

ON THE SOLVABILITY
OF THE CHARACTERISTIC DIRICHLET PROBLEM
FOR LINEAR DEGENERATE PARABOLIC EQUATIONS

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ABSTRACT. We investigate the classical solvability for some classes of linear, degenerate equations in divergence form with prescribed Dirichlet data. Since the boundary value problem is characteristic according to Fichera on a part of the boundary, some typical nonlinear phenomena at these points are observed as boundary gradient blowups of the classical solutions in space directions. The regularity results explain the lack of hypoellipticity for special right-hand sides or boundary data for linear degenerate parabolic equations.

1. INTRODUCTION

In this paper we investigate the classical solvability for linear degenerate parabolic equations in divergence form

$$(1.1) \quad Lu = u_t - (a^{ij}(x, t)u_{x_i})_{x_j} + b^i(x, t)u_{x_i} = 0 \quad \text{in } Q = \Omega \times (-t_1, t_2),$$

$$(1.2) \quad u(x, t) = \varphi(x, t) \quad \text{on the parabolic boundary } \gamma = (\partial\Omega \times [-t_1, t_2]) \cup (\Omega \times \{t = -t_1\}),$$

where Ω is a bounded C^∞ smooth domain in \mathbf{R}^n , $t_1, t_2 = \text{const} > 0$ and the summation convention under repeated indices is understood. A model example of (1.1) is the equation

$$(1.3) \quad L_1 u = u_t - t^k u_{xx} + a_1 t^m u_x = 0 \quad \text{in } Q_1 = (-l, l) \times (-t_1, t_2), \\ u = \varphi_1(x, t) \quad \text{on the parabolic boundary of } Q_1,$$

where $a_1 = \text{const} \neq 0$,

$$(1.4) \quad k, m \in \mathbf{N}, \quad k \text{ is even and } k < 2m + 1.$$

In the case that $k > 2m + 1$, from the local solvability results in [12], [6], it follows that equation (1.3) is not locally solvable for special right-hand sides (or equivalently for some boundary data) even in a distributional sense in \mathcal{D}' .

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Further on we suppose that

$$(1.5) \quad a^{ij}, b^i, \varphi \in C^\infty(\overline{Q}), \quad a^{ij} = a^{ji};$$

$$(1.6) \quad a(t)|\xi|^2 \leq a^{ij}(x, t)\xi_i\xi_j \leq \Lambda a(t)|\xi|^2$$

for every $(x, t) \in Q' \ni \overline{Q}$, $\xi \in \mathbf{R}^n$, with a nonnegative function $a(t) \in C[-t_1, t_2]$, $a(t) > 0$ for $t \neq 0$, $a(0) = 0$, $\Lambda = \text{const} \geq 1$;

$$(1.7) \quad a(t) \text{ has a finite order zero,}$$

i.e. $a(t) \geq a_0|t|^k$ for some positive constant $a_0 > 0$ and some $k \in \mathbf{N}$.

Our main assumption coming from the local solvability results in \mathcal{D}' of equation (1.3) in [7] describes the admissible rate of the degeneracy of the diffusion, i.e. the principal term $a^{ij}(x, t)\xi_i\xi_j$ in comparison with the reaction, i.e. the lower order term $b^i(x, t)\xi_i$:

$$(1.8) \quad |b^i(x, t)| \leq b(t)$$

for every $(x, t) \in \overline{Q}$, $b(t) \in C[-t_1, t_2]$ and $b(t) \leq a^\sigma(t)$ for every $t \in [-t_1, t_2]$ and some $\sigma > \frac{1}{2} - \frac{1}{2k}$. Note that for the model equation (1.3), condition (1.8) follows for $a(t) = t^k$, $b(t) = |a_1|t^m$ and $\sigma = \frac{m}{k} > \frac{1}{2} - \frac{1}{2k}$ from (1.4).

Under the above conditions we will prove existence and uniqueness of a classical $C^\infty(\overline{Q} \setminus (\partial\Omega \times \{t = 0\})) \cap C^{\alpha, \alpha/2}(\overline{Q})$ solution of (1.1), (1.2) for some $\alpha \in (0, 1)$. Example 4.1 in section 4 illustrates for the model equation (1.3) that this result is sharp by constructing an explicit solution of the boundary value problem.

