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NUMBER THEORETIC PROPERTIES OF GENERATING FUNCTIONS RELATED TO DYSON'S RANK FOR PARTITIONS INTO DISTINCT PARTS

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ABSTRACT. Let $Q(n)$ denote the number of partitions of n into distinct parts. We show that Dyson's rank provides a combinatorial interpretation of the well-known fact that $Q(n)$ is almost always divisible by 4. This interpretation gives rise to a new false theta function identity that reveals surprising analytic properties of one of Ramanujan's mock theta functions, which in turn gives generating functions for values of certain Dirichlet L-functions at nonpositive integers.

1. INTRODUCTION AND STATEMENT OF RESULTS

A *partition* λ of a positive integer n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers, written in nonincreasing order, whose sum is n. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we say that λ_i is the ith *part* of the partition, and we write $\ell(\lambda)$ to denote the number of parts of λ. The *rank* of λ is $\lambda_1 - \ell(\lambda)$. For instance, the rank of $(5,3,1,1)$ is $5-4=1$. The *Young diagram* of λ is the partial grid of squares consisting of k rows, aligned at the left, with the ith row containing λ_i squares for each $i \le k$. (See Figure 1.)

Let $p(n)$ denote the number of partitions of n. Ramanujan proved the following famous congruence identities for $p(n)$:

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

Several infinite families of arithmetic congruences for $p(n)$ have been discovered since Ramanujan's time, producing identities such as

$$p(157525693n + 111247) \equiv 0 \pmod{13}.$$

(See [15] for a detailed account of congruences for $p(n)$.)

Identities (1.1)-(1.3) simply begged for a combinatorial explanation. Dyson [10] conjectured that for any m, the number of partitions of $5n+4$ having rank congruent to $m \pmod 5$ is equal to $\frac{1}{5}p(5n+4)$, and the number of partitions of $7n+5$ having rank congruent to $m \pmod 7$ is equal to $\frac{1}{7}p(7n+5)$, thereby providing

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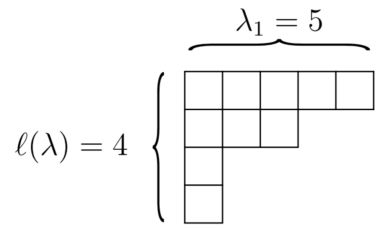

FIGURE 1. The partition $(5, 3, 1, 1)$. Notice that the rank of a partition is the difference between the width and the height of its Young diagram.

a combinatorial interpretation of (1.1) and (1.2) if true. Atkin and Swinnerton-Dyer proved these conjectures in [7]. Interestingly, equation (1.3) does not have a similar combinatorial interpretation given by the rank. Andrews and Garvan later discovered another combinatorial statistic, the crank, that classifies the partitions of $11n+6$ into 11 equal classes determined by the crank modulo 11. (See [5], [11].)

Let $Q(n)$ denote the number of partitions of n into distinct parts. We call such partitions *strict* partitions. For instance, $(5, 3, 2)$ is a strict partition of 10, but $(5, 3, 1, 1)$ is not. Several infinite families of congruence identities have also been shown for Q. (See [13], [14], [17], [18].) In fact, it was shown in [14] that for any prime p, there exist positive integers a and b such that $Q(an+b) \equiv 0 \pmod{p}$ for all positive integers n.

The nearly arithmetic congruence identities modulo 4, first discovered by Rødseth [17], rival (1.1)-(1.3) in their simplicity. The first few such identities are:

$$Q(5n+1) \equiv 0 \pmod{4} \quad \text{if } n \not\equiv 0 \pmod{5}, \tag{1.4}$$

$$Q(7n+2) \equiv 0 \pmod{4} \quad \text{if } n \not\equiv 0 \pmod{7}, \tag{1.5}$$

$$Q(11n+5) \equiv 0 \pmod{4} \quad \text{if } n \not\equiv 0 \pmod{11}, \tag{1.6}$$

$$Q(13n+7) \equiv 0 \pmod{4} \quad \text{if } n \not\equiv 0 \pmod{13}. \tag{1.7}$$

It turns out that $Q(n)$ is also highly divisible by arbitrary powers of 2. Gordon and Ono [12] have shown that for any positive integer j,

$$\lim_{N\to\infty} \frac{\#\{n < N \mid Q(n) \equiv 0 \pmod{2^j}\}}{N} = 1. \tag{1.8}$$

The proof of this fact depends on the theory of Galois representations and is not combinatorial. A simple combinatorial argument due to Franklin [19] shows that $Q(n)$ is divisible by 2 if and only if $n \neq k(3k \pm 1)/2$ for any integer k, thus proving equation (1.8) in the case $j = 1$. Alladi [1] has provided combinatorial interpretations of (1.8) for $j \leq 4$.

