (5, E_8)$. For $p = 3$, the computation of the Brown-Peterson cohomology was done by Kono and Yagita in [Ko-Ya], and Kono and Yagita showed that x_8^a is not in the Chern subring unless a is divisible by 2. In [Ta], Targa showed that x_{2p+2}^a in $CH^*(BPGL_p(\mathbb{C}))_{(p)}$, where $a \leq p - 2$, is not in the Chern subring for an arbitrary odd prime p .

In this paper, we prove the following and generalize the above computation of Kono, Yagita and Targa. Let Q_i be the Milnor operations of degree $2p^i - 1$ which acts on the mod p cohomology of a space.

Theorem 1.1. *For $(p, G) = (p, PU(p))$, let $x = Q_0Q_1x_2$ where x_2 is the generator of $H^2(BG; \mathbb{Z}/p) = \mathbb{Z}/p$. For $(p, G) = (3, F_4), (3, E_6), (3, E_7), (3, E_8), (5, E_8)$, let $x = Q_1Q_2x_4$, where x_4 is the generator of $H^4(BG; \mathbb{Z}/p) = \mathbb{Z}/p$. Then, x^a is not in the Chern subring $Ch_{HZ/p}(G)$ unless a is divisible by $p - 1$.*

This theorem implies that if x comes from the Chow ring through the cycle map, then the Chow ring is not generated by Chern classes. Recall that the motivic cohomology $H^{*,*'}(BG(\mathbb{C}), \mathbb{Z}/p)$ contains $CH^*(BG(\mathbb{C}))/p$ as

$$CH^*(BG(\mathbb{C}))/p = H^{2*,*}(BG(\mathbb{C}), \mathbb{Z}/p).$$

Moreover, the motivic cohomology has the action of Milnor operations Q_i where the degree of Q_i is $(2p^i - 1, p^i - 1)$. We refer the reader to Voevodsky's paper [Vo] for motivic cohomology operations. If there exists an element $x_{4,3}$ in $H^{4,3}(BG(\mathbb{C}), \mathbb{Z}/p)$ corresponding to x_4 in $H^4(BG; \mathbb{Z}/p)$, then $x = Q_1Q_2(x_{4,3})$ is in the Chow ring

$$CH^{p^2+p+1}(BG(\mathbb{C}))/p = H^{2p^2+2p+2, p^2+p+1}(BG(\mathbb{C}), \mathbb{Z}/p)$$

and through the cycle map it maps to x in Theorem 1.1. In [Ya, Lemma 9.6], Yagita proved that if $px_4 \in H^4(BG; \mathbb{Z}_{(p)})$ is a Chern class of some representation, then the element $x_{4,3}$ above exists. In [Sc-Ya], Schuster and Yagita showed that for $(p, G) = (3, F_4)$, $3x_4$ is the Chern class of the complexification of the irreducible representation of F_4 . In this paper, by computing the Chern class of the adjoint representation of E_8 , we prove the following proposition.

Proposition 1.2. *For $(p, G) = (3, F_4), (3, E_6), (3, E_7), (3, E_8)$ and $(5, E_8)$, there exist a complex representation α of G and $\gamma \in \mathbb{Z}_{(p)}^\times$ such that the element $\gamma px_4 \in H^4(BG; \mathbb{Z}_{(p)})$ is a Chern class $c_2(\alpha)$.*

Thus, we have the following result on Chern subrings of Chow rings.

Theorem 1.3. *For $(p, G) = (p, PU(p)), (3, F_4), (3, E_6), (3, E_7), (3, E_8)$ and $(5, E_8)$, the Chow ring $CH^*(BG(\mathbb{C}))_{(p)}$ is not generated by Chern classes.*

Our results seem to support Totaro's conjecture since not only elements in $Ch_{BP}(G)$ but also some elements that are not in the Chern subrings come from Chow rings $CH^*(BG(\mathbb{C}))$ for the above (p, G) .

In §2, we consider Chern classes of elementary abelian p -groups. In §3, we prove Theorem 1.1. In §4, we prove Proposition 1.2. Throughout the rest of this paper, we write $H^*(-)$ for $H^*(-; \mathbb{Z}/p)$ whenever the coefficient \mathbb{Z}/p is clear from the context.

