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THE SYMMETRY PRESERVING REMOVAL LEMMA

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ABSTRACT. In this paper we observe that in the hypergraph removal lemma, the edge removal can be done in such a way that the symmetries of the original hypergraph remain preserved. As an application we prove the following generalization of Szemerédi's Theorem on arithmetic progressions. Let A be an Abelian group with subsets $S_1, S_2, \dots, S_t$ such that the number of arithmetic progressions $x, x+d, \dots, x+(t-1)d$ with $x+(i-1)d \in S_i$ is $o(|A|^2)$. Then we can shrink each S_i by $o(|A|)$ elements such that the new sets don't have any arithmetic progression of the above type.

1. INTRODUCTION

A directed k-uniform hypergraph H on the vertex set V is a subset of V^k such that there is no repetition in the k coordinates. The elements of H will be called **edges**. A **homomorphism** between two directed k-uniform hypergraphs F and H with vertex sets $V(F)$ and $V(H)$ is a map $f : V(F) \mapsto V(H)$ such that $(f(a_1), f(a_2), \dots, f(a_k))$ is in H whenever $(a_1, a_2, \dots, a_k)$ is in F. The **automorphism group** $\mathrm{Aut}(H)$ is the group of bijective homomorphisms $\pi : V(H) \mapsto V(H)$. The **homomorphism density** $t(F, G)$ of F in G is the probability that a map $f : V(G) \mapsto V(H)$ chosen uniformly at random is a homomorphism.

The so-called hypergraph removal lemma ([1], [3], [5], [7], [10])(in the directed setting) says that if the density $t(F, G)$ is small, then a small subset of the edges of G can be removed to make it F-free.

Theorem 1 (Removal Lemma)**.** *For every $k \in \mathbb{N}$, $\epsilon > 0$ and k-uniform directed hypergraph F, there is a constant $\delta = \delta(k, \epsilon, F) > 0$ such that for every k-uniform directed hypergraph G with $t(F, G) \le \delta$ there is a subset $S \subseteq G$ with $|S| \le \epsilon |V(G)|^k$ such that $t(F, G \setminus S) = 0$.*

Using this deep result, we observe that the edge removal can be done in a way that the symmetries of G remain preserved. In other words the set S in Theorem 1 can be assumed to be a union of orbits of $\mathrm{Aut}(G)$.

Theorem 2 (Symmetry Preserving Removal Lemma)**.** *For every $k \in \mathbb{N}$, $\epsilon > 0$ and k-uniform directed hypergraph F there is a constant $\delta_2 = \delta_2(k, \epsilon, F) > 0$ such that for every k-uniform directed hypergraph G with $t(F, G) \le \delta_2$, there is a subset $S \subseteq G$ with $S \le \epsilon |V(G)|^k$ such that $t(F, G \setminus S) = 0$ and furthermore* $\mathrm{Aut}(G) \subseteq \mathrm{Aut}(G \setminus S)$.

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Proof. Let $V = V(G)$. Using the original removal lemma it remains to show that if $S \subseteq V^k$ satisfies $t(F, G \setminus S) = 0$, then there is $S' \subseteq V^k$ which is $\mathrm{Aut}(G)$ invariant, $t(F, G \setminus S') = 0$ and $S' \leq |F||S|$. Such an S' is the union of those orbits O of $\mathrm{Aut}(G)$ on V^k for which $|O|/|O \cap S| \leq |F|$. Assume that $f : V(F) \mapsto V$ is a homomorphism from F to $G \setminus S'$. Then for every fixed $e \in F$ and for a random element $\pi \in \mathrm{Aut}(G)$, the probability is that $\pi(f(e)) \notin G \setminus S'$ is less than $1/|F|$ and so there is some $\pi \in \mathrm{Aut(G)}$ with $\pi(f(F)) \subseteq G \setminus S$, which is a contradiction. □

