PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 1, January 2010, Pages 1–7
S 0002-9939(09)10086-2
Article electronically published on September 4, 2009

A NOTE ON CERTAIN KRONECKER COEFFICIENTS

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(Communicated by Jonathan I. Hall)

ABSTRACT. We prove an explicit formula for the tensor square of an irreducible complex representation of the symmetric group defined by a rectangle of height two. We also describe part of the decomposition for the tensor product of representations defined by rectangles of heights two and four. Our results are deduced, through Schur-Weyl duality, from the observation that certain actions on triple tensor products of vector spaces are multiplicity free.

1. Introduction

Irreducible complex representations of the symmetric group $\mathcal{S}_n$ are well known to be indexed by partitions of n in a natural way (see e.g. [Mc], I.7). We will denote by $[\lambda]$ the representation associated to the partition λ. A major unsolved problem is to find a general rule for the tensor product $[\lambda]\otimes[\mu]$ of two such representations. Equivalently, one would like a general rule for the computation of the Kronecker coefficients, which are defined as the multiplicities appearing in the formula

$$[\lambda]\otimes[\mu]=\bigoplus_{\nu} k_{\lambda\mu\nu}[\nu].$$

In the setting of algebraic complexity theory, a specific instance of this general problem has been brought to the fore: can one compute the tensor square of an irreducible complex representation of the symmetric group defined by a *rectangle* partition? (See [BLMW] for an overview.) If the rectangle has height one this is pretty obvious, since the corresponding representation is the trivial one. In this note we give an answer for the next case, that of a rectangle of height two. (Note that a rectangle of width two would lead exactly to the same answer, since one can pass from a partition to the dual one, in terms of representations of the symmetric group, simply by the product with the sign representation.)

We denote by $\ell(\lambda)$ the length of a partition λ, that is, the number of nonzero parts. A partition is said to be *even* (resp. *odd*) when all its nonzero parts are *even* (resp. *odd*).

Received by the editors October 8, 2008.
2000 *Mathematics Subject Classification.* Primary 20C30, 15A72, 11S90.

Our main result is the following:

Theorem 1. *For any integer n, the tensor product $[n,n]\otimes[n,n]$ is a multiplicity free representation of $\mathcal{S}_{2n}$. Its decomposition into irreducibles is*

$$[n,n]\otimes[n,n]=\bigoplus_{\substack{\lambda\,\mathrm{even},\,|\lambda|=2n\\ \ell(\lambda)\leq 4}}[\lambda]\oplus\bigoplus_{\substack{\mu\,\mathrm{odd},\,|\mu|=2n\\ \ell(\mu)=4}}[\mu].$$

An independent and very different proof of this result can also be found in [GWXZ]. It would be interesting to understand the splitting of $[n,n]\otimes[n,n]$ into its symmetric and skew-symmetric parts.

To state our second result, we introduce the following notation:

$$[\lambda]\otimes_\ell[\mu]=\bigoplus_{\ell(\nu)\leq\ell}k_{\lambda\mu\nu}[\nu].$$

Theorem 2. *For any integer n, the partial tensor product $[2n,2n]\otimes_3[n,n,n,n]$ is a multiplicity free representation of $\mathcal{S}_{4n}$. Its decomposition into irreducibles is*

$$[2n,2n]\otimes_3[n,n,n,n]=\bigoplus_{\substack{|\lambda|=2n\\ \lambda_2+\lambda_3-\lambda_1\geq 0\text{ and even}}}[2\lambda].$$