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2. CHERN CLASSES OF ELEMENTARY ABELIAN p -GROUPS

In this section, we investigate the total Chern class of a finite dimensional complex representation $\rho : A_n \rightarrow GL_m(\mathbb{C})$ of an elementary abelian p -group A_n of rank n .

First, we recall the cohomology of BA_n . The mod p cohomology of an elementary abelian p -group is a polynomial tensor exterior algebra

$$\mathbb{Z}/p[t_1, \dots, t_n] \otimes \Lambda(dt_1, \dots, dt_n).$$

The elements $dt_1, \dots, dt_n \in H^1(BA_n)$ correspond to the dual of the basis of $\pi_1(BA_n) = H_1(BA_n)$. The elements t_1, \dots, t_n are obtained from dt_1, \dots, dt_n by applying the Milnor operation Q_0 . For the mod p cohomology of a space, there exists an action of Milnor operations Q_0, Q_1, Q_2, \dots and reduced power operations $\varphi^0 = 1, \varphi^1, \varphi^2, \dots$. The action of Milnor operations on the mod p cohomology of an elementary abelian p -group is given by

$$Q_i(dt_k) = t_k^i, \quad Q_i t_k = 0, \quad Q_i(x \cdot y) = Q_i(x) \cdot y + (-1)^{\deg x} x \cdot Q_i(y).$$

The action of reduced power operations is given by

$$\varphi^i dt_k = 0, \quad \varphi^i t_k = \begin{cases} t_k^p & (i = 1) \\ 0 & (i \geq 2), \end{cases} \quad \varphi^j(x \cdot y) = \sum_{i=0}^j \varphi^{i-j} x \cdot \varphi^j y.$$

Second, we recall the invariant theory of finite general linear groups and special linear groups. The action of Milnor operations commutes with the action of the general linear group $GL_n(\mathbb{Z}/p)$ since the action of the general linear group on the mod p cohomology comes from the one on the elementary abelian p -group A_n . For the sake of notational simplicity, we write V_n for the subspace spanned by t_1, \dots, t_n ,

$$V_n = \mathbb{Z}/p\{t_1, \dots, t_n\}.$$

Let

$$\begin{aligned} SM_n &= H^*(BA_n)^{SL_n(\mathbb{Z}/p)}, & M_n &= H^*(BA_n)^{GL_n(\mathbb{Z}/p)}, \\ SD_n &= \mathbb{Z}/p[t_1, \dots, t_n]^{SL_n(\mathbb{Z}/p)}, & D_n &= \mathbb{Z}/p[t_1, \dots, t_n]^{GL_n(\mathbb{Z}/p)}, \end{aligned}$$

respectively. Kameko and Mimura [Ka-Mi] gave a simpler description for SM_n, M_n using Milnor operations. For the Dickson invariants SD_n, D_n and the Mui invariants SM_n, M_n above, we refer the reader to [Ka-Mi] and its references. Let us define $c_{n,i}$ for $n = 1, \dots, n - 1$ as follows: Consider the polynomial

$$f_n(X) = \prod_{v \in V_n} (X + v)$$

in $\mathbb{Z}/p[t_1, \dots, t_n][X]$. We define $(-1)^{n-i} c_{n,i}$ to be the coefficient of $X^{p^{n-i}}$ in $f_n(X)$. We define e_n by $e_n = Q_0 \cdots Q_{n-1}(dt_1 \cdots dt_n)$. Then, we have the following. For a ring R and for a finite set $\{a_1, \dots, a_r\}$, we denote by $R\{a_1, \dots, a_r\}$ a free R -module with the basis $\{a_1, \dots, a_r\}$.

Proposition 2.1. *The following hold:*

- (1) $c_{n,0} = e_n^{p-1}$.
- (2) $f_n(X) = X^{p^n} - c_{n,n-1}X^{p^{n-1}} + \cdots + (-1)^n c_{n,0}X$.
- (3) SD_n is a polynomial algebra $\mathbb{Z}/p[e_n, c_{n,n-1}, \dots, c_{n,1}]$.
- (4) D_n is also a polynomial algebra $\mathbb{Z}/p[c_{n,n-1}, \dots, c_{n,1}, c_{n,0}]$.

