

NEBENHÜLLE AND THE GLEASON PROBLEM

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ABSTRACT. This article concerns the Gleason property as a local phenomenon. We prove that there always exists an open set where the domain $D \Subset \mathbb{C}^2$ has the Gleason \mathcal{B} property whenever the boundary of the Nebenhülle of D coincides with a C^2 smooth part of the boundary bD ; here \mathcal{B} is either one of the Banach algebras, H^∞ or A . As an easy consequence of this, we see that if the extremal boundary points are C^2 -smooth, then D has the Gleason \mathcal{B} property close to those points. Also a $\bar{\partial}$ -problem for locally supported forms is solved.

1. INTRODUCTION AND NOTATIONS

This article addresses some aspects of the Gleason problem. The problem was first studied by Andrew Gleason. In his article, [Gle64], Gleason studied the Banach algebra $A(B(0, 1))$ consisting of those holomorphic functions defined on the unit ball in \mathbb{C}^n which can be continuously continued up to the boundary. He asked whether the algebra $A(B(0, 1))$ was finitely generated. He showed that if this was the case, then the maximal ideal consisting of functions vanishing at the origin is generated by the coordinate functions.

The question whether these ideals in algebras of holomorphic functions are generated by the coordinate functions has been named the Gleason problem.

For the time being, let $\mathcal{B}(D)$ denote an algebra of holomorphic functions on an open set $D \subset \mathbb{C}^n$.

We say that a domain $D \subset \mathbb{C}^n$ has the Gleason \mathcal{B} property at $p \in D$ if for every $f \in \mathcal{B}(D)$ such that $f(p) = 0$ there exist functions $f_1, f_2, \dots, f_n \in \mathcal{B}(D)$ such that

$$f(z) = \sum (z_j - p_j) f_j(z)$$

for all $z \in D$.

Many authors have addressed the Gleason problem. The first one to solve Gleason's original question was Leizenzon; the details can be found in [Hen71]. Leizenzon actually proved that the algebra $A(B(0, 1))$ has the Gleason A property at every point in the unit ball, $B(0, 1)$.

When a domain $D \Subset \mathbb{C}^n$ has the Gleason \mathcal{B} property at every point in D , we simply say that D has the Gleason \mathcal{B} property.

The Gleason \mathcal{B} property has been studied for different algebras of holomorphic functions defined on different types of domains; this has been done in various ways,

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for example by explicit methods (see e.g. [Hen71], [Jak84], [BF95] or [LW02]) and by $\bar{\partial}$ -solutions (see e.g. [Bea80], [FØ83], [Ceg93] and [CCF07]). The usual obstacle is the properties (or rather the lack of properties) the domain in question possesses.

Some counterexamples (see e.g. [OA87] and [BF91]) are known; in these examples it is shown that there is a subset of the domain where the domain does not have the Gleason \mathcal{B} property.

In view of these results, we have studied the Gleason problem locally in rather general domains. In this paper we show that under mild assumptions on the boundary there will always exist points where the Gleason \mathcal{B} property holds.

In the remainder of this article D will be a bounded domain in \mathbb{C}^n . We will study $\mathcal{B}(D)$, which will denote one of the Banach algebras, $H^\infty(D)$ (the bounded holomorphic functions on D) or $A(D)$ (holomorphic functions on D which can be continuously continued to the boundary of D).

The boundary of D will be denoted bD , and the set of strictly pseudoconvex boundary points will be denoted $S(bD)$. A point in space will have the form $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, and $B(p, r)$ is the open ball with center at p and radius r . By \mathcal{Z}_g we denote the zero set of the function g .

We now present the main result of this paper, Theorem 1, which we state here, but postpone the proof until section 4.