2. Schur-Weyl duality and multiplicity free actions

In order to prove the previous two theorems we will restate them in terms of representations of general linear groups in a quite standard way. Recall the statement of the Schur-Weyl duality between representations of symmetric groups and of general linear groups (see e.g. [Ho]): let V be any finite-dimensional complex vector space, and n any integer. Then the $\mathcal{S}_n\times GL(V)$-module $V^{\otimes n}$ decomposes as

$$V^{\otimes n}=\bigoplus_{|\lambda|=n}[\lambda]\otimes S_\lambda V,$$

where $S_\lambda V$ denotes the Schur module of weight λ, which is an irreducible polynomial representation of $GL(V)$. For three vector spaces U,V,W, a straightforward consequence of taking $\mathcal{S}_n$-invariants in $(U\otimes V\otimes W)^{\otimes n}$ is that for three partitions λ,μ,ν of n, the multiplicity of $S_\lambda U\otimes S_\mu V\otimes S_\nu W$ inside $Sym^n(U\otimes V\otimes W)$ is equal to the Kronecker coefficient $k_{\lambda\mu\nu}$. In particular, if U and V have respective dimensions du and dv, with $(u,v)=1$, we deduce that

$$Sym(U\otimes V\otimes W)^{SL(U)\times SL(V)}=\bigoplus_{n\geq 0}\bigoplus_{|\lambda|=nuv}k_{(nv)^u,(nu)^v,\lambda}S_\lambda W.$$

Here $(nv)^u$ denotes the rectangular partition with u parts all equal to nv, so that the corresponding Schur module of U consists of $SL(U)$-invariants. This shows that Kronecker coefficients involving rectangular partitions are closely related to invariant theory. Indeed our two theorems above will be translated into the statements that two invariant algebras $Sym(U\otimes V\otimes W)^{SL(U)\times SL(V)\times N}$ are polynomial algebras, where N is a group of strictly upper triangular matrices in $SL(W)$.

Note that the complete invariant algebra $Sym(U\otimes V\otimes W)^{SL(U)\times SL(V)\times SL(W)}$ is then also a polynomial algebra. This happens when the dimensions of the three spaces are either $(n,2,2)$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$. In the terminology of [Ka], these cases correspond to θ-groups defined by the triple nodes of the Dynkin diagrams of type D_{n+2}, E_6, E_7 and E_8. The two cases we examine in this note are thus related to D_6 and E_7, respectively.

As is well known, multiplicity free actions of reductive groups can be detected by the existence of an open orbit for a Borel subgroup. We use this principle in the following setting: let G and H be two reductive groups with finite-dimensional representations V and W. Let B denote a Borel subgroup of H and N its unipotent radical. Suppose that $G \times B$ acts on $V \otimes W$ with an open orbit $\mathcal{O}$. Let $X_1, \dots, X_r$ denote the boundary components of $\mathcal{O}$, that is, the irreducible hypersurfaces in its complement. Applying [Br], Proposition 3 of Chapter 3, we are led to the following conclusions:

- $X_1, \dots, X_r$ have equations $f_1, \dots, f_r$ which are semi-invariants of B with linearly independent weights $\mu_1, \dots, \mu_r$; in particular r cannot exceed the rank of H.
- The algebra $\mathbb{C}[V \otimes W]^{G\times N}$ is a polynomial algebra over $f_1, \dots, f_r$.

That $f_1, \dots, f_r$ are semi-invariants of B of weights $\mu_1, \dots, \mu_r$ means that $f_i(bx) = \mu_i(b) f_i(x)$ for all $x \in V \otimes W$ and $b \in B$. Moreover, as an H-module, the algebra of G-invariant functions on $V \otimes W$ is multiplicity free:

$$\mathbb{C}[V \otimes W]^G = \bigoplus_{\mu \in \mathbb{Z}_+\mu_1 + \cdots + \mathbb{Z}_+\mu_r} W_\mu$$

if W_μ denotes the irreducible H-module of highest weight μ. Indeed, such a component of $\mathbb{C}[V \otimes W]^G$ can be detected by its one-dimensional subspace of N-invariants.

In order to prove our two theorems, we will therefore just need to prove that the corresponding actions have open orbits and to identify the boundary components.

3. Proof of Theorem 1

Let U and V be two-dimensional vector spaces.