(5) M_n is a free D_n -module

$$D_n\{1, e_n^{p-2}dt_1 \dots dt_n, e_n^{p-2}Q_{i_1} \dots Q_{i_r}(dt_1 \dots dt_n)\} \quad \text{and}$$

(6) SM_n is a free SD_n -module

$$SD_n\{1, dt_1 \dots dt_n, Q_{i_1} \dots Q_{i_r}(dt_1 \dots dt_n)\},$$

where $0 \leq i_1 < \dots < i_r \leq n - 1, 1 \leq r \leq n - 1$.

Third, we consider Chern classes. Any finite dimensional complex representation of an abelian group is a direct sum of 1-dimensional complex representations. (See Serre’s book [Se, Theorem 9].) Therefore, the total Chern class $c(\rho)$ is a product of $c(\lambda)$ ’s where $c(\lambda) = 1 + v, v \in V_n$. Thus, the Chern classes are in $\mathbb{Z}/p[t_1, \dots, t_n]$ instead of $H^*(BA_n)$. Let us consider the total Chern class $c(\text{reg})$ of the regular representation $\text{reg} : A_n \rightarrow GL_p(\mathbb{C})$. It is clear that $c(\text{reg}) \in M_n$.

Proposition 2.2. *It follows that*

$$c(\text{reg}) = \prod_{v \in V_n \setminus \{0\}} (1 + v) = 1 - c_{n,n-1} + \dots + (-1)^n c_{n,0} \in D_n.$$

For a group W acting on $V_n \setminus \{0\}$, we say that the action of W is transitive on $V_n \setminus \{0\}$ if and only if for each u, v in $V_n \setminus \{0\}$, there exists $w \in W$ such that $wu = v$. We investigate the total Chern class $c(\rho)$ when the image of the induced homomorphism $B\rho^* : H^*(BGL_m(\mathbb{C})) \rightarrow \mathbb{Z}/p[t_1, \dots, t_n]$ is invariant under a certain group action.

Lemma 2.3. *Let $\rho : A_n \rightarrow GL_m(\mathbb{C})$ be a complex representation of an elementary abelian p -group A_n of rank n . Suppose that a subgroup W of $GL_n(\mathbb{Z}/p)$ acts on A_n in the obvious manner. Suppose that the total Chern class $c(\rho)$ is in $\mathbb{Z}/p[t_1, \dots, t_n]^W$ and suppose that the action of W on $V_n \setminus \{0\}$ is transitive. Then, $c(\rho) = c(\text{reg})^a$ for some $a \geq 0$.*

Proof. Suppose that

$$c(\rho) = \prod_{v \in V_n \setminus \{0\}} (1 + v)^{\mu(v)}.$$

The non-negative integer $\mu(v)$ is the divisibility of $c(\rho)$ by $1 + v$. In other words, $c(\rho)$ is divisible by $(1+v)^{\mu(v)}$ but not divisible by $(1+v)^{\mu(v)+1}$. In order to prove the lemma, it suffices to show that $\mu(v)$ is a constant function of $v \in V_n \setminus \{0\}$. Suppose that $\mu(u) < \mu(v)$ for some $u, v \in V_n \setminus \{0\}$. Let $w \in W$ be an element such that $wv = u$. Then, since w acts trivially on $c(\rho)$, we have

$$c(\rho) = wc(\rho) = \prod_{v' \in V_n \setminus \{0\}} (w(1 + v'))^{\mu(v')} = \left(\prod_{v' \in V_n \setminus \{0, v\}} (1 + wv')^{\mu(v')} \right) (1 + u)^{\mu(v)}.$$

This implies that $\mu(u) \geq \mu(v)$, a contradiction. Hence, we have the desired result. □

By Proposition 2.2 and Lemma 2.3, we have the following result:

Proposition 2.4. *Let G be a compact connected Lie group and let A_n be an elementary abelian p -subgroup of G . Suppose that the Weyl group of A_n , that is, the quotient of the normalizer of A_n in G by the centralizer of A_n in G , acts transitively on $V_n \setminus \{0\}$. Then, $B\eta^*(Ch_{H\mathbb{Z}/p}(G)) \subset D_n$, where $\eta : A_n \rightarrow G$ is the inclusion of A_n into G .*