We show that Dyson's rank also provides a combinatorial interpretation of (1.4)-(1.7), and more generally of (1.8) for $j = 2$, as follows. Define $T(m, k; n)$ to be the number of strict partitions of n having rank congruent to $m \pmod{k}$.

Theorem 1.1. *Let n be a positive integer. We have*

$$T(0, 4; n) = T(1, 4; n) = T(2, 4; n) = T(3, 4; n)$$

if and only if $24n + 1$ has a prime divisor $p \not\equiv \pm 1 \pmod{24}$, and the largest power of p dividing $24n + 1$ is p^e where e is odd.

TABLE 1. The strict partitions of 12 and of 16 sorted by rank.

Rank (mod 4)	Partitions of 12
0	(10, 2), (7, 4, 1), (7, 3, 2)
1	(11, 1), (8, 3, 1), (7, 5), (5, 4, 2, 1)
2	(9, 2, 1), (8, 4), (6, 3, 2, 1), (5, 4, 3)
3	(12), (9, 3), (6, 5, 1), (6, 4, 2)

Rank (mod 4)	Partitions of 16
0	(14, 2), (11, 4, 1), (11, 3, 2), (10, 6) (8, 5, 2, 1), (8, 4, 3, 1), (7, 6, 3), (7, 5, 4)
1	(15, 1), (12, 3, 1), (11, 5), (9, 4, 2, 1) (8, 7, 1), (8, 6, 2), (8, 5, 3), (6, 4, 3, 2, 1)
2	(13, 2, 1), (12, 4), (10, 3, 2, 1), (9, 6, 1) (9, 5, 2), (9, 4, 3), (6, 5, 4, 1), (6, 5, 3, 2)
3	(16), (13, 3), (10, 5, 1), (10, 4, 2) (9, 7), (7, 6, 2, 1), (7, 5, 3, 1), (7, 4, 3, 2)

To illustrate Theorem 1.1, we sort the partitions of 12 and of 16 by rank in Table 1. Notice that $24 \cdot 12 + 1 = 289 = 17^2$, and so $n = 12$ does not satisfy the conditions of Theorem 1.1. On the other hand, $24 \cdot 16 + 1 = 385 = 5 \cdot 77$, so $n = 16$ satisfies the conditions with $p = 5$.

Notice that if n satisfies the conditions of Theorem 1.1, then $Q(n) \equiv 0 \pmod 4$. It is easily shown that this set of integers contains those of the form $pn + \frac{p^2-1}{24}$, $n \not\equiv 0 \pmod p$, for all primes $p > 3$ not congruent to $\pm s1 \pmod{24}$, thus proving (1.4)-(1.7) combinatorially via the rank.

Theorem 1.1 reveals fascinating properties of the generating functions related to the ranks of strict partitions. For $|z| \le 1$ and $|q| < 1$, define

$$G(z, q) := 1 + \sum_{s=1}^{\infty} \frac{q^{s(s+1)/2}}{(1 - zq)(1 - zq^2) \cdots (1 - zq^s)}.$$

Let $Q(n, r)$ denote the number of partitions of n into distinct parts with rank r. A combinatorial argument shows that

$$G(z, q) = \sum_{n,r} Q(n, r) z^r q^n,$$

where n and r range from 0 to ∞.

The next theorem shows that the specializations of this series at fourth roots of unity z have elegant and useful q-series expansions.

Theorem 1.2. *Let $z, q \in \mathbb{C}$ with $|z| \le 1$, $|q| < 1$. Then*

$$\begin{aligned} G(i, q) &= \sum_{k=0}^{\infty} i^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} i^{k-1} q^{k(3k-1)/2}, \\ G(-i, q) &= \sum_{k=0}^{\infty} (-i)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{k(3k-1)/2}, \end{aligned}$$

where we define $Q(0) = 1$.

The functions $G(1,q)$ and $G(-1,q)$ are both related to automorphic forms in the variable τ where $q = e^{2\pi i\tau}$ (we use this notation throughout). Since we have that

$$qG(1,q^{24}) = \frac{\eta(48\tau)}{\eta(24\tau)},$$

where $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the usual classical weight 1/2 modular form of Dedekind, it follows that $G(1,q)$ is essentially a weight 0 modular form. The work of Andrews, Dyson, and Hickerson shows that $G(-1,q)$ is related to the Fourier expansion of a Maass cusp form that has eigenvalue 1/4 with respect to the hyperbolic Laplacian operator $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$, where $\tau = x+iy$. (See [9].)