Lemma 1. *Consider the action of $SL(U) \times SL(V)$ on the flag variety $\mathcal{F}(U \otimes V)$. The generic isotropy group of this action is a product of $\mathbb{Z}_2$ by the 8-element quaternion group.*

Proof. Consider a general flag $W_1 \subset W_2 \subset W_3 \subset U \otimes V$. The projective line $\mathbb{P}W_2 \subset \mathbb{P}(U \otimes V)$ meets the quadric $Q = \mathbb{P}U \times \mathbb{P}V$ in two general points, which means that W_2 has a basis of the form $u_0 \otimes v_0, u_1 \otimes v_1$, where u_0, u_1 is a basis of U and v_0, v_1 is a basis of V. Multiplying, if necessary, one of these vectors by a scalar, we may suppose that W_1 is the line in W_2 generated by $u_0 \otimes v_0 + u_1 \otimes v_1$. Finally, W_3 is the kernel of a general linear form ϕ vanishing on W_2. Since in terms of the dual basis, $W_2^\perp = \langle u_0^\vee \otimes v_1^\vee, u_1^\vee \otimes v_0^\vee \rangle$, we can suppose that $\phi = u_0^\vee \otimes v_1^\vee - u_1^\vee \otimes v_0^\vee$. This means that W_3 is generated by W_2 and $u_0 \otimes v_1 + u_1 \otimes v_0$.

Now it is straightforward to compute the stabilizer of our flag explicitly and to identify it with the product of $\mathbb{Z}_2$ by a copy of the 8-element quaternion group. $\square$

In fact, the only important thing to us is that this stabilizer is finite because of the following corollary. Let W be a four-dimensional vector space and B a Borel subgroup in $GL(W)$.

Corollary 1. *The group $SL(U) \times SL(V) \times B$ has an open orbit in $U \otimes V \otimes W$.*

Proof. Consider a tensor $T\in U\otimes V\otimes W$ as a morphism $\phi_T : W^\vee\to U\otimes V$. For a generic T this morphism is injective and maps the flag defining B (or rather the orthogonal flag) to a generic flag in $U\otimes V$. By Lemma 1, $SL(U)\times SL(V)$ has an open orbit in the flag variety $\mathcal{F}(U\otimes V)$. Also, once the image flag is fixed, it is clear that B acts transitively on the set of compatible injections. □

As we explained above, the next step is to describe the boundary components of the open orbit. Let us denote by $F=(W_1\subset W_2\subset W_3\subset W)$ the flag whose stabilizer is the Borel subgroup B of $GL(W)$, and by $F^\perp$ the orthogonal flag in $W^\vee$.

As in the proof of Lemma 1, we denote by $\phi_T : W^\vee\to U\otimes V$ the morphism defined by the tensor $T\in U\otimes V\otimes W$. We can describe the boundary components of the open orbit in $U\otimes V\otimes W$ by the following codimension one conditions:

(1) ϕ_T is not an isomorphism. The corresponding boundary component X_1 is the complement of the $SL(U)\times SL(V)\times GL(W)$-orbit. It is just the quartic hypersurface of equation $f_1=\det\phi_T$. This equation is a weight vector in $S_{22}U^\vee\otimes S_{22}V^\vee\otimes\bigwedge^4 W^\vee=\bigwedge^4(U\otimes V)^\vee\otimes\bigwedge^4 W^\vee\subset S^4(U\otimes V\otimes W)^\vee$. This means that the weight μ_1 of f_1 is, written as a sequence of three partitions, $\mu_1=(22,22,1111)$.

(2) $\phi_T(\mathbb{P}W_3^\perp)$ belongs to the quadric Q. The corresponding boundary component X_2 is defined by the condition that $q(\phi_T(w^\vee))=0$ if $w^\vee$ generates $W_3^\perp$ and q denotes an equation of Q. Thus an equation f_2 of X_2 is a highest weight vector in $\bigwedge^2 U^\vee\otimes\bigwedge^2 V^\vee\otimes S_2W^\vee$. It has degree two and weight $\mu_2=(11,11,2)$.