Linear degenerate elliptic and parabolic equations appear in simple models of real diffusion, filtration processes, and Brownian motion (see, for example, [1], [13], [8], [9]). They were investigated with different methods, for example, with energy methods by Kohn and Nirenberg [8] or with regularization of the equation and the Bernstein technique of a priori estimates by Oleinik and Radkevich [13]. According to the pioneering papers of Fichera [3], [4], some part of the boundary should be free from boundary data if the equation is a degenerate one. More precisely, for general linear second order degenerate elliptic equations

$$(1.9) \quad \sum_{i,j=1}^n a^{ij}u_{x_i x_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u = f(x) \quad \text{in } \Omega,$$

$$\sum a^{ij}(x)\xi^i\xi^j \geq 0 \quad \text{for every } \xi \in \mathbf{R}^n,$$

Fichera introduced the function

$$\beta(x) = \sum_{i=1}^n (b^i(x) - \sum_{j=1}^n a_{x_j}^{ij}(x))\nu^i,$$

where $\nu(x)$ is the outer unit normal to $\partial\Omega$. Then the Dirichlet data should be prescribed on the parts of the boundary $\partial\Omega$ where $\sum a^{ij}(x)\nu^i\nu^j > 0$ or $\sum a^{ij}(x)\nu^i\nu^j = 0$ but $\beta > 0$. The sign of the Fichera function $\beta(x)$ is invariant under every smooth nondegenerate change of the independent variables. What does happen with the solution when the data are prescribed on the rest of the boundary, where the Dirichlet problem is characteristic? The best explanation of the phenomena which appear is given from the theory of viscosity solutions developed by M. Crandall and P.-L. Lions (for more details see [2], [5]). If, according to Fichera, the characteristic part of the boundary is with nonzero $(n-1)$ -dimensional measure, then this part of the

topological boundary de facto is an “interior domain” for equation (1.9). More precisely, from the definition of a weak viscosity Dirichlet problem (see Definition 7.4 in [2]) the solution detaches from the data. For example, the same behavior of the solution can be observed by the solutions of the heat equation on the upper base of the cylinder. Note that this part of the boundary for the heat equation is a characteristic one according to the theory of Fichera.

When the characteristic set on the boundary has zero measure in \mathbf{R}^n , then it is not clear what happens with the solution. It is not realistic to expect that the solution detaches from the smooth boundary data exactly at one characteristic point on the boundary. Our conjecture is that, in general, the normal to the boundary derivative of the solution blows up at this point. There is a simple example of Krylov (see Remark 4.1 in [9]) which confirms our conjecture.

Example 1.1.

$$(1.10) \quad \begin{aligned} u_t - \Delta u = f(x, t) \quad \text{in } B = \left\{ \sum_{i=1}^n x_i^2 + t^2 < 1 \right\}, \\ u = 0 \quad \text{on } \partial B. \end{aligned}$$

The north pole $(0, 1)$ of the unit ball B is a characteristic point for the Dirichlet problem (1.10) and should be free from boundary data. When $f \in C^2(\overline{B})$ and $n \geq 3$ the classical solution of (1.10) has bounded second derivatives in \overline{B} . However, for $n = 1$ and $f(0, 1) \neq 0$ the time derivative $|u_t(x, t)|$ blows up for $(x, t) \rightarrow (0, 1)$.

Let us point out that equation (1.3) for $a_1 = 0$ was considered by Helffer [7] and Matsuzawa [11]. They proved C^∞ hypoellipticity for L_1 and therefore local solvability for the adjoint operator L_1^* . When $a_1 \neq 0$, it was shown in [6] and [12] that for $k > 2m + 1$ equation (1.3) is locally nonsolvable in the set of the ultradistributions, while for $k \leq 2m + 1$ the operator L_1^* is locally solvable and L_1 is C^∞ hypoelliptic.

In this paper we connect the local solvability results with the global weak and classical solvability of (1.1), (1.2). More precisely, we show that the solution is C^∞ smooth except for the characteristic points $\partial\Omega \times \{t = 0\}$, and higher regularity results at these points depend on some additional conditions on the boundary data and the rate of the degeneracy of the diffusion and the reaction terms in equation (1.1).

The paper is organized in the following way. In section 2 the basic definitions, notation and the main results are given, while section 3 deals with the proof of the main results. In section 4 the solution of the model equation (1.3) is explicitly found.

2. BASIC DEFINITIONS AND MAIN RESULTS

In order to formulate the results let us recall some definitions of Banach spaces in [10]. $W_2^{1,0}(Q)$ is the Hilbert space with a scalar product

$$(u, v)_{W_2^{1,0}(Q)} = \int_Q (uv + u_{x_k} v_{x_k}) dx dt;$$

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$$(u, v)_{W_2^{1,1}(Q)} = \int_Q (uv + u_{x_k} v_{x_k} + u_t v_t) dx dt;$$

$L_{q,r}(\overline{Q})$, $1 \leq q \leq \infty$, $1 \leq r \leq \infty$, in the Banach space of all measurable functions in \overline{Q} with a finite norm

$$\|u\|_{L_{q,r}(\overline{Q})} = \left(\int_{-t_1}^{t_2} \left(\int_{\Omega} |u(x,t)|^q dx \right)^{r/q} dt \right)^{1/r};$$