We end this section by recalling the following fact:

Proposition 2.5. *The action of $SL_n(\mathbb{Z}/p)$ on $V_n \setminus \{0\}$ is transitive for $n \geq 2$.*

Thus, in order to prove Theorem 1.1, it suffices to show that there exists an elementary abelian p -subgroup A_n whose Weyl group is $SL_n(\mathbb{Z}/p)$ and that $B\eta^*(x) \notin D_n$. This is what we do in the next section.

3. CHERN SUBRINGS

In this section, we prove Theorem 1.1 by examining the cohomology of a non-toral elementary abelian p -subgroup of G . There exist non-toral elementary abelian p -subgroups in a compact connected Lie group if the integral homology of the Lie group has p -torsion. These non-toral elementary abelian p -subgroups and their Weyl groups are known for $(p, G) = (p, PU(n)), (3, F_4), (3, E_6), (3, E_7), (3, E_8)$, and $(5, E_8)$. We refer the reader to Andersen et al. [A-G-M-V] and its references. In this paper, we use the following results for $(p, G) = (p, PU(p)), (3, F_4)$ and $(5, E_8)$ only:

Proposition 3.1. *The following hold:*

(1) *For $(p, G) = (p, PU(p))$, there exists a non-toral elementary abelian p -subgroup A_2 of rank 2 such that its Weyl group in G is the special linear group $SL_2(\mathbb{Z}/p)$.*

(2) *For $(p, G) = (3, F_4), (5, E_8)$, there exists a non-toral elementary p -subgroup A_3 of rank 3 such that its Weyl group in G is the special linear group $SL_3(\mathbb{Z}/p)$.*

Let $\eta : A_n \rightarrow G$ be the inclusion of a non-toral elementary abelian p -subgroup in G . In [Ka-Ya], we computed the image of the induced homomorphism

$$B\eta^* : H^*(BG) \rightarrow SM_n$$

for $(p, G) = (p, PU(p))$, $n = 2$ and for $(p, G) = (3, F_4), (3, E_6), (3, E_7), (5, E_8)$, $n = 3$. Since we wish to include the case $(p, G) = (3, E_8)$ in Theorem 1.1, instead of making use of the computation of the image of $B\eta^*$, we use the following result, which is also used in the computation of the image of $B\eta^*$:

Proposition 3.2. *The following hold:*

(1) *The induced homomorphism*

$$H^2(BPU(p)) \rightarrow SM_2^2 = \mathbb{Z}/p\{dt_1 dt_2\}$$

is an isomorphism.

(2) *For $(p, G) = (3, F_4)$ and $(5, E_8)$, the induced homomorphism*

$$H^4(BG) \rightarrow SM_3^4 = \mathbb{Z}/p\{Q_0(dt_1 dt_2 dt_3)\}$$

is an isomorphism.

Now, we prove Theorem 1.1 for $(p, G) = (3, E_8)$. As we mentioned at the end of the previous section, it suffices to show that $B\eta^*(x) \notin D_3$. There is a sequence of inclusions

$$F_4 \rightarrow E_6 \rightarrow E_7 \rightarrow E_8,$$

and the induced homomorphisms

$$H^4(BF_4) \leftarrow H^4(BE_6) \leftarrow H^4(BE_7) \leftarrow H^4(BE_8) = \mathbb{Z}/p$$

are isomorphisms. Recall that we denote the generator of $H^4(BE_8; \mathbb{Z}/3)$ by x_4 . We define $x \in H^{26}(BE_8; \mathbb{Z}/3)$ by $x = Q_1 Q_2(x_4)$. Since the induced homomorphism

maps x_4 to $Q_0(dt_1 dt_2 dt_3)$ by Proposition 3.2, it maps x to $e_3 = Q_0 Q_1 Q_2(dt_1 dt_2 dt_3)$ in SD_3 . Since D_3 is a polynomial algebra generated by $e_3^{p-1}, c_{3,1}$ and $c_{3,2}$ and since SD_3 is a polynomial algebra generated by $e_3, c_{3,1}, c_{3,2}$, it is clear that e_3^a is in D_3 if and only if a is divisible by $p-1$. Thus, we have Theorem 1.1 for $(p, G) = (3, E_8)$. Theorem 1.1 for the other (p, G) 's can be proved in the same manner.