(3) $\phi_T(\mathbb{P}W_2^\perp)$ is a tangent line to Q. This is the case if and only if $\phi_T(W_2^\perp)$ is generated by vectors of the form $u_0\otimes v_0$ and $u_0\otimes v_1+u_1\otimes v_0$. Considered as a line ℓ in $\bigwedge^2 W^\vee$, this means that $W_2^\perp$ is mapped by ϕ_T to the line generated by $u_0^2\otimes(v_0\wedge v_1)\oplus(u_0\wedge u_1)\otimes v_0^2$ in $\bigwedge^2(U\otimes V)=S^2U\otimes\bigwedge^2 V\oplus\bigwedge^2 U\otimes S^2V$. Since the map $S^2(S^2U)\to S_{22}U$ kills any tensor of the form $(u^2)^2$, we deduce that ϕ_T maps ℓ^2 to zero in $S_{22}U\otimes S_{22}V$. This implies that an equation f_3 of the corresponding boundary component X_3 is a highest weight vector in $S_{22}U^\vee\otimes S_{22}V^\vee\otimes S_{22}W^\vee$. It has degree four and its weight is $\mu_3=(22,22,22)$.

(4) $\phi(\mathbb{P}W_1^\perp)$ is a tangent plane to Q. This is similar to the case of X_2, up to duality. A hyperplane H in $U\otimes V$ defines a line in $\bigwedge^3(U\otimes V)=U\otimes V\otimes(\bigwedge^2 U\otimes\bigwedge^2 V)$, hence a line ℓ in $U\otimes V$ (the orthogonal line with respect to the polarity defined by Q). This hyperplane is tangent to the quadric Q if and only if ℓ is contained in Q. This means that X_4 is defined by the condition that the composition

$$S^2(\bigwedge^3 W^\vee)\to S^2(\bigwedge^3(U\otimes V))=S^2(U\otimes V)\otimes(\bigwedge^2 U\otimes\bigwedge^2 V)^2\to(\bigwedge^2 U\otimes\bigwedge^2 V)^3$$

vanishes. Hence an equation f_4 of X_4 is a highest weight vector in $S_{33}U^\vee\otimes S_{33}V^\vee\otimes S_{222}W^\vee$. It has degree six and weight $\mu_4=(33,33,222)$.

The four weights of f_1,f_2,f_3,f_4 are linearly independent. Since the rank of $GL(W)$ is four, we must have found all the boundary components, and we can conclude that

$$\mathbb{C}[U\otimes V\otimes W]^{SL(U)\times SL(V)\times N}=\mathbb{C}[f_1,f_2,f_3,f_4],$$

where N denotes the unipotent radical of B. This implies that $\mathbb{C}[U \otimes V \otimes W]$ contains a copy of $S_{n,n}U^\vee \otimes S_{n,n}V^\vee \otimes S_\lambda W^\vee$ if and only if λ is a nonnegative linear combination of the components of μ_1, μ_2, μ_3 and μ_4 on W, that is, the weights $(1111), (2), (22)$ and (222). Moreover, in that case the multiplicity is equal to one.

Rephrasing this result via Schur-Weyl duality we get the statement of Theorem 1.

4. Proof of Theorem 2

For the proof of Lemma 2 we need to briefly recall the principle of *castling transforms*, introduced by Sato and Kimura [SK]. Consider a G-module M of dimension m and a vector space N of dimension n. Suppose that $m > n$. A tensor T in $M \otimes N$ can be identified to a linear map $\phi_T : N^\vee \to M$. If ϕ_T is injective, in particular for a generic T, the stabilizer of T in $G \times GL(N)$ is canonically isomorphic to the stabilizer in G of the image of ϕ_T, considered as a point of the Grassmannian $G(n, M)$. But this Grassmannian is isomorphic with $G(n-m, M^\vee)$, and the generic stabilizer of the action of $G \times GL(N)$ on $M \otimes N$ is therefore isomorphic with the generic stabilizer of the action of $G \times GL(P)$ on $M^\vee \otimes P$, for P a vector space of dimension $m-n$. Replacing $M \otimes N$ by $M^\vee \otimes P$ is precisely what Sato and Kimura call a castling transform. In the case where $M \otimes N$ is prehomogeneous and $m < 2n$, $M^\vee \otimes P$ is also prehomogeneous but of smaller dimension.