$V_2(Q)$ is the Banach space of all elements of $W_2^{1,0}(Q)$ having a finite norm

$$\|u\|_{V_2(Q)} = \operatorname{ess\,sup}_{-t_1 \leq t \leq t_2} \|u(\cdot, t)\|_{L_2(\Omega)} + \|\nabla_x u\|_{L_2(Q)};$$

$V_2^{1,0}(Q)$ is the Banach space of all elements of $V_2(Q)$ that are continuous in t in the norm of $L_2(\Omega)$, i.e.

$$\|u(\cdot, t+h) - u(\cdot, t)\|_{L_2(\Omega)} \rightarrow 0 \quad \text{when } h \rightarrow 0,$$

with a norm

$$\|u\|_{V_2^{1,0}(Q)} = \sup_{-t_1 \leq t \leq t_2} \|u(\cdot, t)\|_{L_2(\Omega)} + \|\nabla_x u\|_{L_2(Q)};$$

and $C^{\alpha, \alpha/2}(\overline{Q})$, $\alpha \in (0, 1)$, is the Banach space of all continuous functions which are Hölder continuous with exponent α on x and Hölder continuous with exponent $\frac{\alpha}{2}$ with respect to t .

A zero over the spaces means the closure of C_0^∞ functions which are zero on the parabolic boundary in the corresponding norms.

We will use the standard definition of generalized solutions (see, for example, sec. 3, ch. 3, in [10]).

Definition 2.1. A function $u(x, t) \in V_2(Q)$ is a generalized solution of (1.1), (1.2) if $u(x, t) - \varphi(x, t) \in \mathring{V}_2(Q)$, and the identity

$$(2.1) \quad \begin{aligned} & \int_{\Omega} (u(x, \tau) - \varphi(x, \tau)) \eta(x, \tau) dx - \int_{Q_\tau} (u - \varphi) \eta_t dx dt \\ & + \int_{Q_\tau} (a^{ij} (u - \varphi)_{x_i} \eta_{x_j} + b^i (u - \varphi)_{x_i} \eta) dx dt = - \int_{Q_\tau} L\varphi \cdot \eta dx dt \end{aligned}$$

holds for every $-t_1 \leq \tau \leq t_2$, $Q_\tau = Q \cap \{t \leq \tau\}$ and for every $\eta \in \mathring{W}_2^{1,1}(Q)$.

Let us also recall the compatibility condition of the data on $\partial\Omega \times \{t = -t_1\}$ (see (5.6)-(5.8) in [10]). These conditions consist in the fact that the derivatives $\frac{\partial^p u}{\partial t^p}$ for $(x, t) \in \partial\Omega \times \{t = -t_1\}$, which can be determined for $t = -t_1$ by means of equation (1.1) and the initial conditions, must satisfy the boundary condition (1.2).

Theorem 2.2. *Suppose (1.5)-(1.8) hold and the compatibility conditions of the data on $\partial\Omega \times \{t = -t_1\}$ up to infinite order are satisfied. Then problem (1.1), (1.2) has a unique classical solution $u(x, t) \in C^\infty(\overline{Q} \setminus \{\partial\Omega \times \{t = 0\}\}) \cap C^{\alpha, \alpha/2}(\overline{Q})$ for some $\alpha \in (0, 1)$.*

Remark 2.3. The result in Theorem 2.2 holds for arbitrary real positive constants $k \notin \mathbf{N}$ if the condition for σ in (1.8) is replaced with the inequality $\sigma > \frac{1}{2} - \frac{1}{2([k]+1)}$, where $[k] \in \mathbf{N}$, $[k] \leq k$.

Remark 2.4. The increasing regularity of the solutions at the characteristic points $\partial\Omega \times \{t = 0\}$ can be proved for more general equations in nondivergence form,

$$(2.2) \quad u_t - a^{ij}(x, t)u_{x_i x_j} + b^i(x, t)u_{x_i} = 0 \quad \text{in } Q, \quad u = \varphi \quad \text{on } \gamma,$$

under additional assumptions such as $\varphi(x, t)/a(t) \in C^\infty(\partial\Omega \times [-t_1, t_2])$, $b^i(x, t)/a(t) \in C^\infty(\overline{Q})$. In this case, from the proof of Theorem 2.2, it follows that $u(x, t) \in C^\infty(\overline{Q})$.

