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STOÏLOW FACTORIZATION FOR QUASIREGULAR MAPPINGS IN ALL DIMENSIONS

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ABSTRACT. We generalize to higher dimensions the classical Stoïlow factorisation theorem; we show that any quasiregular mapping f of the Riemann n-sphere $\hat{\mathbb{R}}^n \approx \mathbb{S}^n$ can be written in the form $f = \varphi \circ h$, where $h : \mathbb{S}^n \to \mathbb{S}^n$ is quasiconformal and φ is a uniformly quasiregular mapping, hence rational with respect to some bounded measurable conformal structure.

1. INTRODUCTION

The classical Stoïlow theorem in the complex plane states that a discrete open map $f : \Omega \to \mathbb{R}^2$, where $\Omega \subset \mathbb{R}^2$ is a domain, can be factorized in the following way: there is an analytic function φ and a homeomorphism h such that

$$f = \varphi \circ h$$

[S, p. 120]. Here *discrete* simply means that the preimage of a point is discrete in the domain Ω. Planar quasiregular mappings are discrete and open (a fact usually proved via Stoïlow's Theorem) and the factorization $f = \varphi \circ h$ shows that φ is analytic and h is quasiconformal; see [LV, p. 247]. The factorization theorem is used to parameterize solutions to Beltrami equations and has analogues for other first order PDEs [AIM].

In higher dimensions $n \geq 3$ it is not known if a discrete open map is even locally topologically equivalent to a quasiregular map, and the rigidity theory, Liouville's Theorem [Re], [IM], shows the analytic functions (defined as solutions to Cauchy–Riemann systems) to be the restrictions of Möbius transformations. Thus such a factorization theorem is not possible. However, we shall see here that if we slightly relax the analyticity condition in a natural way, such a factorization theorem is true, at least for quasiregular mappings. The basic facts concerning quasiregular mappings in space can be found in the monographs [R, IM].

A quasiregular map $\varphi : \mathbb{S}^n \to \mathbb{S}^n$ with a uniform bound on the distortion of all its iterates $\varphi \circ \varphi, \ldots, \varphi \circ \cdots \circ \varphi, \ldots$ is called uniformly quasiregular (uqr). Such maps are always rational with respect to some measurable Riemannian structure. This means that there is a bounded measurable $G : \mathbb{S}^n \to S(n)$, the space of $n \times n$ symmetric positive definite matrices of determinant 1, such that

$$D^t\varphi G(\varphi(x)) D\varphi = J(x,\varphi)^{2/n} G(x), \qquad \text{a.e. } \mathbb{S}^n. \tag{1.1}$$

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The space of $W^{1,n}(\mathbb{S}^n)$ solutions to this nonlinear PDE forms a semigroup analogous to the analytic functions and quasiconformally conjugate to the rational function in two dimensions. Because of Rickman's version of Montel's Theorem [R] there is a reasonably complete Fatou/Julia theory associated with the iteration of uqr mappings; see [HM, MMP]. There are also strong restrictions on the geometry and topology of closed manifolds admitting nontrivial uqr mappings [MMP, BHM]; for instance they cannot be negatively curved. The Fatou set $\mathcal{F}(\varphi)$ of a uqr-mapping φ is the open set where the iterates form a normal family (that is, have locally uniformly convergent subsequences). The Julia set $\mathcal{J}(\varphi)$ is the complement of the Fatou set $\mathcal{J} = \mathbb{S}^n \setminus \mathcal{F}$. If the degree of $\varphi \geq 2$, the only interesting case for us, then the Julia set is nonempty, closed and a completely invariant set.

2. Results

In this paper we prove the following variant of Stoïlow's theorem:

Theorem 2.1. *Suppose $f : \mathbb{S}^n \to \mathbb{S}^n$ is a nonconstant quasiregular mapping, $n \geq 2$. Then there exist a uniformly quasiregular mapping φ whose Julia set is a Cantor set and a quasiconformal mapping $h : \mathbb{S}^n \to \mathbb{S}^n$ such that $f = \varphi \circ h$.*

Our proof is based on a modification of the "trapping method", used in [M1] to produce uqr mappings with prescribed branch set and also in [P] to prove the existence of uqr maps on spherical space forms. Indeed we will show a little more in that the uqr mapping is structurally stable (or generic), there is a single attracting fixed point, there are no relations between critical points, and the Julia set is ambiently quasiconformally equivalent to the middle-thirds Cantor set (so is not wild). Before we prove this main result we discuss some consequences.

2.1. **Uniqueness.** Classically the factorization (for quasiregular maps of $\mathbb{S}^2$) is unique up to Möbius transformation. If $\varphi \circ f = \psi \circ g$, then there is a Möbius transformation Φ so that $\varphi \circ \Phi = \psi$. Clearly this statement cannot hold in higher dimensions if φ and ψ are merely assumed uqr. However if we fix the invariant conformal structure, then we can make uniqueness statements up to a finite dimensional Lie group.