Theorem 1. *Let $D \subset \mathbb{C}^2$ be a bounded domain such that $bD \in \mathbb{C}^2$. If $\mathcal{B} \in \{H^\infty, A\}$, then there always exist points in D where we can solve the Gleason \mathcal{B} problem.*

With this theorem it is possible to modify the counterexample in [BF91] to show that there exists a domain in which there is an open subset where the domain does not have the Gleason \mathcal{B} property but in another open subset of the domain the Gleason \mathcal{B} property holds true; see [Car08a].

The outline of the paper is the following: In section 2 we show that the Gleason \mathcal{B} property holds true for points close to strictly pseudoconvex boundary points of a certain kind. In section 3 we solve a simple $\bar{\partial}$ -problem for locally supported forms. This is finally used in section 4 to give a more general description of domains where we can prove a local Gleason \mathcal{B} property.

2. A LOCAL SOLUTION TO THE GLEASON PROBLEM

In this section the domain D will be a bounded domain in \mathbb{C}^2 . The interesting result of this section is the following, rather technical, proposition.

Proposition 1. *Let $D \subset \mathbb{C}^2$ be a bounded domain. Let $\xi \in bD$ and assume that there exists a function $\kappa_\xi \in H(\tilde{D})$ for some pseudoconvex domain $\tilde{D} \ni D$ such that κ_ξ is a peaking function for $A(D)$ at ξ . Furthermore assume that there is a relatively open neighborhood V of ξ in bD such that V is C^1 smooth and if*

$$\lambda \in L_{(0,1)}^\infty(\bar{D}) \cap \ker \bar{\partial} \text{ and} \\ \text{supp}(\lambda) \cap bD \Subset V,$$

then there exists a function $u \in C(\bar{D})$ such that $\bar{\partial}u = \lambda$. Then there is a neighborhood O_ξ of ξ such that D has the Gleason \mathcal{B} property at p for all $p \in O_\xi \cap D$.

Remark 1. The peaking functions κ_ξ mentioned above do exist for a big set of strictly pseudoconvex boundary points. In fact, we shall later see that all bounded domains in \mathbb{C}^n with a C^2 -smooth boundary have uncountably many such points.

To prove this proposition we start with a lemma.

Lemma 1. *Let $g_1(z) = 1 - \kappa_\xi(z)$, where κ_ξ , D and \tilde{D} satisfy the hypothesis of Proposition 1. There exist an affine linear function $g_2 \in H(\mathbb{C}^n)$ and a neighborhood $\omega \subset \tilde{D}$ of ξ such that*

$$\mathcal{Z}_{g_1} \cap \mathcal{Z}_{g_2} \cap \omega = \{\xi\}.$$

Proof. First we make a complex affine change of coordinates (we denote this mapping as σ , and let $Tf = f \circ \sigma^{-1}$ for all functions f on D) which takes ξ to $(0, 0)$ and such that the outward unit normal at the origin is $(0, i)$. We call the translated domain D_T .

Let D_2 be the one dimensional slice $\{z_2 : (0, z_2) \in D_T\}$. Then κ_ξ peaks at the origin and is therefore non-constant, so $Tg_1(0, z_2)$ has an isolated zero at $(0, 0)$.

Especially, there exists a neighborhood U of 0 such that $Tg_1(0, z_2) = 0$ only when $z_2 = 0$.

Defining $Tg_2(z_1, z_2) := z_1$ gives that

$$\mathcal{Z}_{Tg_1} \cap \mathcal{Z}_{Tg_2} \cap U = \{0\}.$$

Let g_2 be the reversed complex linear change of coordinates of Tg_2 and let ω be the preimage of U under the same transformation intersected with \tilde{D} . This completes the proof. \square

Now we prove Proposition 1. In the part of the proof where the Gleason property is shown we follow the ideas in [Bea80].

In the following we let ν_ξ denote the outward unit normal vector at ξ . Also, for a set $D \subset \mathbb{C}^2$ and a constant r , we define D_r as

$$D_r = \{z \in \mathbb{C}^2 : z + r\nu_\xi \in D\}.$$

Proof of Proposition 1. Let g_1, g_2 and ω be as in Lemma 1.