3. PROOF OF THE MAIN RESULTS

The first step in the proof of Theorem 2.2 is to regularize equation (1.1); i.e. for every sufficiently small positive constant $0 < \epsilon \leq \epsilon_0$ we consider the problem

$$(3.1) \quad \begin{aligned} L^\epsilon v^\epsilon &= v_t^\epsilon - ((a^{ij}(x, t) + \epsilon \delta_i^j) v_{x_i x_j}^\epsilon)_{x_j} + b^i(x, t) v_{x_i}^\epsilon = f^\epsilon(x, t) \quad \text{in } Q, \\ v^\epsilon &= 0 \quad \text{on } \gamma, \end{aligned}$$

where $\delta_i^j = 1$ for $i = j$, $\delta_i^j = 0$ for $i \neq j$, and $f^\epsilon(x, t) = -L^\epsilon \varphi$. Let

$$(3.2) \quad \begin{aligned} a^\epsilon &\in C^\infty[-t_1, t_2], \quad a(t) < a^\epsilon(t) \leq a(t) + \epsilon \quad \text{for every } 0 < \epsilon \leq \epsilon_0 \quad \text{and} \\ \|a^\epsilon - a\|_{C[-t_1, t_2]} &\rightarrow 0 \quad \text{when } \epsilon \rightarrow 0. \end{aligned}$$

We change the variable t :

$$(3.3) \quad s = h^\epsilon(t) = \int_0^t a^\epsilon(\tau) d\tau, \quad h^\epsilon \in C^\infty[-t_1, t_2].$$

Since $h^\epsilon(t)$ is a strictly monotone function $h^\epsilon : [-t_1, t_2] \rightarrow [-T_1, T_2]$, the inverse function $(h^\epsilon)^{-1}(s)$ is well defined in $[-T_1, T_2]$. For convenience, further on we will omit the index ϵ .

Let us consider the problem

$$(3.4) \quad Pv = v_s - (A^{ij}(x, s)v_{x_i x_j})_{x_j} + B^i(x, s)v_{x_i} = F(x, s) \quad \text{in } G = \Omega \times [-T_1, T_2],$$

$$(3.5) \quad v(x, s) = 0 \quad \text{on } \Gamma = (\partial\Omega \times [-T_1, T_2]) \cup \{\Omega \times \{s = -T_1\}\},$$

where

$$(3.6) \quad \begin{aligned} A^{ij}(x, s) &= (a^{ij}(x, h^{-1}(s)) + \epsilon \delta_i^j)/A(s), \quad B^i(x, s) = b^i(x, h^{-1}(s))/A(s), \\ A(s) &= a^\epsilon(h^{-1}(s)), \quad B(s) = b(h^{-1}(s)), \quad \phi(x, s) = \varphi(x, h^{-1}(s)), \\ F(x, s) &= f(x, h^{-1}(s))/A(s) = -P\phi(x, s). \end{aligned}$$

From (1.5)-(1.8), (3.2), (3.6), easy calculations give us

$$(3.7) \quad A^{ij} \in C^\infty(\overline{G}), \quad A^{ij} = A^{ji};$$

$$(3.8) \quad |\xi|^2 \leq A^{ij}(x, s)\xi_i \xi_j \leq 2\Lambda|\xi|^2 \quad \text{for every } (x, s) \in \overline{G}, \quad \xi \in \mathbf{R}^n \quad \text{and } 0 < \epsilon \leq \epsilon_0;$$

$$(3.9) \quad |B^i(x, s)| \leq B(s)/A(s) \quad \text{for every } (x, s) \in \overline{G}.$$

Our second step is to apply to problems (3.4), (3.5) Theorems 4.1 and 4.2 in chapter 3 in [10] in order to prove existence of a generalized solution $v \in \mathring{V}_2^{1,0}(G)$. Then from Theorem 7.1 in chapter 3 in [10] we show that every solution $v \in \mathring{V}_2^{1,0}(G)$ satisfies the estimate

$$(3.10) \quad \operatorname{ess\,sup}_{\overline{G}} |v(x, t)| \leq C_1$$

with a constant C_1 independent of ϵ and finally from Theorem 10.1 in chapter 3 in [10] that

$$(3.11) \quad \|v\|_{C^{\alpha, \alpha/2}(\overline{G})} \leq C_2$$

for some $\alpha \in (0, 1)$, with constants α, C_2 independent of ϵ .