Theorem 2.2. *Let G be bounded measurable conformal structure on $\mathbb{S}^n$. Then there is a closed finite dimensional Lie group Γ of quasiconformal homeomorphisms of $\mathbb{S}^n$ with the following property: If $g, h : \mathbb{S}^n \to \mathbb{S}^n$ are quasiconformal mappings such that*

$$\varphi \circ g = \psi \circ h : \mathbb{S}^n \to \mathbb{S}^n \tag{2.1}$$

and both φ and ψ are rational with respect to G, then there is $\gamma \in \Gamma$ such that

$$\varphi \circ \gamma = \psi : \mathbb{S}^n \to \mathbb{S}^n.$$

Proof. The space of homeomorphic solutions to the equation (1.1) is a closed Lie group Γ, [M2, M3]. Let us suppose that both φ and ψ preserve this structure. Let $\gamma = g \circ h^{-1}$. Rewrite (2.1) as $\varphi \circ \gamma = \psi$. Using the local injectivity of quasiregular mappings away from the closed measure 0 branch set and the fact that quasiregular mappings preserve sets of measure 0, we find that locally, away from the closed set of measure zero $\psi^{-1}(B_\varphi)$, where B_φ is the branch set of φ, we have

$$\gamma = \varphi^{-1} \circ \psi.$$

The right hand side above clearly solves (1.1) where defined. Thus γ is a quasiconformal homeomorphism which solves (1.1) away from a set of measure 0. Hence $\gamma \in \Gamma$ and the result follows. □

2.2. **Two dimensions.** We remark that in two dimensions the space of generic uqr mappings that our factorization produces can be described. Indeed in two dimensions any uqr mapping is quasiconformally conjugate to a rational (analytic) mapping.

2.3. **Smooth uqr mappings.** It will be clear from our construction that if in Theorem 2.1 the map f is smooth of class $C^k(\mathbb{S}^n)$, then the quasiconformal homeomorphism h, and consequently the uqr mapping φ, can be chosen to be smooth of the same class. Typically one does not expect branched (not locally injective) quasiregular mappings to be smooth; however there are examples of M. Bonk and J. Heinonen [BM] of a quasiregular map $f : \mathbb{S}^3 \to \mathbb{S}^3$ which is $C^{3-\epsilon}(\mathbb{S}^3)$ for every $\epsilon > 0$. R. Kaufman, J. Tyson and J. Wu extended the results of [BM] to higher dimensions, [KTW]. The following theorem (which was certainly known to Bonk and Heinonen) is a consequence.

Theorem 2.3 (Smooth uqr mappings of $\mathbb{S}^n$)**.**

- *For each $\epsilon > 0$, there is a $C^{3-\epsilon}(\mathbb{S}^3)$ uniformly quasiregular mapping φ whose Julia set is a Cantor set.*
- *For each $\epsilon > 0$, there is a $C^{2-\epsilon}(\mathbb{S}^4)$ uniformly quasiregular mapping φ whose Julia set is a Cantor set.*
- *For each $n \geq 5$, there is an $\epsilon = \epsilon(n) > 0$ and a $C^{1+\epsilon}(\mathbb{S}^n)$ uniformly quasiregular mapping φ whose Julia set is a Cantor set.*

3. Proof of Theorem 2.1

Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a nonconstant quasiregular map of degree $2 \leq d < \infty$. The result is trivial if $d = 1$ and since $\mathbb{S}^n$ is compact, discreteness implies the degree is finite. Choose a point $x_0 \in \mathbb{S}^n$ having the following properties:

(1) x_0, $f(x_0)$ and $f^{-1}\{x_0\}$ do not meet the branch set
$$B_f = \{x \in \mathbb{S}^n \mid f \text{ is not a local homeomorphism at } x\}.$$
(2) There is a small ball $U_0 = B(x_0, r)$ about x_0 such that $f^{-1}U_0$ has components $U_1, \ldots, U_d$ pairwise disjoint and such that $f : U_i \to U_0$ is injective.
(3) $f(U_0)$ is disjoint from $\bigcup_{i=0}^{d} U_i$.

This situation can always be arranged since almost every point of $\mathbb{S}^n$ has the above properties [R]. Let $\{x_1, \ldots, x_d\} = f^{-1}\{x_0\}$ and let a,$b > 0$ so small that $2b < a$ and

(1) $B(x_i, a) \subset U_i$, $i = 0, \ldots, d$,
(2) $B(f(x_0), a) \subset f(U_0)$,
(3) $B(x_0, b) \subset \bigcap_{i=1}^{d} f\,(B(x_i, a))$,
(4) $f\,(B(x_0, b)) \subset B\,(f(x_0), a)$.