This prompts one to ask if the functions $G(z,q)$ have interesting analytic properties when z is an arbitrary root of unity. Theorem 1.2 shows that these series have a simple form when $z = i$ and when $z = -i$. In fact, they are examples of *false theta functions*. To demonstrate this, we first recall some necessary background and notation. A *Dirichlet character* of order a is a map $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying

- $\chi(n+a) = \chi(n)$ for any integer n,
- $\chi(mn) = \chi(m)\chi(n)$ for any integers m, n, and
- $\chi(n) = 0$ for any n such that $\gcd(a,n) > 1$.

The eight Dirichlet characters of order 24 are shown in Table 2.

A *theta function* is a function $\theta(z;\tau)$, where z is a fixed complex number and the domain of τ is the complex upper half-plane $\mathcal{H}$, of the form

$$\theta(z;\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi inz}e^{2\pi in^2\tau} = \sum_{n=-\infty}^{\infty} e^{2\pi inz}q^{n^2}.$$

Several variants of these functions are also called theta functions if they satisfy certain modular transformation laws. In particular, if χ is an even Dirichlet character of order a, then

$$\sum_{n=-\infty}^{\infty}\chi(n)q^{n^2}$$

is a modular form of weight 1/2 over the congruence subgroup $\Gamma_0(4a^2)$ of the full modular group $PSL_2(\mathbb{Z})$. Moreover, these theta functions essentially form a basis of all modular forms of weight 1/2 by a classical theorem due to Serre and Stark [16].

Table 2. The nonzero values of the 8 Dirichlet characters of order 24.

n	1	5	7	11	13	17	19	23
$\chi_0(n)$	1	1	1	1	1	1	1	1
$\chi_1(n)$	1	1	−1	−1	1	1	−1	−1
$\chi_2(n)$	1	−1	1	−1	−1	1	−1	1
$\chi_3(n)$	1	−1	−1	1	−1	1	1	−1
$\chi_4(n)$	1	−1	1	−1	1	−1	1	−1
$\chi_5(n)$	1	−1	−1	1	1	−1	−1	1
$\chi_6(n)$	1	1	1	1	−1	−1	−1	−1
$\chi_7(n)$	1	1	−1	−1	−1	−1	1	1

Notice that, by Theorem 1.2,

$$qG(i,q^{24}) = \sum_{k=0}^{\infty} i^k q^{(6k+1)^2} + \sum_{k=1}^{\infty} i^{k-1} q^{(6k-1)^2}.$$

This closely resembles the theta functions described above, but the coefficients cannot be written as a linear combination of even Dirichlet characters. Thus, we have encountered a false theta function.

False theta function identities can be used to obtain generating functions for the values of Dirichlet L-functions at nonpositive integers. This was first observed by Andrews, Ono and Jiménez-Urroz [4], and by Zagier [20]. Here we show that the identities in Theorem 1.2 also may be used in this way. We first recall the definition of L-functions. Given a Dirichlet character χ, the corresponding Dirichlet L-function is a generalization of the Riemann ζ-function defined by

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1.9}$$

Each L-function has a meromorphic continuation to the entire complex plane. In Section 2.3, we use our expressions for $G(\pm i, z)$ to obtain the following.

Theorem 1.3. *We have*

(1.10)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L(\chi_6, -2n) t^n = e^{-t} + \sum_{n=1}^{\infty} \frac{(1-i)e^{-(12n^2+12n+1)t}}{2\prod_{r=1}^{n}(1-ie^{-24rt})} + \frac{(1+i)e^{-(12n^2+12n+1)t}}{2\prod_{r=1}^{n}(1+ie^{-24rt})}$$

$$= e^{-t} + e^{-t} \sum_{n=1}^{\infty} \frac{e^{-24nt}}{\prod_{r=1}^{n}(1+e^{-48rt})} \tag{1.11}$$

and

(1.12)
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L(\chi_7, -2n) t^n = e^{-t} + \sum_{n=1}^{\infty} \frac{(1+i)e^{-(12n^2+12n+1)t}}{2\prod_{r=1}^{n}(1-ie^{-24rt})} + \frac{(1-i)e^{-(12n^2+12n+1)t}}{2\prod_{r=1}^{n}(1+ie^{-24rt})}.$$

The L-values at negative integers can also be obtained using generalized Bernoulli numbers. The Bernoulli numbers $B_{n,\chi}$ associated with the Dirichlet character χ of order a are defined by the generating function

$$\sum_{m=1}^{a} \chi(m) \frac{te^{mt}}{e^{at}-1} = \sum_{t=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

