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SOLUTION TO FARHADI–GHAHRAMANI'S MULTIPLIER PROBLEM

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ABSTRACT. We answer, in the negative, the main question raised by H. Farhadi and F. Ghahramani in their article *Involutions on the second duals of group algebras and a multiplier problem*, Proc. Edinburgh Math. Soc. (2) **50** (2007), no. 1, 153–161.

Let G be a locally compact group, and consider its group algebra $L_1(G)$ with convolution product. Let $\mathcal{A} = L_1(G)^{**}$, endowed with the first Arens product, which we shall denote by $*$. We write $\cdot$ for the first Arens product on $\mathcal{A}^{**}$, as well as for the canonical left action of $\mathcal{A}^{**}$ on $\mathcal{A}^*$, and of $\mathcal{A}^*$ on $\mathcal{A}$. Recall that these are defined as follows, for $m, n \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$:

$$\begin{aligned} \langle m \cdot n, f \rangle &= \langle m, n \cdot f \rangle, \\ \langle n \cdot f, a \rangle &= \langle n, f \cdot a \rangle, \\ \langle f \cdot a, b \rangle &= \langle f, a * b \rangle. \end{aligned}$$

Analogously, one defines the second Arens product starting from the product $b * a$ (but we shall not need this here).

In [3], noting that for any $N \in L_1(G)^{****}$ the map

$$R_N(f) := N \cdot f \quad (f \in L_1(G)^{***})$$

is a right $L_1(G)$-module homomorphism on $L_1(G)^{***}$, the authors ask the following question [3, Problem 4.1]:

$$(*) \qquad \begin{array}{l} \textit{Is every } w^*\textit{-continuous surjective right } L_1(G)\textit{-module map} \\ R : L_1(G)^{***} \to L_1(G)^{***} \textit{ equal to } R_N \textit{ for some } N \in L_1(G)^{****}\textit{?} \end{array}$$

In the present paper we shall show:

Theorem 1. *The answer to the above multiplier problem is negative for all infinite, countable, discrete, abelian groups (e.g., for $G = \mathbb{Z}$).*

To prepare the proof, let us first consider $\mathcal{A} = \ell_1(G)^{**}$ for an arbitrary discrete group G. Denote by $\mathcal{B}_{\mathcal{C},r}(\mathcal{A}^*)$ the algebra of all bounded linear right $\mathcal{C}$-module maps on $\mathcal{A}^*$, where $\mathcal{C}$ is a (closed) subalgebra of $\mathcal{A}$, and by $\mathcal{B}^{\sigma}_{\mathcal{C},r}(\mathcal{A}^*)$ the subalgebra of w^*-continuous module maps.

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Since $\mathcal{A}$ is unital, by [8, Proposition 4.1], the map R_N induces an isometric algebra isomorphism

$$\mathcal{A}^{**} = (\mathcal{A}^* \cdot \mathcal{A})^* \cong \mathcal{B}_{\mathcal{A},r}(\mathcal{A}^*).$$

So we need to find a surjective map

$$\Phi \in \mathcal{B}^{\sigma}_{\ell_1(G),r}(\ell_1(G)^{***}) \setminus \mathcal{B}_{\ell_1(G)^{**},r}(\ell_1(G)^{***}) \tag{1}$$

(which, as we shall see, exists whenever G is infinite, countable, and abelian). Note that, as is easily checked, for any $N \in \ell_1(G)^{****}$ the map R_N is in fact a right $\ell_1(G)^{**}$-module homomorphism, i.e., belongs to $\mathcal{B}_{\ell_1(G)^{**},r}(\ell_1(G)^{***})$.

For later use we shall give below some background on semigroup compactifications; for a full meal, we refer the reader to [5] and [1]. Recall that the spectrum of the commutative C^*-algebra $\ell_\infty(G)$ is the Stone-Čech compactification βG, which is a compact right topological semigroup with the first Arens product (inherited from $\ell_1(G)^{**}$). Denoting the closure in βG of a subset S of G by $\overline{S}$, one defines the *growth* or *remainder* of S as

$$S^* := \overline{S} \cap (\beta G \setminus G);$$

then S^* is compact, and G^* is a compact right topological semigroup. An element $s \in \beta G$ is called *left cancellable* if left multiplication by s is injective on βG.