We want the intersection of the zero sets of g_1 and g_2 to be inside the domain, so we translate the zero sets of the functions. This is done by an affine transformation, $g_j(\cdot) \rightarrow h_j(\cdot + w)$, $j = 1, 2$.

Since the functions $g_j \in H(\tilde{D})$ we can ensure that the functions h_j still have the desired properties on D .

Let $\delta > 0$ be so small that if we set $p := \xi - r\nu_\xi$, where $0 < r < \delta$ is arbitrary, we get that

$$h_j(z) := g_j(z + r\nu_\xi), \quad j = 1, 2,$$

will satisfy that $h_j \in A(D)$, $\mathcal{Z}_{h_1} \cap \mathcal{Z}_{h_2} \cap D = \{p\}$ and furthermore that $\mathcal{Z}_{h_1} \cap bD \Subset V$.

Now we cover the domain with three open sets. Let $\varepsilon_0 > 0$ be so small that the ball

$$U_0 := B(p, 2\varepsilon_0) \Subset \omega_r \cap D.$$

Choose $\varepsilon_1 > 0$ so that $\mathcal{Z}_{h_2} \cap U_2 = \emptyset$, where

$$U_2 = \left(\{z \in \mathbb{C}^2 : d_e(z, \mathcal{Z}_{h_1}) < 2\varepsilon_1\} \setminus \overline{B(p, \varepsilon_0)} \right) \cap \tilde{D}_r,$$

where \tilde{D} is the domain from the hypothesis. If necessary, shrink ε_1 so that $U_2 \cap bD \Subset V$. Let

$$U_1 = \left(\{z \in \mathbb{C}^2 : d_e(z, \mathcal{Z}_{h_1}) > \varepsilon_1\} \setminus \overline{B(p, \varepsilon_0)} \right) \cap \tilde{D}_r.$$

Obviously we have $D \Subset U_0 \cup U_1 \cup U_2$.

Let $\{\phi_j\}_{j=0}^2$ be a partition of unity subordinate to the covering $U_0 \cup U_1 \cup U_2$ of D . This construction gives that $\phi_0 \equiv 1$ on $B(p, \varepsilon_0)$, $\phi_1 \equiv 1$ on the set

$$\left(\{z \in \mathbb{C}^2 : d_e(z, \mathcal{Z}_{h_1}) > 2\varepsilon_1\} \setminus \overline{B(p, 2\varepsilon_0)}\right) \cap \tilde{D}_r$$

and $\phi_2 \equiv 1$ on the set

$$\left(\{z \in \mathbb{C}^2 : d_e(z, \mathcal{Z}_{h_1}) < \varepsilon_1\} \setminus \overline{B(p, 2\varepsilon_0)}\right) \cap \tilde{D}_r.$$

Given $f \in \mathcal{B}(D)$ arbitrary such that $f(p) = 0$. On U_0 we have by the Oka-Hefers Lemma that we can write $f(z) = f_1^0(z)(z_1 - p_1) + f_2^0(z)(z_2 - p_2)$, where $f_k^0 \in \mathcal{B}(D \cap U_0)$, $k = 1, 2$. (One can solve the problem on a slightly bigger ball and thereby get a solution continuous up to the boundary.) By the choice of r we have that $h_j \in H(\tilde{D}_r)$ and $D \subset\subset \tilde{D}_r$. Using the Oka-Hefers Lemma again we get that

$$h_j(z) = h_1^j(z)(z_1 - p_1) + h_2^j(z)(z_2 - p_2)$$

and $h_k^j \in A(D)$, $j, k = 1, 2$. Now let

$$f_k^j(z) = \frac{f(z)h_k^j(z)}{h_j(z)}.$$

Since $h_j \neq 0$ on U_j we get that $f_k^j \in \mathcal{B}(U_j \cap D)$, $j = 1, 2$, $k = 1, 2$ and

$$f(z) = f_1^j(z)(z_1 - p_1) + f_2^j(z)(z_2 - p_2)$$

on $\overline{U_j} \cap \overline{D}$. So $F_1(z) = \sum_{j=0}^2 \phi_j(z)f_1^j(z)$ and $F_2(z) = \sum_{j=0}^2 \phi_j(z)f_2^j(z)$ give a smooth solution to the problem.