It is well-known that

$$L(\chi, 1-k) = -\frac{B_{k,\chi}}{k}$$

for any positive integer k. The right hand side of (1.10) is, as a power series in t,

$$2 + 46t + 3985t^2 + \frac{1743623}{3}t^3 + \cdots,$$

which matches the values given by the Bernoulli numbers for χ_6. As another illustration, the right hand side of (1.12) is

$$-48t - 3984t^2 - 581208t^3 - \cdots.$$

In addition to being a false theta function, $G(i,q)$ is related to the famous mock theta functions of Ramanujan, which Bringmann and Ono [8] recently have established are the holomorphic parts of certain weight 1/2 harmonic Maass forms. One famous such function is

$$R(z,q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-zq^k)(1-z^{-1}q^k)}. \tag{1.13}$$

The coefficient of $z^m q^n$ in $R(z,q)$ is the number of partitions of n having rank m. Thus, evaluating $R(z,q)$ at roots of unity z is useful in obtaining congruence relations for $p(n)$ via the rank.

Replacing q by $1/q$ in (1.13), we obtain the following theorem.

Theorem 1.4. *We have*

$$R(i,1/q) = R(-i,1/q) = \frac{1-i}{2}G(i,q) + \frac{1+i}{2}G(-i,q)$$

or alternatively,

$$\begin{aligned} qR(i,1/q^{24}) &= \sum_{n=0}^{\infty}(-1)^n\left(q^{(12n+1)^2} + q^{(12n+5)^2} + q^{(12n+7)^2} + q^{(12n+11)^2}\right) \\ &= q + q^{25} + q^{49} + q^{121} - q^{169} - q^{289} - q^{361} - q^{529} + q^{625} + \cdots . \end{aligned}$$

Thus, the analytic behavior of the false theta functions $G(\pm i,q)$ gives the behavior of $R(\pm i;q)$ for q outside the unit disk! This is a remarkable connection between the rank generating functions of strict and unrestricted partitions.

2. Proofs

We now present the proofs of the main results.

2.1. $Q(n)$ mod 4 via the rank.

Let $\mathcal{D}$ denote the set of all strict partitions (partitions having distinct parts), and let $\mathcal{P}$ denote the set of all (unrestricted) partitions. Let $\mathcal{D}_n$ denote the set of all strict partitions of n.

Define a *pentagonal partition* to be a partition of the form $(2k, 2k-1, \ldots, k+1)$ or $(2k-1, 2k-2, \ldots, k)$ for some positive integer k. The former is a partition of $k(3k+1)/2$, and the latter is a partition of $k(3k-1)/2$. Numbers of the form $k(3k\pm 1)/2$ are called *pentagonal numbers.* An example of each type of pentagonal partition is shown in Figure 2.

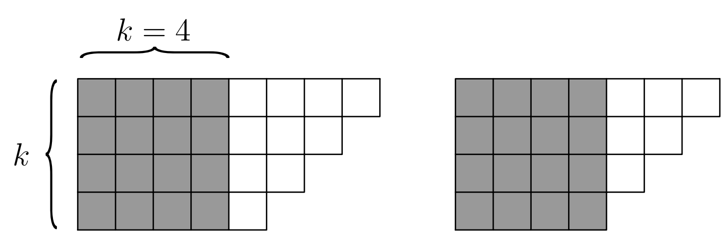

Figure 2. The two types of pentagonal partitions of length k, shown here for $k = 4$.

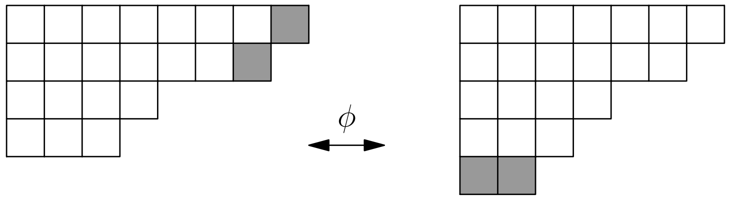

FIGURE 3. Franklin's Involution $\phi : \mathcal{D}'_n \to \mathcal{D}'_n$.

Let $\mathcal{D}'_n$ denote the set of all strict partitions of n that are not pentagonal partitions. For any partition λ, let $m(\lambda)$ be the largest index m such that $\lambda_1 = \lambda_2 + 1 = \lambda_3 + 2 = \cdots = \lambda_m + m - 1$. Also let $s(\lambda)$ denote the smallest part of λ.

Given a partition λ, the *conjugate partition* of λ, denoted λ', is the partition formed by interchanging the rows and columns of its Young diagram.

To prove Theorem 1.1, we first provide a necessary and sufficient condition for the equalities $T(0,4;n) = T(2,4;n)$ and $T(1,4;n) = T(3,4;n)$ to hold.