For $m \in \mathcal{A}$, write λ_m for left multiplication by m in $\mathcal{A}$. We note the following.

Lemma 2. *If m is a left cancellable element of βG, then $\lambda_m : \ell_1(G)^{**} \to \ell_1(G)^{**}$ is an isometry.*

Proof. This can be shown by transferring *verbatim* the argument given in [1, Proposition 4.4] for right multiplication to our situation (the compact right topological semigroup V in [1, Proposition 4.4] is βG in our case). □

We are now ready to derive Theorem 1.

Proof. Let G be an infinite, countable, discrete, abelian group, and recall that we write $\mathcal{A} = \ell_1(G)^{**}$ (endowed with the first Arens product). Then, by [5, Section 8.4], there exists $m \in G^* \subseteq \mathcal{A}$ such that m is left cancellable in βG. In view of Lemma 2, $\Phi := \lambda_m^* \in \mathcal{B}^{\sigma}(\mathcal{A}^*)$ is surjective. To establish (1), we need to show that

(i) Φ is a right $\ell_1(G)$-module map, and
(ii) Φ is *not* a right $\ell_1(G)^{**}$-module map.

To see (i), let $H \in \mathcal{A}^*$, $a \in \ell_1(G) \subseteq \mathcal{A}$ and $b \in \mathcal{A}$. Then we have

$$\langle \Phi(H \cdot a), b \rangle = \langle H, a * m * b \rangle.$$

But $a \in \ell_1(G)$, and since G is abelian, $\ell_1(G)$ is contained in the algebraic centre of $\ell_1(G)^{**}$. Hence a commutes with m, and we get

$$\langle \Phi(H \cdot a), b \rangle = \langle H, m * a * b \rangle = \langle \Phi(H) \cdot a, b \rangle,$$

as desired.

As for (ii), assume towards a contradiction that Φ is a right $\mathcal{A}$-module map. Then we obtain for all $H \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$ that

$$\langle H, a * m * b \rangle = \langle \Phi(H \cdot a), b \rangle = \langle \Phi(H) \cdot a, b \rangle = \langle H, m * a * b \rangle.$$

Thus, we have $a * m * b = m * a * b$ for all $a, b \in \mathcal{A}$, whence (choosing $b = \delta_e$, the unit of $\mathcal{A}$) we see that m lies in the algebraic centre $Z_a(\mathcal{A})$ of $\mathcal{A} = \ell_1(G)^{**}$. But since $\ell_1(G)$ is commutative, $Z_a(\mathcal{A})$ equals the first topological centre $Z_t^1(\mathcal{A})$ of $\mathcal{A}$, i.e., the set of all elements $n \in \mathcal{A}$ such that λ_n is w^*-continuous on $\mathcal{A}$. By [7, Theorem 1], $Z_t^1(\ell_1(G)^{**}) = \ell_1(G)$. Hence we obtain that $m \in \ell_1(G)$, which contradicts the fact that $m \in G^*$. □

Remark 3. The motivation for question (∗), i.e., [3, Problem 4.1], stems from its link to a long-standing open problem raised by Duncan and Hosseiniun [2, p. 323] asking if, for a locally compact group G, the natural involution on $L_1(G)$ extends to an involution on its bidual. Indeed, Farhadi–Ghahramani show in [3, Theorem 4.2] that the latter has a negative answer for all infinite, locally compact groups G satisfying (∗). Note that the existence of an extension of the involution forces G to be discrete (cf. [4, Theorem 2] or [3, Proposition 3.1]). Now, on the one hand, our Theorem 1 shows that there are discrete groups which violate (∗). On the other hand, [3, Theorem 3.2(a)] answers Duncan–Hosseiniun's question in the negative for all (discrete) groups which admit an infinite, amenable subgroup. It follows from work by Ol′shanskiĭ [9] that there are infinite, countable, discrete groups without infinite, amenable subgroups (cf. [6, Remark 11]). So, the Duncan–Hosseiniun problem remains open for this class of groups, and the approach proposed through [3, Theorem 4.2] may very well prove fruitful since these groups, being of course radically different from the ones appearing in Theorem 1, may satisfy (∗).

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