For this purpose it is enough to check that conditions (1.2), (7.1) and (7.2) in chapter 3 in [10] are satisfied, because (1.3)-(1.6) follow from (7.1), (7.2). Assumption (1.2) is exactly (3.8) with $\nu = 1$, $\mu_1 = 2\Lambda$. As for (7.1), (7.2), for $q = \infty$, $1 < r < \min(1 + 1/k, 1 + (2\sigma - 1 + 1/k)/(1 + 1/k))$, $\kappa_1 = (r - 1)/r$ and $\mu_1 = (C_3 + C_4)^{1/r}$ (C_3 and C_4 are defined below), we have from (1.8), (3.2), (3.9) the estimates

$$\begin{aligned} \|(B^i)^2\|_{L^\infty, r(G)}^r &= \int_{-T_1}^{T_2} \sup_{\overline{\Omega}} |B^i(x, s)|^{2r} ds \leq \int_{-t_1}^{t_2} b^{2r}(t) a^{1-2r}(t) dt \\ &\leq \int_{-t_1}^{t_2} a^{2r(\sigma-1)+1}(t) dt \leq C_3 \end{aligned}$$

and

$$\begin{aligned} C_3 &= (t_1 + t_2) \left(\max_{[-t_1, t_2]} a(t) \right)^{2r(\sigma-1)+1} < \infty \quad \text{when } 2r(\sigma - 1) + 1 \geq 0, \\ C_3 &= a_0^{2r(\sigma-1)+1} \int_{-t_1}^{t_2} |t|^{2rk(\sigma-1)+k} dt < \infty \quad \text{when } 2r(\sigma - 1) + 1 < 0 \end{aligned}$$

because $2rk(\sigma - 1) + k > -1$;

$$\begin{aligned} \|F\|_{L^\infty, r(G)}^r &= \int_{-T_1}^{T_2} \sup_{\overline{\Omega}} |F(x, s)|^r ds = \int_{-t_1}^{t_2} \sup_{\overline{\Omega}} |L^\epsilon \varphi(x, t)|^r (a^\epsilon(t))^{1-r} dt \\ &\leq \{2n[1 + \|a^{ij}\|_{L^\infty(\overline{Q})} + \|\nabla_x a^{ij}\|_{L^\infty(\overline{Q})} + \|b^i\|_{L^\infty(\overline{Q})}]\|\varphi\|_{C^{2,1}(\overline{Q})}\}^r \\ &\quad \cdot a_0^{1-r} \int_{-t_1}^{t_2} t^{(1-r)k} dt = C_4 < \infty. \end{aligned}$$

Here the constants C_3, C_4 are independent of ϵ . Let us check that $u^\epsilon(x, t) = \varphi(x, t) + v^\epsilon(x, h^\epsilon(t)) \in V_2^{1,0}(Q) \cap C^{\alpha, \alpha/2}(\overline{Q})$ is a generalized solution of (1.1), (1.2). If $\eta(x, t) \in \mathring{W}_2^{1,1}(Q)$, then $\zeta(x, s) = \eta(x, (h^\epsilon)^{-1}(s)) \in \mathring{W}_2^{1,1}(G)$, and hence for every $-T_1 < R \leq T_2$ we have from Definition 2.1 the identity

$$\begin{aligned} &\int_{\Omega} v^\epsilon(x, R)\zeta(x, R) dx - \int_{G_R} v^\epsilon(x, s)\zeta_s(x, s) dx ds \\ &\quad + \int_{G_R} (A^{ij}(x, s)v_{x_i}^\epsilon \zeta_{x_j} + B^i(x, s)v_{x_i}^\epsilon \zeta) dx ds = \int_{G_R} F^\epsilon(x, s)\zeta(x, s) dx ds, \end{aligned}$$

$$G_R = G \cap \{s \leq R\}.$$

After the change of variable (3.3) we obtain for every $-t_1 < r \leq t_2$, $r = (h^\epsilon)^{-1}(R)$ the equality

$$(3.12) \quad \begin{aligned} & \int_{\Omega} (u^\epsilon(x, r) - \varphi(x, r)) \eta(x, r) dx - \int_{Q_r} (u^\epsilon - \varphi) \eta_t dx dt \\ & + \int_{Q_r} [(a^{ij}(x, t) + \epsilon \delta_i^j)(u^\epsilon - \varphi)_{x_i} \eta_{x_j} + b^i(x, t)(u^\epsilon - \varphi)_{x_i} \eta] dx dt \\ & = - \int_{Q_r} L^\epsilon \varphi \cdot \eta dx dt. \end{aligned}$$

Moreover, easy calculations give us for every fixed $\epsilon > 0$ the estimates

$$\|u^\epsilon\|_{V_2^{1,0}(Q)} < \infty, \quad \lim_{\tau \rightarrow 0} \|u^\epsilon(\cdot, t + \tau) - u^\epsilon(\cdot, t)\|_{L_2(\Omega)} = 0.$$

Since $\|h^\epsilon\|_{C^1[-t_1, t_2]} \leq (t_1 + t_2)(\max_{[-t_1, t_2]} a(t) + 1) = C_5$, C_5 is independent of ϵ , we have from (3.11) the inequality

$$(3.13) \quad \|u^\epsilon\|_{C^{\alpha, \alpha/2}(\overline{Q})} \leq \|\varphi\|_{C^{\alpha, \alpha/2}(\overline{Q})} + C_2 C_5 = C_6,$$

where C_6 is independent of ϵ .