Lemma 2.1. *If $n \neq k(3k \pm 1)/2$ for any k, we have*

$$T(1,4;n) = T(3,4;n) \text{ and } T(0,4;n) = T(2,4;n).$$

Otherwise, if $n = k(3k+1)/2$, then

$$T(k,4;n) = T(k+2,4;n) + 1 \text{ and } T(k+1,4;n) = T(k+3,4;n).$$

If $n = k(3k-1)/2$, then

$$T(k-1,4;n) = T(k+1,4;n) + 1 \text{ and } T(k,4;n) = T(k+2,4;n).$$

Proof. We require an involution $\phi : \mathcal{D}'_n \to \mathcal{D}'_n$, commonly known as *Franklin's Involution*, defined as follows. Let $\lambda \in \mathcal{D}'_n$, and let $m = m(\lambda)$ and $s = s(\lambda)$. If $s \leq m$, define $\phi(\lambda)$ to be the partition formed by removing the part s from the partition and increasing each of the first s parts by 1. If $s > m$, define $\phi(\lambda)$ to be the partition formed by decreasing each of the first m parts of λ by 1 and inserting a part of size m into the partition. (See Figure 3.) Notice that these operations are not well defined on pentagonal partitions.

It is easily verified that ϕ is an involution. Furthermore, for any nonpentagonal partition λ, the rank of $\phi(\lambda)$ differs from the rank of λ by ± 2. Thus, if $n \neq k(3k \pm 1)/2$, we have $T(1,4;n) = T(3,4;n)$ and $T(0,4;n) = T(2,4;n)$.

If $n = k(3k+1)/2$, there is one pentagonal partition of n, namely $(2k, 2k-1, \ldots, k+1)$, and the rank of this partition is k. Thus $T(k,4;n) = T(k+2,4;n)+1$ and $T(k+1,4;n) = T(k+3,4;n)$.

If $n = k(3k-1)/2$, there is one pentagonal partition of n, namely $(2k-1, 2k-2, \ldots, k)$, and the rank of this partition is $k-1$. Thus $T(k-1,4;n) = T(k+1,4;n)+1$ and $T(k,4;n) = T(k+2,4;n)$. This completes the proof. □

Now, notice that $T(0,4;n) + T(2,4;n) = T(0,2;n)$ and $T(1,4;n) + T(3,4;n) = T(1,2;n)$. Thus, by Lemma 2.1, in order to find exactly when $T(m,4;n) = \frac{1}{4}Q(n)$ for $m = 0,1,2,3$ it suffices to find the difference between the number of partitions of n having even rank, $T(0,2;n)$, and the number having odd rank, $T(1,2;n)$. Let $S(n) = T(0,2;n) - T(1,2;n)$ be this difference. An explicit formula for the function $S(n)$ has already been obtained [3], and we state this result below.

Theorem 2.1. *We have $S(n) = T(24n+1)$, where the function $T(m)$ is defined on the set of integers $m \neq 1$ congruent to $1 \pmod 6$ as follows. Write m in the form $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where each p_i is either a prime congruent to $1 \pmod 6$ or the negative of a prime congruent to $5 \pmod 6$. Then $T(m) = T(p_1^{e_1})T(p_2^{e_2}) \cdots T(p_k^{e_k})$, where*

$$T(p^e) = \begin{cases} 0 & \text{if } p \not\equiv 1 \pmod{24} \text{ and } e \text{ is odd}, \\ 1 & \text{if } p \equiv 13 \text{ or } 19 \pmod{24} \text{ and } e \text{ is even}, \\ (-1)^{e/2} & \text{if } p \equiv 1 \pmod{24} \text{ and } e \text{ is even}, \\ e+1 & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = 2, \\ (-1)^e(e+1) & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = -2. \end{cases}$$

It follows that $S(n) = 0$ if and only if $24n+1$ has a prime divisor $p \not\equiv \pm 1 \pmod{24}$, and the largest power of p dividing $24n+1$ is p^e for some odd positive integer e. Suppose n is a pentagonal number. Then $24n+1 = 24(k(3k\pm 1)/2) + 1 = (6k \pm 1)^2$ for some k, which cannot have a prime raised to an odd power in its prime factorization since it is a perfect square. Thus, if $S(n) = 0$, then n is not a pentagonal number, and so by Lemma 2.1, $T(0,4;n) = T(2,4;n)$ and $T(1,4;n) = T(3,4;n)$. Furthermore, if $S(n) = 0$, then $T(0,4;n) + T(2,4;n) = T(0,2;n) = T(1,2;n) = T(1,4;n) + T(3,4;n)$ by the definition of $S(n)$. Thus $S(n) = 0$ if and only if $T(0,4;n) = T(1,4;n) = T(2,4;n) = T(3,4;n)$. This proves Theorem 1.1.