The argument given for the symmetry preserving removal lemma is very general. It applies to various modified versions of the removal lemma. An important such version is the t-partite removal lemma, where t is a fixed natural number. The vertex set of a t-partite k-uniform hypergraph is a t-tuple $\{V_i\}_{i=1}^t$ of finite sets. An edge of a t-partite hypergraph is an element from $\prod_{i=1}^k V_{a_i}$, where $a_1, a_2, \ldots, a_k$ are k distinct numbers between 1 and t. Let G_1, G_2 be two t-partite k-uniform hypergraphs with vertex sets $\{V_i\}_{i=1}^t$ and $\{W_i\}_{i=1}^t$. A homomorphism from G_1 to G_2 is a t-tuple of maps $\{\phi_i : V_i \to W_i\}_{i=1}^t$ such that $(\phi_{a_1}(r_1), \phi_{a_2}(r_2), \ldots, \phi_{a_k}(r_k)) \in \prod_{i=1}^k W_{a_i}$ is an edge in G_2 whenever $(r_1, r_2, \ldots, r_k) \in \prod_{i=1}^k V_{a_i}$ is an edge in G_1. An automorphism is a bijective homomorphism from G_1 to G_1, and the homomorphism density $t(G_1, G_2)$ is the probability that a random t-tuple of maps $\{\phi_i : V_i \to W_i\}_{i=1}^t$ is a homomorphism.

We give an example of an application of the symmetry preserving removal lemma, and then we generalize it in the next chapter.

Example 1. Let S be a subset of a group T. The Cayley graph $\mathrm{Cy}(T, S) \subseteq T \times T$ is the collection of pairs (a, b) with $ab^{-1} \in S$. The automorphism group of $\mathrm{Cy}(T, S)$ contains T with the action $(a, b) \mapsto (ag, bg)$. Clearly any subset of $T \times T$ invariant under this action of G is a Cayley graph itself. This means that the T-orbits of edges in Cy(T,S) correspond to elements of S.

We apply the symmetry preserving removal lemma to

$$F = \{(1, 2), (1, 3), (2, 3)\}$$

with $V(F) = \{1, 2, 3\}$ and for $G = \mathrm{Cy}(T, S)$. A homomorphism from F to G is a map $f : \{1, 2, 3\} \mapsto T$ such that $a = f(1)f(2)^{-1}$, $b = f(2)f(3)^{-1}$ and $c = f(1)f(3)^{-1}$ are all in S. Consequently, the number of such homomorphisms is the number

$$|T||\{(a, b, c)|ab = c\ ,\ \ a, b, c \in S\}|.$$

The symmetry preserving removal lemma yields that if $ab = c$ has $o(|T|^2)$ solutions in S, then one can remove $o(|T|)$ elements from S such that in the new set there is no solution to $ab = c$. This was first proved by Ben Green [2] for Abelian groups and generalized to groups by Král, Serra and Vena [4].

2. Cayley hypergraphs

In this section we describe a potential way of generalizing Cayley graphs to the hypergraph setting and then discuss the symmetry preserving removal lemma on such graphs.

Definition 2.1. Let $G_1, G_2, \ldots, G_t$ be finite groups and let H be a subgroup of $\prod_{i=1}^t G_i$. The group H acts on each G_i by $(h_1, h_2, \ldots, h_t)g = h_i g$, where $(h_1, h_2, \ldots, h_t) \in H$ and $g \in G_i$. A t-partite k-uniform hypergraph T on the vertex set $\{G_i\}_{i=1}^t$ is called a **Cayley hypergraph** if its automorphism group contains H with the previous action.

This definition is very general, so we will start by analyzing a special case. Assume that all the groups $G_1, G_2, \ldots, G_t$ are isomorphic to an Abelian group A. Furthermore, to get something interesting we want to assume that H is not too big and not too small. Let $C = \{C_1, C_2, \ldots, C_r\}$ be a collection of k-element subsets of $\{1, 2, \ldots, t\}$. Each set C_i defines a projection $p_i : H \mapsto A^k$ to the coordinates in C_i. Assume that the factor group $A^{C_i}/p_i(H) \cong A$ and let $\psi_i : A^{C_i} \to A$ be a homomorphism with kernel $p_i(H)$. Now we pick subsets $S_i \subseteq A$ for $1 \leq i \leq r$ and we define the graph $H_{k,t}(A, \{S_i\}, C)$ as

$$\bigcup_{i=1}^{r} \psi_i^{-1}(S_i),$$

where $\psi_i^{-1}(S_i)$ is the union of cosets in A^{C_i} of $p_i(H)$ representing an element in S_i. Note that the way we produced $H_{k,t}(A, \{S_i\}, C)$ guarantees that its automorphism group contains H as a subgroup.