Let $\eta \in C_0^\infty(Q)$ and $r = t_2$ in (3.12) so that after integration by parts the identity

$$(3.14) \quad \int_Q u^\epsilon \eta_t dx dt - \int_Q u^\epsilon [((a^{ij}(x, t) + \epsilon \delta_i^j) \eta_{x_j})_{x_i} + (b^i(x, t) \eta)_{x_i}] dx dt = 0$$

holds.

From (3.13), the Arzelà-Ascoli Theorem and the Cantor diagonalization process, there exists a subsequence $\epsilon_N \rightarrow 0$ such that $u^{\epsilon_N}(x, t) \rightarrow u(x, t) \in C^{\alpha, \alpha/2}(\overline{Q})$ when $\epsilon_N \rightarrow 0$. After the limit $\epsilon_N \rightarrow 0$ in (3.14) we obtain that $u(x, t)$ satisfies the same equality with $\epsilon = 0$; i.e., $u(x, t)$ is a generalized solution of (1.1) in $\mathcal{D}'(Q)$.

The third step in the proof is to apply the hypoellipticity results in [7], [11], [13].

Theorem 1.1 (Helffer [7]). *Suppose that the operator*

$$(3.15) \quad L_2 = \frac{\partial}{\partial t} - A_{2m} \left(x, t, \frac{1}{i} \frac{\partial}{\partial x_k} \right) + \sum_{j=0}^{2m-1} A_j \left(x, t, \frac{1}{i} \frac{\partial}{\partial x_k} \right) \quad \text{in } Q$$

with $C^\infty(\overline{Q})$ coefficients $A_j(x, t, \xi)$ which are homogeneous polynomials of order j with respect to $\xi \in \mathbf{R}^n$ satisfies the following assumptions:

(H1)

$$\Re \left(\int_{-t'}^t A_{2m}(x, s, \xi) ds \right) \geq c |t - t'|^{k+1} |\xi|^{2m}$$

for every $(x, t, t', \xi) \in \Omega \times (-t_1, t_2)^2 \times \mathbf{R}^n$, $t > t'$;

(H2) there exist constants θ, τ such that for every $0 \leq j \leq 2m$ and for every $(\alpha, \beta) \in \mathbf{N}^n \times \mathbf{N}^n$ satisfying

$$|\alpha| + |\beta| + j > 0, \quad (|\alpha| + j)\theta + (|\beta| + j)\tau \leq 1$$

the estimates

$$\begin{aligned} & \left| \frac{\partial^{\alpha+\beta}}{\partial x^\beta \partial \xi^\alpha} A_{2m-j}(x, t, \xi) \right| \\ & \leq C_{\alpha\beta j} |\Re A_{2m}|^{1-(|\alpha|+j)\theta - (|\beta|+j)\tau} |\xi|^{2m[(|\alpha|+j)\theta + (|\beta|+j)\tau] - |\alpha| - j} \end{aligned}$$

hold for every $(x, t, \xi) \in Q \times \mathbf{R}^n$ and for some constants $C_{\alpha\beta j}$;

(H3) θ and τ satisfy the inequality

$$2m(\tau + \theta) \frac{k}{k+1} < 1.$$

If $u \in \mathcal{D}'(Q)$ is a generalized solution of $L_2 u = 0$, then $u \in C^\infty(Q)$.

Assumptions (H1)-(H3) in Theorem 1.1 in [7] hold for operator L with $m = 1$, $k \in \mathbf{N}$, $A_2 = a^{ij}(x, t)\xi_i\xi_j$, $A_1 = \sum_{j=1}^n (\sum_{h=1}^n a_{x_h}^{hj}(x, t) + b^j(x, t)) \xi_j$ and $A_0 = 0$. We have in fact

$$\begin{aligned} \Re \left(\int_{t'}^t A_2(x, s, \xi) ds \right) &= \int_{t'}^t a^{ij}(x, s)\xi_i\xi_j ds \\ &\geq a_0|\xi|^2 \int_{t'}^t |s|^k ds \geq \frac{a_0}{k+1} |\xi|^2 |t - t'|^{k+1}. \end{aligned}$$