To analyze the generating functions that arise in studying $S(n)$ and other functions related to the rank, we first recall some standard notation. For any positive integer n, we define

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

and

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

The *Sylvester triangle* of a partition λ is the largest partition of the form $(s, s-1, \ldots, 3, 2, 1)$ such that $s - i + 1 \leq \lambda_i$ for $i = 1, 2, \ldots, s$. An example is shown in Figure 4. Notice that if λ is a strict partition, then λ has the same number of parts as its Sylvester triangle.

We proceed to prove Theorem 1.2, which we restate below. Recall that

$$G(z,q) = 1 + \sum_{s=1}^{\infty} \frac{q^{s(s+1)/2}}{(zq;q)_s}$$

for $|z| \leq 1$ and $|q| < 1$.

Theorem 2.2. *Let $z, q \in \mathbb{C}$ with $|z| \leq 1$, $|q| < 1$. Then*

$$\begin{aligned} G(i,q) &= \sum_{k=0}^{\infty} i^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} i^{k-1} q^{k(3k-1)/2}, \\ G(-i,q) &= \sum_{k=0}^{\infty} (-i)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{k(3k-1)/2}, \end{aligned}$$

where we define $Q(0) = 1$.

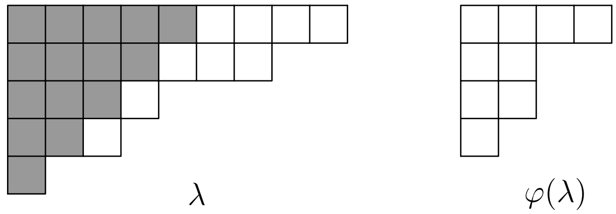

FIGURE 4. A partition λ with its Sylvester triangle shaded and its image under φ.

Proof. We first provide an elementary combinatorial proof of the identity

$$G(z,q) = \sum_{n,r} Q(n,r) q^n z^r.$$

Similar identities have already appeared in the literature (see, for instance, [3]), but we state the proof here for the reader's enjoyment.

Let $Q(n,r,s)$ denote the number of strict partitions with rank r and exactly s parts, and let $p(n,r,s)$ denote the number of partitions of n with largest part at most s and exactly r parts. It is easily verified combinatorially that for any positive integer s,

$$\sum_{n,r} p(n,r,s) q^n z^r = \frac{1}{(zq;q)_s}. \tag{2.1}$$

We now define a map $\varphi : \mathcal{D} \to \mathcal{P}$ as follows. Suppose λ is a partition of n into s distinct parts.

By removing the Sylvester triangle from λ, we are left with a partition $\nu = (\lambda_1 - s, \lambda_2 - (s-1), \ldots, \lambda_s - 1)$ of $n - s(s+1)/2$. We define $\varphi(\lambda)$ to be the conjugate partition ν' of ν. Notice that ν' has at most s parts, and the number of parts of ν' is equal to the rank of λ.

For each nonnegative integer s, φ is a bijection from the set of partitions of n into exactly s distinct parts to the set of partitions ν' having largest part at most s. Hence

$$\sum_{n,r,s} Q(n,r,s) q^n z^r = \sum_s q^{s(s+1)/2} \sum_{n,r} p(n,r,s) q^n z^r, \tag{2.2}$$

where the variables range over all nonnegative integers. Note that

$$\sum_s Q(n,r,s) = Q(n,r). \tag{2.3}$$

By (2.1), (2.2), and (2.3), we have

$$\sum_{n,r} Q(n,r) q^n z^r = 1 + \sum_{s=1}^{\infty} q^{s(s+1)/2} \frac{1}{(zq;q)_s} = G(z,q).$$

Setting $z = i$, we have

$$\begin{aligned} G(i,q) &= \sum_{n,r} Q(n,r) i^r q^n \\ &= \sum_n [T(0,4;n) + iT(1,4;n) - T(2,4;n) - iT(3,4;n)]q^n \\ &= \sum_n [(T(0,4;n) - T(2,4;n)) + i(T(1,4;n) - T(3,4;n))]q^n \\ &= \sum_{k=0}^{\infty} i^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} i^{k-1} q^{k(3k-1)/2}, \end{aligned}$$

where the last equality follows from Lemma 2.1. Analogously,

$$G(-i,q) = \sum_{k=0}^{\infty} (-i)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{k(3k-1)/2}.$$