Let U, V, W be complex vector spaces of respective dimension two, four and three. Let B be a Borel subgroup of $GL(W)$.

Lemma 2. *The group $SL(U) \times SL(V) \times B$ has an open orbit in $U \otimes V \otimes W$.*

Proof. We claim that the generic isotropy group of the action of $SL(U) \times SL(V) \times GL(W)$ on $U \otimes V \otimes W$ is a copy of SL_2, up to a finite group. To check this, we observe that $U \otimes V \otimes W$ is, according to the terminology of Sato and Kimura, a nonreduced prehomogeneous vector space, which means that it is related to smaller prehomogeneous spaces of the same type by certain castling transforms.

In order to apply this process to the case we are interested in, we first observe that the generic stabilizers of $SL(U) \times SL(V) \times GL(W)$ and $SL(U) \times GL(V) \times SL(W)$ on $U \otimes V \otimes W$ are equal, up to a finite group. Applying a castling transform with $G = SL(U) \times SL(W)$ acting on $U \otimes W$, we deduce that this generic stabilizer is the same as the generic stabilizer of the action of $SL(U) \times GL(Q) \times SL(W)$ on $U^\vee \otimes Q \otimes W^\vee$, where Q has dimension $2 \times 3 - 4 = 2$. Up to a finite group, this is also the generic stabilizer of the action of $SL(U) \times SL(Q) \times GL(W)$, and after a new castling transform, we deduce that this is also the generic stabilizer of the action of $SL(U) \times SL(Q) \times GL(R)$ on $U \otimes Q^\vee \otimes R$, where now R has dimension $2 \times 2 - 3 = 1$. Let us identify U and Q, which are both two-dimensional. Then it is easy to see that the identity map $I \in U \otimes Q^\vee$ has generic stabilizer, so that this generic stabilizer is just a copy of SL_2 embedded diagonally in $SL(U) \times SL(Q)$.

We can keep track of this generic stabilizer along our two castling transforms. We start from the point in $U \otimes U^\vee = End(U)$ defined by I. The corresponding point in $U^\vee \otimes U \otimes W^\vee$, where W is identified with the orthogonal to I in $End(U)$ (the hyperplane $End_0(U)$ of traceless matrices), is just the graph of the embedding of $End_0(U)$ in $End(U)$. Its isotropy is the image of $SL(U)$ in $SL(U^\vee) \otimes SL(U) \otimes SL(W^\vee)$ given by the natural action of $SL(U)$ on each of the three spaces. Now we make our second castling transform to get a point in $U \otimes V \otimes W$, where V is now identified with the kernel of the natural evaluation map $End_0(U) \otimes U \to U$. The corresponding stabilizer is again a copy of $SL(U)$ embedded in $SL(U)\otimes SL(V)\otimes SL(W)$ through its natural action on U,V and W.

We can now check our claim: it can be translated into the assertion that the image of $SL(U)$ into $GL(W)$, where $W=End_0(U)$, does not intersect a general Borel subgroup. But this is straightforward: such a Borel subgroup is defined by a line, generated by a generic traceless matrix m, and a hyperplane containing it, which can be defined as the orthogonal to a generic traceless matrix n orthogonal to m. For an element of SL_2, preserving m and n forces it to belong to the intersection of two tori, and this intersection is finite. $\square$

Now we identify in $U\otimes V\otimes W$ the boundary components of the open orbit of $SL(U)\times SL(V)\times B$. They can be described in terms of the following codimension one conditions on a tensor $T\in U\otimes V\otimes W$, which will be best expressed in terms of certain auxiliary morphisms.