As for conditions (H2), (H3) we choose $\theta = 0$, $\tau = \frac{1}{2} + \frac{1}{2k} - \epsilon_1$ for some sufficiently small $\epsilon_1 > 0$. For all indices j, α, β satisfying the inequalities

$$(3.16) \quad 2k \left(\frac{1}{2} + \frac{1}{2k} - \epsilon_1 \right) / (k+1) = 1 - \frac{2\epsilon_1 k}{k+1} < 1, \quad |\alpha| + |\beta| + j > 0, \quad (|\beta| + j) \left(\frac{1}{2} + \frac{1}{2k} - \epsilon_1 \right) \leq 1,$$

one has to check the estimates in (H2). Condition (3.16) is satisfied only by the indices

$$(3.17) \quad \begin{aligned} &\text{i) } j = 1 \text{ and } \alpha = \beta = 0 \text{ or } |\alpha| = 1, \beta = 0, \\ &\text{ii) } j = 0, \beta = 0 \text{ and } |\alpha| = 1 \text{ or } |\alpha| = 2, \\ &\text{iii) } j = 0, |\beta| = 1 \text{ and } \alpha = 0, |\alpha| = 1 \text{ or } |\alpha| = 2. \end{aligned}$$

In case (3.17)_i the estimate in (H2) is

$$(3.18) \quad \begin{aligned} &\left| \frac{\partial^{\alpha+\beta}}{\partial x^\beta \partial \xi^\alpha} \sum_j \left(\sum_h a_{x_h}^{hj} + b^j \right) \xi_j \right| \\ &\leq K_1 |\xi|^2 [(|\beta|+1) \left(\frac{1}{2} + \frac{1}{2k} - \epsilon_1 \right) - |\alpha| - 1] |a^{ij}(x, t)\xi_i\xi_j|^{1 - (|\beta|+1) \left(\frac{1}{2} + \frac{1}{2k} - \epsilon_1 \right)}. \end{aligned}$$

Since from (1.6) and (1.8) we have the inequalities

$$(3.19) \quad \begin{aligned} &\left| \sum_{j,h} a_{x_h}^{hj}(x, t)\xi_j \right| \leq K_0 \left(\sum_{i,j} a^{ij}\xi_i\xi_j \right)^{1/2} \leq K_0 \Lambda |\xi| a^{1/2}(t) \leq K_0 \Lambda |\xi| (a(t))^{\frac{1}{2} - \frac{1}{2k} + \epsilon_1}, \\ &\left| \sum_j b^j(x, t)\xi_j \right| \leq n |\xi| a^\gamma(t) \leq n |\xi| (a(t))^{\frac{1}{2} - \frac{1}{2k} + \epsilon_1}, \end{aligned}$$

for $0 < \epsilon_1 \leq \min\left(\frac{1}{2k}, \gamma - \frac{1}{2} + \frac{1}{2k}\right)$, then (3.18) holds in case (3.17)_i with $K_1 = K_0 \Lambda + n$.

In the case that $j = 0$ one has to prove the estimates

$$(3.20) \quad \left| \frac{\partial^{\alpha+\beta}}{\partial x^\beta \partial \xi^\alpha} a^{ij}(x, t)\xi_i\xi_j \right| \leq K_2 |\xi|^{2|\beta| \left(\frac{1}{2} + \frac{1}{2k} - \epsilon_1 \right)} |a^{ij}(x, t)\xi_i\xi_j|^{1 - |\beta| \left(\frac{1}{2} + \frac{1}{2k} - \epsilon_1 \right)}.$$

Since we have $|a^{hj}(x, t)\xi_j| \leq \Lambda |\xi| a(t)$, $|a^{hj}(x, t)| \leq n^2 \Lambda a(t)$, $|a_{x_i}^{hj}(x, t)\xi_h\xi_j| \leq K_0 \Lambda |\xi|^2 (a(t))^{\frac{1}{2}} \leq K_0 \Lambda |\xi|^2 (a(t))^{\frac{1}{2} - \frac{1}{2k} + \epsilon_1}$, using (3.19) we obtain (3.20) in the cases (3.17)_{ii} and (3.17)_{iii} with $K_2 = n^2 \Lambda + K_0 \Lambda$.

According to Theorem 1.1 the generalized solution $u(x, t)$ of (1.1) belongs to the class $C^\infty(Q) \cap C^{\alpha, \alpha/2}(\overline{Q})$. From the standard local regularity theory for uniformly parabolic equations (see [10]) it follows that

$$u(x, t) \in C^\infty(\overline{Q} \setminus (\partial\Omega \times \{t = 0\})) \cap C^{\alpha, \alpha/2}(\overline{Q}), \quad \alpha \in (0, 1).$$