Next we define a modification $\tilde{f}$ as follows. On $\mathbb{S}^n \backslash \bigcup_{i=0}^{d} B(x_i, a)$ we set $\tilde{f} = f$. For $1 \leq i \leq d$, we set $\tilde{f}|_{B(x_i,b)}$ to be a translation onto $B(x_0, b)$ and $\tilde{f}|_{B(x_0,b)}$: $B(x_0, b) \to B(f(x_0), b)$ a translation. For the annular regions $B(x_i, a) \backslash B(x_i, b)$ there exist quasiconformal extensions for each $i = 0, 1, \ldots, d$ by application of Sullivan's Annulus Theorem [TV, Thm. 3.17]. In the smooth setting these annular regions will be flat and the smooth version of the annulus theorem may be applied; the topological complications in dimension 4 will not arise with a careful (indeed generic) choice of parameters. Thus $\tilde{f}|_{\partial B(x_i,a)} = f$ and $\tilde{f}|_{\partial B(x_i,b)}$ is a translation. The conditions above imply that the map $\tilde{f} : \mathbb{S}^n \to \mathbb{S}^n$ is well defined and quasiregular. Next, denote by Φ a conformal mapping that exchanges $B(x_0, b)$ with its complement and set

$$\tilde{g} = \Phi \circ \tilde{f} : \mathbb{S}^n \to \mathbb{S}^n.$$

It is shown in [M1] that the mapping $\tilde{g}$ as well as all its iterates are uniformly quasiregular. The set $B(x_0, b)$ is a conformal trap, where all the points, whose neighbourhood is distorted, land only after the next iterate under g. In particular this happens to all the points in the branch set. Note that also $B_{\tilde{g}} = B_f$. The Julia set of the mapping g is a Cantor set in $\bigcup_{i=1}^{d} B(x_i, b)$ quasiconformally equivalent to the middle-thirds Cantor set.

To gain the Stoïlow factorization we simply write

$$g = \tilde{f} \circ \Phi.$$

First note that we can write $\tilde{f} = f \circ \tilde{h}$, where $\tilde{h}$ is a quasiconformal mapping so that

(1) $\tilde{h}|_{B(x_i,a)} = f|^{-1}_{\tilde{f}B(x_i,a)} \circ \tilde{f}|_{B(x_i,a)}$ for every $i = 0, \ldots, d$,

(2) $\tilde{h}|_{\mathbb{S}^n \setminus (\bigcup_{i=0}^{d} B(x_i,a))} = \mathrm{Id}\,|_{\mathbb{S}^n \setminus (\bigcup_{i=0}^{d} B(x_i,a))}$.

Since $g = f \circ \tilde{h} \circ \Phi$ we see, by defining a quasiconformal map $h = \Phi \circ \tilde{h}^{-1}$, that $f = g \circ h$. It is now enough to show that the mapping g is indeed uniformly quasiregular. To deduce this, we note that for every $n \in \mathbb{N}$,

$$g^{n+1} = \tilde{f} \circ \tilde{g}^n \circ \Phi$$

holds. Since $\tilde{g}$ is uqr the same holds also for g. Moreover, if the distortion of iterates $\tilde{g}^n$ is bounded by K, then the distortion of the iterates g^n is bounded by $K \cdot K_{\tilde{f}}$, independently of n.

Let us also briefly compare the properties of uqr mappings $\tilde{g}$ and g. For the mapping $\tilde{g}$ the set $B(x_0, b)$ is the trap containing an attractive fixed point. The same role for the mapping g is given by the set $T := B(f(x_0), b)$. The following steps illustrate how the different regions of the sphere are trapped.

(1) The set T is first inverted inside $B(x_0, b)$ under Φ and then translated inside T by $\tilde{f}$ conformally. Therefore $g^n|T$ is conformal for every n.

(2) Similarly the whole set $\mathbb{S}^n \setminus B(x_0, b)$ is inverted to $B(x_0, b)$ and then translated conformally onto T. So in fact it is precisely the exterior of the trap for $\tilde{g}$ that becomes the trap for g.

(3) The sets $\Phi\,(B(x_i, b))$ for $i = 1, \ldots, d$ are inverted to $B(x_i, b)$ and mapped further conformally onto $B(x_0, b)$ by $\tilde{f}$. There one can see the Julia set evolving inside $\bigcup_{i=1}^{d} \Phi B(x_i, b) \subset B(x_0, b)$. The Julia set for g is a Cantor set as it is for $\tilde{g}$.

(4) The points in $B(x_0, b) \setminus \left(\bigcup_{i=1}^{d} \Phi(B(x_i, b)) \cup \Phi(B(f(x_0), b)\right))$ are inverted to $\mathbb{S}^n \setminus \left(\bigcup_{i=0}^{d} B(x_0, b) \cup B(f(x_0), b)\right)$ and stay in $\mathbb{S}^n \setminus B(x_0, b)$ under $\tilde{f}$ and possibly pick up some distortion. After this step the situation is as in step (2).

(5) The points in $\Phi(B(f(x_0), b))$ are mapped onto T under inversion and kept out from $B(x_0, b)$ by $\tilde{f}$, hence back in step (2).

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