(1) First recall that $SL(U)\times SL(V)\times GL(W)$ has itself an open orbit whose complement is an irreducible hypersurface X_1 of degree 12 [Ka]. An equation f_1 of this hypersurface can be obtained as follows. Consider the morphism $\psi^T: U^\vee\otimes W^\vee\to V$ induced by T. Taking its second wedge power, we get an induced map
$$\Psi^T:\bigwedge^2 U^\vee\otimes S^2W^\vee\hookrightarrow\bigwedge^2(U^\vee\otimes W^\vee)\stackrel{\wedge^2\psi^V}{\longrightarrow}\bigwedge^2 V$$
between two vector spaces of the same dimension, six. We can thus let $f_1=\det\Psi^T$, an invariant of degree 12 and weight $\mu_1=(66,3333,444)$.

(2) Restricting ψ^T, we can define a morphism
$$\psi_1^T: U\otimes W_1^\perp\to V$$
between two vector spaces of the same dimension, four. We can thus define another boundary component X_2 by the condition that this is not an isomorphism. An equation of this hypersurface is $f_2=\det\psi_1^T$, a semi-invariant of degree 4 and weight $\mu_2=(22,1111,22)$.

(3) To describe our next boundary component, we need to recall that there exists an invariant nondegenerate skew-symmetric form of S^3U, or equivalently an equivariant morphism
$$\omega:\bigwedge^2(S^3U)\to(\bigwedge^2 U)^3.$$
Now consider the morphism
$$\phi_T:\bigwedge^3 W^\vee\longrightarrow S^3U\otimes\bigwedge^3 V\simeq S^3U\otimes\bigwedge^4 V\otimes V^\vee$$
induced by T. (Note that $\bigwedge^3 W$ and $\bigwedge^4 V$ are both one dimensional.) Taking its square, we get a map
$$\Phi_T:S^2(\bigwedge^3 W^\vee)\to\bigwedge^2(S^3U)\otimes(\bigwedge^4 V)^2\otimes\bigwedge^2 V^\vee\to(\bigwedge^2 U)^3\otimes(\bigwedge^4 V)^2\otimes\bigwedge^2 V^\vee,$$
where the last arrow is induced by ω. The image of Φ_T defines, up to scalar, a skew-symmetric form ω_T on V. On the other hand, we can restrict the morphism Ψ^T to the line $\bigwedge^2 U^\vee\otimes S^2W_2^\perp\subset\bigwedge^2 U^\vee\otimes S^2W^\vee$.

Its image is, up to scalar, an element of Ω_T of $\bigwedge^2 V$. We can thus define a boundary component X_3 by the condition that the natural pairing $\langle \omega_T, \Omega_T \rangle = 0$. An equation f_3 of this hypersurface has degree 8 and weight $\mu_3 = (44, 2222, 422)$.

Since we have found three boundary components and $\dim(U \otimes V \otimes W) - \dim(SL(U) \times SL(V) \times B) = 3$, we must have found all the boundary components and we can conclude that

$$\mathbb{C}[U \otimes V \otimes W]^{SL(U) \times SL(V) \times B} = \mathbb{C}[f_1, f_2, f_3].$$

The weights of f_1, f_2, f_3 are independent, as expected, and we deduce that $\mathbb{C}[U \otimes V \otimes W]$ contains a component $S_{n,n}U \otimes S_{n,n}V \otimes S_\lambda W$ if and only if $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a nonnegative linear combination of the components of μ_1, μ_2 and μ_3 on W, that is, $(422), (444)$ and (422). Because of the identity

$$\lambda = \frac{\lambda_1 - \lambda_2}{2}(422) + \frac{\lambda_3 + \lambda_2 - \lambda_1}{4}(444) + \frac{\lambda_2 - \lambda_3}{2}(22),$$

this is equivalent to the conditions that λ be even and that $\lambda_3 - \lambda_1 - \lambda_2$ be a nonnegative multiple of four. Moreover, in that case the multiplicity is equal to one. Rephrasing this result via Schur-Weyl duality we get the statement of Theorem 2.

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