Uniqueness. Suppose $u, v \in C^\infty(Q) \cap C^{\alpha, \alpha/2}(\overline{Q})$ are two classical solutions of (1.1), (1.2). Then $w = u - v$ is a classical solution of (1.1) with zero boundary data. If $w \not\equiv 0$, without loss of generality, let us suppose that w attains its positive maximum at some point $(x_0, t_0) \in \overline{Q} \setminus \gamma$. Then the function $z(x, t) = e^{-ct}w(x, t)$, for $c = \text{const} > 0$, satisfies the equation

$$(3.21) \quad \begin{aligned} z_t + cz - (a^{ij}(x, t)z_{x_j})_{x_i} + b^i(x, t)z_{x_i} &= 0 \quad \text{in } Q, \\ z &= 0 \quad \text{on } \gamma, \end{aligned}$$

and attains its positive maximum at some point $(x_1, t_1) \in \overline{Q} \setminus \gamma$. Since $z(x_1, t_1) > 0$, $z_t(x_1, t_1) \geq 0$, $\nabla_x z(x_1, t_1) = 0$ and $a^{ij}(x_1, t_1)z_{x_i x_j}(x_1, t_1) \leq 0$ (see [14]), this contradicts equations (3.21).

4. EXPLICIT SOLUTION FOR THE MODEL EQUATION

In this section we will give an explicit formula for the solutions of the model equation (1.3) for $k = m = p$.

Example 4.1.

$$(4.1) \quad \begin{aligned} u_t - t^p u_{xx} + at^p u_x &= 0 \quad \text{in } Q = (-1, 1) \times (-1, 1), \\ u(-1, t) &= A(t), \quad u(1, t) = B(t), \quad -1 \leq t \leq 1, \\ u(x, -1) &= g(x), \quad -1 \leq x \leq 1, \\ A(-1) &= g(-1), \quad B(-1) = g(1). \end{aligned}$$

By means of the Green function, the solution of (4.1) can be explicitly written as

$$\begin{aligned} u(x, t) &= \sqrt{\frac{p+1}{4\pi}} \int_0^2 \frac{g(y-1) \exp(-a(y-1)/2)}{(t^{p+1}+1)^{1/2}} \\ &\cdot \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(p+1)(x+1-y+4n)^2}{4(t^{p+1}+1)}\right) - \exp\left(-\frac{(p+1)(x+1+y+4n)^2}{4(t^{p+1}+1)}\right) \right] dy \\ &+ \frac{(p+1)^{3/2} \exp(a/2)}{2\sqrt{\pi}} \int_0^{(t^{p+1}+1)/(p+1)} \frac{A((p+1)s-1)^{1/(p+1)} \exp(a^2 s/4)}{(t^{p+1}+1-s(p+1))^{3/2}} \\ &\cdot \sum_{n=-\infty}^{\infty} (x+1+4n) \exp\left(-\frac{(p+1)(x+1+4n)}{4(t^{p+1}+1-s(p+1))}\right) ds \\ &+ \frac{(p+1)^{3/2} \exp(-a/2)}{4\sqrt{\pi}} \int_0^{(t^{p+1}+1)/(p+1)} \frac{B((p+1)s-1)^{1/(p+1)} \exp(a^2 s/4)}{(t^{p+1}+1-s(p+1))^{3/2}} \\ &\cdot \sum_{n=-\infty}^{\infty} \left[(x-1+4n) \exp\left(-\frac{(p+1)(x-1+4n)^2}{4(t^{p+1}+1-s(p+1))}\right) \right. \\ &\left. + (x+3+4n) \exp\left(-\frac{(p+1)(x+3+4n)^2}{4(t^{p+1}+1-s(p+1))}\right) \right] ds. \end{aligned}$$

Simple computations show that $u(x, t) = u_1(x, t) + u_2(x, t) + u_3(x, t)$, where $u_1 \in C^\infty(\overline{Q})$, $u_2 \in C^\infty(\overline{Q} \setminus \{(-1, 0)\}) \cap C^{2/(p+1)}(\overline{Q})$, $u_3 \in C^\infty(\overline{Q} \setminus \{(1, 0)\}) \cap C^{2/(p+1)}(\overline{Q})$, assuming that the compatibility conditions up to infinite order are satisfied at the points $(-1, -1)$ and $(1, -1)$. Here

$$u_2 = \frac{(p+1)^{3/2} \exp(a^2/2)}{2\sqrt{\pi}} \int_0^{\frac{t^{p+1}+1}{p+1}} \frac{A((p+1)s-1)^{1/(p+1)} \exp(a^2 s/4)}{(t^{p+1}+1-s(p+1))^{3/2}} (x+1) \cdot \exp\left(-\frac{(p+1)(x+1)^2}{4(t^{p+1}+1-s(p+1))}\right) ds$$

and u_3 is defined in a similar way.

Finally, let us mention some open problems. From Example 1.1 and some other counterexamples of Krylov, it is known that the dimension of the space and the curvature of the boundary at the characteristic points play an important role for the rate of the singularities at these points. What is the precise dependence and the sharp singularities of the solution?

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