The next step will be to find a function u such that

$$f_1(z) = F_1(z) + u(z)(z_2 - p_2) \text{ and } f_2(z) = F_2(z) - u(z)(z_1 - p_1)$$

are in $\mathcal{B}(D)$.

Define λ as follows:

$$\lambda = \begin{cases} \frac{-\bar{\partial}F_1}{z_2 - p_2} & \text{when } \{z_2 \neq p_2\} \cup B(p, \varepsilon_0), \\ \frac{\bar{\partial}F_2}{z_1 - p_1} & \text{otherwise.} \end{cases}$$

This makes λ well defined since $\frac{-\bar{\partial}F_1}{z_2 - p_2} = \frac{\bar{\partial}F_2}{z_1 - p_1}$ on the intersection of their domains of definition. Since $\partial^2 \equiv 0$ we have that λ is a $\bar{\partial}$ -closed $(0, 1)$ -form on D , whose coefficients are bounded on D .

The support of λ is contained in $\overline{U_i} \cap \overline{U_j}$, $i \neq j$. Hence we have

$$\text{supp}(\lambda) \cap bD \Subset V.$$

By the hypothesis there exists a function

$$u \in C(\overline{D})$$

such that $\bar{\partial}u = \lambda$. With this u we in fact have that $f_1, f_2 \in \mathcal{B}(D)$. Also

$$f(z) = f_1(z)(z_1 - p_1) + f_2(z)(z_2 - p_2).$$

So D has the Gleason \mathcal{B} property at $p = \xi - r\nu_\xi$ for $0 < r < \delta$.

Since the boundary, bD , close to ξ is C^1 smooth, we can show the same result for $p = \zeta - r\nu_\zeta$, where $\zeta \in bD$ is close enough to ξ and ν_ζ is the normal vector calculated at ζ with $0 < r < \delta/2$.

So in fact we have shown that there exists a neighborhood O_ξ of ξ in D with the Gleason \mathcal{B} property. \square

3. NEBENHÜLLE AND A LOCAL $\bar{\partial}$ -SOLUTION

In this section we will show a $\bar{\partial}$ -result for certain types of forms; to do this we remind the reader of the definition of the Nebenhülle. The following definition is equivalent to the one given in [DF77].

Definition 1. The Nebenhülle of a domain D is defined as

$$N(D) = \text{interior} \left(\bigcap D_\alpha \right),$$

where the intersection is taken over all strictly pseudoconvex domains, D_α , with smooth boundaries such that $D \Subset D_\alpha$.

To prove Proposition 3 we need the following proposition, which was proved in [Car08b]. We state the result here, since it is referred to several times in this paper.

Proposition 2 (Proposition 7, [Car08b]). *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Let $V \subset S(bD) \cap b(N(D))$ be an open set which is C^k smooth, where $k \geq 2$.*

Assume that K is a nonempty, compact subset of V . Then there exists a bounded strictly pseudoconvex domain $\hat{D} \subset \mathbb{C}^n$ with C^k regular boundary such that

- (1) $D \subset \hat{D}$,
- (2) $K \subset b\hat{D}$.

It is important to notice that it is only V that has to have C^k smoothness. This property is not necessary on the whole boundary bD . Now we are ready to state the promised $\bar{\partial}$ -result.