This completes the proof. □

2.2. **The relation between $R(\pm i, q)$ and $G(\pm i, q)$.** To prove Theorem 1.4, we require the following identity given in Ramanujan's "lost" notebook:

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} \frac{q^n}{(-aq;q)_n(-a^{-1}q;q)_n} &= (1+a)\sum_{n=0}^{\infty} a^{3n} q^{\frac{n(3n+1)}{2}} (1 - a^2 q^{2n+1}) \\ &\quad - \frac{a\sum_{n=0}^{\infty}(-1)^n a^{2n} q^{\frac{n(n+1)}{2}}}{(-aq;q)_\infty(-a^{-1}q;q)_\infty}. \end{aligned} \tag{2.4}$$

Andrews [2] noted that by substituting $a = i$ in (2.4) and taking the real part of both sides, we obtain the identity

$$1 + \sum_{n=1}^{\infty} \frac{q^n}{(-q^2;q^2)_n} = \sum_{n=0}^{\infty} (-1)^n q^{n(6n+1)}(1+q^{4n+1}) + \sum_{n=0}^{\infty} (-1)^n q^{(2n+1)(3n+2)}(1+q^{4n+3}). \tag{2.5}$$

Notice that the left hand side of (2.5) is equal to $R(i, 1/q)$. Replacing q by q^{24} in 2.5 and multiplying by q, we obtain, by Theorem 1.2,

$$\begin{aligned} qR(i, 1/q^{24}) &= \sum_{n=0}^{\infty} (-1)^n \left(q^{(12n+1)^2} + q^{(12n+5)^2} + q^{(12n+7)^2} + q^{(12n+11)^2} \right) \\ &= \frac{1-i}{2} qG(i, q^{24}) + \frac{1+i}{2} qG(-i, q^{24}), \end{aligned}$$

and the result follows.

2.3. **Exponential generating functions for Dirichlet L-values.** In order to prove Theorem 1.3, we first prove the following.

Lemma 2.2. *Let χ_6 and χ_7 denote the Dirichlet characters of order 24 given in Table 2, and let $0 \le t < 1$. We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{1+i}{2} L(\chi_6, -2n) + \frac{1-i}{2} L(\chi_7, -2n) \right) t^n = e^{-t} + \sum_{n=1}^{\infty} \frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1 - ie^{-24rt})} \tag{2.6}$$

and
(2.7)
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{1-i}{2}L(\chi_6,-2n)+\frac{1+i}{2}L(\chi_7,-2n)\right)t^n = e^{-t}+\sum_{n=1}^{\infty}\frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1+ie^{-24rt})}.$$

Proof. Define $H:\mathbb{R}\to\mathbb{C}$ by

$$H(t)=e^{-t}+\sum_{n=1}^{\infty}\frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1-ie^{-24rt})}.$$

By Theorem 1.2, for $t>0$,

$$H(t)=e^{-t}G(i,e^{-24t})=\sum_{k=0}^{\infty}i^k e^{-(6k+1)^2t}+\sum_{k=1}^{\infty}i^{k-1}e^{-(6k-1)^2t}. \tag{2.8}$$

Notice that

$$\frac{(1+i)}{2}\chi_6(6k+1)+\frac{(1-i)}{2}\chi_7(6k+1)=i^k$$

and

$$\frac{(1+i)}{2}\chi_6(6k-1)+\frac{(1-i)}{2}\chi_7(6k-1)=i^{k-1},$$

and that $\chi_6(n)=\chi_7(n)=0$ when n is not of the form $6k+1$ or $6k-1$. Thus (2.8) becomes

$$H(t)=\sum_{n=0}^{\infty}\left(\frac{(1+i)}{2}\chi_6(n)+\frac{(1-i)}{2}\chi_7(n)\right)e^{-n^2t}.$$

Now, let $F:\{s\in\mathbb{C}:\mathrm{Re}(s)>0\}\to\mathbb{C}$ be defined by $F(s)=\int_0^\infty H(t)t^{s-1}dt$. For any s with $\mathrm{Re}(s)>0$, we have

$$F(s) = \int_0^\infty\sum_{n=0}^{\infty}\left(\frac{(1+i)}{2}\chi_6(n)+\frac{(1-i)}{2}\chi_7(n)\right)e^{-n^2t}t^{s-1}dt \tag{2.9}$$

$$= \sum_{n=0}^{\infty}\left(\frac{(1+i)}{2}\chi_6(n)+\frac{(1-i)}{2}\chi_7(n)\right)\int_0^\infty e^{-n^2t}t^{s-1}dt \tag{2.10}$$