4. PROOF OF PROPOSITION 1.2

In this section, we prove Proposition 1.2 by computing the second Chern class of the adjoint representation of the exceptional Lie group $\alpha : E_8 \rightarrow SO(248)$. Similar computations were done in [Sc-Ya] for the irreducible representation $F_4 \rightarrow SO(26)$.

Since the induced homomorphisms

$$H^4(BF_4; \mathbb{Z}_{(3)}) \leftarrow H^4(BE_6; \mathbb{Z}_{(3)}) \leftarrow H^4(BE_7; \mathbb{Z}_{(3)}) \leftarrow H^4(BE_8; \mathbb{Z}_{(3)}) = \mathbb{Z}_{(3)}$$

are isomorphisms, if $3x_4$ in $H^4(BE_8; \mathbb{Z}_{(3)})$ is a Chern class, so it is in $H^4(BG; \mathbb{Z}_{(3)})$ for $G = F_4, E_6, E_7$. So, it suffices to show the proposition for $G = E_8$.

Let $\alpha : E_8 \rightarrow SO(248)$ be the adjoint representation of E_8 . By the construction of the exceptional Lie group E_8 in [Ad], there exists a homomorphism $\beta : \text{Spin}(16) \rightarrow E_8$ such that the induced representation $\alpha \circ \beta$ is the direct sum of $\lambda_{16}^2 : \text{Spin}(16) \rightarrow SO(120)$ and $\Delta_{16}^+ : \text{Spin}(16) \rightarrow SO(128)$. See [Ad, Corollary 7.3] and [Mi-Ni, p. 143]. Let T^8 be the maximal torus of $\text{Spin}(16)$. Let T^1 be the first factor of T^8 and $\eta : T^1 \rightarrow \text{Spin}(16)$ the inclusion of T^1 into $\text{Spin}(16)$. Denote by $R(G)$ the complex representation ring of G . The complexification of λ_{16}^2 corresponds to the second elementary symmetric function of $z_1^2 + z_1^{-2}, \dots, z_8^2 + z_8^{-2}$ in $R(T^8)$ and the complexification of Δ_{16}^+ corresponds to $\sum_{\varepsilon_1 \cdots \varepsilon_8 = 1} z_1^{\varepsilon_1} \cdots z_8^{\varepsilon_8}$ in $R(T^8)$,

where $\varepsilon_r = \pm 1$ for $r = 1, \dots, 8$.

So, the restriction of the complexification of λ_{16}^2 to T^1 corresponds to

$$2^2 \binom{7}{2} + 2 \binom{7}{1} (z_1^2 + z_1^{-2}) = 84 + 14(z_1^2 + z_1^{-2}) \quad \text{in } R(T^1).$$

The restriction of the complexification of Δ_{16}^+ to T^1 corresponds to

$$2^6 (z_1 + z_1^{-1}) = 64(z_1 + z_1^{-1}) \quad \text{in } R(T^1).$$

Therefore, the total Chern class of the complexification of $\alpha \circ \beta \circ \eta$ is

$$\{(1 + 2u)(1 - 2u)\}^{14} \{(1 + u)(1 - u)\}^{64} = 1 - 120u^2 + \cdots \in \mathbb{Z}[u] = H^*(BT^1; \mathbb{Z}),$$

where u is the generator of $H^2(BT^1; \mathbb{Z}) = \mathbb{Z}$. Since $120 = 2^3 \cdot 3 \cdot 5$, the Chern class $c_2(\alpha)$ represents $\gamma p x_4$ for $p = 3, 5$ in $H^4(BE_8; \mathbb{Z}_{(p)})$, where γ is a unit in $\mathbb{Z}_{(p)}$ and x_4 is the generator of $H^4(BE_8; \mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}$. This completes the proof of Proposition 1.2.

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