Proposition 3. *Suppose D , K , and V satisfy the hypothesis in Proposition 2. Assume that f is a $\bar{\partial}$ closed $L^2_{loc}(D)$ form such that $\text{supp}(f) \cap bD \subset K$. Then there exists a $(p, q-1)$ form u such that $\bar{\partial}u = f$ on D and such that:*

- (1) *If $f \in L^\infty_{(p,q)}(D)$ and $V \in C^2$, then we can choose a form $u \in C_{(p,q-1)}(\bar{D})$.*
- (2) *If $V \in C^\infty$ and $f \in C^\infty_{(p,q)}(\bar{D})$, then we can find $u \in C^\infty_{(p,q-1)}(\bar{D})$.*

Proof. Assume $f \in L^\infty(D)$. Let \hat{D} be the strictly pseudoconvex domain from Proposition 2. Extend f trivially to \hat{D} . By Propositions 6.2, 6.3, and 6.4 in [Øvr71] we get a solution

$$u \in C_{(p,q-1)}(\bar{\hat{D}}),$$

and the restriction of u to \bar{D} is our solution to (1).

We prove (2) analogously but refer to page 229 in [Koh77] instead. \square

Remark 2. If \bar{D} is holomorphically convex, then D has a Stein neighborhood basis and thereby $N(D) = D$, so D satisfies the hypothesis in Proposition 2. In that case, (1) in Proposition 3 will be a special case of Theorem 1.1.b in [BR80].

Remark 3. If $bD \in C^\infty$, then (2) in Proposition 3 is a special case of Theorem 1.1 in [Bea80].

4. THE GLEASON PROBLEM

In this section we show that it is possible to solve the Gleason problem near a point $\xi \in bN(D) \cap S(bD)$ in D where the boundary is C^2 smooth. Throughout this section we assume that D and V satisfy the hypothesis of Proposition 2.

First we show that we can find a special type of peaking function.

Lemma 2. For every point $\xi \in V$ there exists a function $\kappa_\xi \in H(\tilde{D})$ such that $\kappa_\xi(\xi) = 1$ and such that $|\kappa_\xi(z)| < 1$ if $z \in \overline{D} \setminus \{\xi\}$, where $D \Subset \tilde{D}$ and \tilde{D} is strictly pseudoconvex with a smooth boundary.

Proof. From Proposition 2 we choose \hat{D} strictly pseudoconvex such that $D \subset \hat{D}$ and $\xi \in b\hat{D}$, $b\hat{D} \in C^k$.

With this it follows that there exists a function $\kappa_\xi \in H(\overline{\hat{D}})$ with the desired properties. (See for example Theorem VI.1.13, page 222 in [Ran86].)

Since \hat{D} has a Stein neighborhood basis we may assume that $\kappa_\xi \in H(\tilde{D})$, where $D \subset \hat{D} \Subset \tilde{D}$ and \tilde{D} is strictly pseudoconvex with smooth boundary. \square

Remark 4. Every $\xi \in V$ is a peak point to $A(D)$.

In the following, remember that $\mathcal{B} \in \{H^\infty, A\}$.

Proposition 4. Let $D \subset \mathbb{C}^2$ be a bounded pseudoconvex domain and let V satisfy the hypothesis of Proposition 2. For every $\xi \in V$ there is a neighborhood O_ξ of ξ such that D has the Gleason \mathcal{B} property at p when $p \in O_\xi \cap D$.

Proof. This follows from Lemma 2, Proposition 3, and Proposition 1. \square

We denote the extremal points of the convex hull of D as $E(\text{conv}(D))$.

Corollary 1. If $D \subset \mathbb{C}^2$ is a bounded domain and if there exists a nonempty open set

$$V \subset bD \cap E(\text{conv}(D))$$

such that $V \in C^2$, then one can solve the Gleason \mathcal{B} problem close to V .

Proof. The convex hull has trivial Nebenhülle. \square

Theorem 1. Let $D \subset \mathbb{C}^2$ be a bounded domain such that $bD \in C^2$. If $\mathcal{B} \in \{H^\infty, A\}$, then there always exist points in D where we can solve the Gleason \mathcal{B} problem.

Proof. There will be points of the boundary that coincide with the extremal points of its convex hull and these are of smoothness C^2 . \square

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