since the integral and sum are absolutely convergent for $\mathrm{Re}(s)>0$. Recall that the Γ function is commonly defined as $\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt$. Substituting $u=n^2t$ in the integral in each summand in (2.10), we find

$$\begin{aligned}F(s) &= \sum_{n=0}^{\infty}\left(\frac{(1+i)}{2}\chi_6(n)+\frac{(1-i)}{2}\chi_7(n)\right)\frac{1}{n^{2s}}\int_0^\infty e^{-u}u^{s-1}dt\\ &= \Gamma(s)\left(\frac{(1+i)}{2}\sum_{n=0}^{\infty}\frac{\chi_6(n)}{n^{2s}}+\frac{(1-i)}{2}\sum_{n=0}^{\infty}\frac{\chi_7(n)}{n^{2s}}\right)\\ &= \Gamma(s)\left(\frac{(1+i)}{2}L(\chi_6,2s)+\frac{(1-i)}{2}L(\chi_7,2s)\right).\end{aligned}$$

It is well-known ([6], p. 250) that Γ has an analytic continuation to $\mathbb{C}\backslash\{n\in\mathbb{Z}:n\le 0\}$, with poles of order 1 at the nonpositive integers, defined as follows. For any negative integer n and any s with $n<\mathrm{Re}(s)\le n+1$, $\Gamma(s)=\frac{1}{s(s+1)\cdots(s+n-1)}\Gamma(s+n)$. It is easily verified that the residue of Γ at the negative integer k is $(-1)^n/n!$.

Using the analytic continuations of $L(\chi_6, s)$ and $L(\chi_7, s)$, we can extend $F(s)$ to a meromorphic function on $\mathbb{C}$ that has poles of order 1 at the nonpositive integers and is analytic elsewhere. Moreover, the residue at the pole $s = -n$ of $F(s)$ is

$$\frac{(-1)^n}{n!}\left(\frac{(1+i)}{2}L(\chi_6, -2n) + \frac{(1-i)}{2}L(\chi_7, -2n)\right).$$

Define the complex numbers $a(n)$ by $H(t) = \sum_{n=0}^{\infty} a(n)t^n$, since H is analytic. Then, for any positive integer N,

$$\begin{aligned}\int_0^\infty H(t)t^{s-1}dt &= \int_0^1 \sum_{n=0}^{\infty} a(n)t^{n+s-1}dt + \int_1^\infty H(t)t^{s-1}dt \\ &= \sum_{n=0}^{N} \frac{a(n)}{n+s} + \sum_{n=N+1}^{\infty} \frac{a(n)}{n+s} + \int_1^\infty H(t)t^{s-1}dt.\end{aligned}$$

Since $\sum_{n=N+1}^{\infty} \frac{a(n)}{n+s} + \int_1^\infty H(t)t^{s-1}dt$ is an analytic function of s in the half-plane $\mathrm{Re}(s) > N$, the residue of the pole at $s = -n$ is $a(n)$. Thus

$$a(n) = \frac{(-1)^n}{n!}\left(\frac{(1+i)}{2}L(\chi_6, -2n) + \frac{(1-i)}{2}L(\chi_7, -2n)\right)$$

for all n, and equation (2.6) follows.

The proof of equation (2.7) is analogous. □

Using (2.6) and (2.7) to solve for $\sum \frac{(-1)^n}{n!}L(\chi_6, -2n)t^n$ and $\sum \frac{(-1)^n}{n!}L(\chi_7, -2n)t^n$, we obtain equations (1.10) and (1.12) of Theorem 1.3.

To prove equality (1.11) of Theorem 1.3, note that by Theorem 1.4,

$$qR(i, 1/q^{24}) = \sum_{n=0}^{\infty} \chi_6(n)q^{n^2}.$$

Replacing q by e^{-t}, an argument identical to that for Lemma 2.2 gives the result.

3. Future work

Given the fascinating properties of the functions $G(z, q)$ and $R(z, q)$ when z is a fourth root of unity, it is natural to ask whether Theorems 1.2 and 1.4 are specializations of a more general phenomenon that occurs when z is an arbitrary root of unity. Understanding the behavior of the coefficients of these functions at mth roots of unity may also unlock more information about the distribution of the rank function modulo m, for both strict and unrestricted partitions.

Dyson's rank does not provide a combinatorial interpretation of the fact that $Q(n)$ is usually divisible by 8 in the same manner as it does for 2 and 4. Thus, it may also be fruitful to investigate generalizations of Dyson's rank in order to find a partition statistic that, when taken modulo 2^j, classifies the partitions of n into 2^j equal classes. More generally, perhaps an analog of the crank function that applies to $Q(n)$ is waiting to be discovered.

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