The symmetry preserving removal lemma for t-partite hypergraphs directly implies the following lemma.

Lemma 2.1 (Cayley Hypergraph Removal Lemma)**.** *For every k, t natural numbers, t-partite k-uniform hypergraph F and $\epsilon > 0$, there exists a constant $\delta > 0$ such that if*

$$t(F, H_{k,t}(A, \{S_i\}, C)) \leq \delta,$$

then there are subsets S_i' in A of size at most $\epsilon|A|$ such that $t(F, H_{k,t}(A, \{S_i \setminus S_i'\}, C)) = 0$.

Example 2. This example uses an idea by Solymosi [8], who showed that the Hypergraph Removal Lemma implies Szemerédi's theorem on arithmetic progressions (even in a multi-dimensional setting). Let t be a natural number, $k = t - 1$, and A be an Abelian group. We define H to be the subgroup in A^t of the elements $(a_1, a_2, \ldots, a_t)$ with $\sum_{i=1}^t a_i = 0$ and $\sum_{i=1}^t (i-1)a_i = 0$. Now set

$$C = \{\{1, 2, \ldots, i-1, i+i, \ldots, t\}\}_{i=1}^t$$

and

$$\psi_i(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_t) = \sum_{j=1}^{t} (j-i)a_i.$$

The functions ψ_i are computed in such a way that $\ker(\psi_i)$ is the projection of H to the coordinates in C_i.

Let F be the complete t-partite $(t-1)$-uniform hypergraph on t points. Lemma 2.1 applied to F and the above hypergraph $H_{t-1,t}(A, \{S_i\}, C)$ implies that if the system

$$x_i = \sum_{j=1}^{t} (j-i)a_i \in S_i$$

has $o(|A|^t)$ solutions, then we can delete $o(|A|)$ elements from each S_i such that the previous system has no solution. It is clear that $x_1, x_2, \ldots, x_t$ form a t-term arithmetic progression and in fact that any such progression with $x_i \in S_i$ gives rise to $|A|^{t-2}$ solutions of the previous system. Using this we obtain the following:

Theorem 3 (Diagonal Szemerédi Theorem). *For every $\epsilon > 0$ and natural number t there exists $\delta = \delta(\epsilon, t) > 0$ such that if A is an Abelian group, $S_1, S_2, \ldots, S_t$ are subsets in A and there are at most $\delta|A|^2$ pairs $x, d \in A$ with $x + (i-1)d \in S_i$ $(1 \leq i \leq t)$, then we can shrink each S_i by at most $\epsilon|A|$ elements such that the new sets don't have such a configuration.*

This theorem implies Szemerédi's theorem [9] in the following way. Assume that S is a subset of A not containing any nontrivial k-term arithmetic progression. Let us apply Theorem 3 to $S = S_i,\ 1 \leq i \leq t$. We have that if $x + (i-1)d \in S_i$ for every $1 \leq i \leq t$, then d is 0. This means that the condition of Theorem 3 holds with $\delta = |S|/|A|^2 \leq 1/|A|$. Now for every $\epsilon > 0$, if $|A| \geq 1/\delta(\epsilon, t)$ we obtain that every S_i contains a subset Q_i of size at most $\epsilon|A|$ such that the system $\{S_i \setminus Q_i\}_{i=1}^t$ does not contain any "diagonal arithmetic progression". In particular, $S \setminus \bigcup_{i=1}^t Q_i$ has to be empty, and so $|S| \leq t\epsilon|A|$.

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